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Characterization of Frequency Stability: Analysis of the Modified Allan Variance and Properties of Its Estimate

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Abstract—An analytical expression for the modified Allan variance is given for each component of the model usually considered to describe the frequency or phase fluctuations in frequency standards. The relation between the Allan variance and the modified Allan variance is specified and compared with that of a previously published analysis. The uncertainty on the estimate of the modified Allan variance calculated from a finite set of measurement data is discussed.

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I. Introduction

Many works [1]-[5] have been devoted to the characterization of the frequency stability of ultrastable frequency sources and have shown that the frequency noise of a generator can be easily characterized by means of the "two-sample variance" [2] of frequency fluctuations, which is also known as the "Allan variance" [2] in the special case where the dead time between samples is zero.

An algorithm for frequency measurements has been developed by J. J. Snyder [6], [7]. It increases the resolution of frequency meters, in the presence of white phase noise. It has been considered in detail by D. W. Allan and J. A. Barnes [8]. They have defined a function called the "modified Allan variance" and they have analyzed its properties for the commonly encountered components of phase or frequency fluctuations [3]. For that purpose, the authors of [8] have expressed the modified Allan variance in terms of the autocorrelation of the phase fluctuations. For each noise component, they have computed the modified Allan variance and deduced an empirical expression for the ratio between the modified Allan variance and the Allan variance.

In this paper, we show that the analytical expression of this ratio can be obtained directly, even for the noise components for which the autocorrelation of phase functions is not defined from the mathematical point of view. We give the theoretical expressions and compare them with those published in [8].

The precision of the estimate of the modified Allan variance is discussed and results related to white phase and white frequency noises are presented.

II. Background and Definitions

In the time domain, the characterization of frequency stability is currently achieved by means of the two-sample variance [2] $\langle\sigma_y^2(2, T, \tau)\rangle$ of fractional frequency fluctuations. It is defined as

$$\langle\sigma_y^2(2, T, \tau)\rangle = \tfrac{1}{2}\langle(\bar{y}_{k+1} - \bar{y}_k)^2\rangle \tag{1}$$

where the quantity $\bar{y}_k$ is the average value of the fractional frequency fluctuations $y(t)$ over the time interval $(t_k, t_k + \tau)$ such that

$$\bar{y}_k = \frac{1}{\tau}\int_{t_k}^{t_k+\tau} y(t)\, dt. \tag{2}$$

In (2), t_k represents the moment at which the kth observation time interval starts. We have

$$t_k = t_0 + kT, \qquad T \geqslant \tau \tag{3}$$

where t_0 is an arbitrary time origin, k is a positive integer, and T is the time interval between the beginning of two successive observations.

In all the following, we assume that the dead time between samples is zero. We then have

$$T = \tau. \tag{4}$$

In this special case, the two-sample variance is well known as the Allan variance $\sigma_y^2(\tau)$

$$\sigma_y^2(\tau) = \langle\sigma_y^2(2, \tau, \tau)\rangle. \tag{5}$$

The relation between the Allan variance and $y(t)$ can be expressed as

$$\sigma_y^2(\tau) = \frac{1}{2\tau^2}\left\langle\left\{\int_{t_k+\tau}^{t_k+2\tau} y(t)\, dt - \int_{t_k}^{t_k+\tau} y(t)\, dt\right\}^2\right\rangle. \tag{6}$$

Equation (6) shows that $\sigma_y^2(\tau)$ is proportional to the true variance of the output of a linear filter with input signal $y(t)$ and impulse response $h_1(t)$ in Fig. 1.

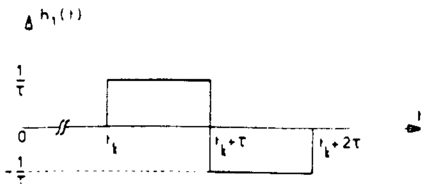

Fig. 1. Variations with time of the linear filter impulse response which represents the signal processing for the Allan variance calculation.

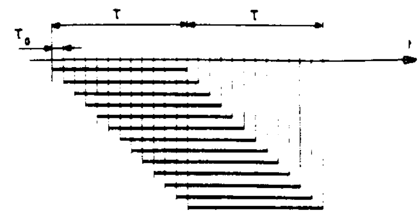

Fig. 2. Illustration of the algorithm considered for the measurement of periodic signal frequency.

The fractional frequency fluctuations $y(t)$ are actually well described by a conventional model which consists of a set of five independent noise processes [2]. Taking into account the finite bandwidth of the processed signal and assuming a single pole filter, the one-sided power spectral density $S_y(f)$ of $y(t)$ can be written as

$$S_y(f) = h_\alpha \frac{f^\alpha}{1 + \left(\frac{f}{f_c}\right)^2} \tag{7}$$

where coefficients h_α do not depend on f. The integer α equals 2, 1, 0, -1, and -2. f_c is the 3-dB bandwidth of the hardware filter.

III. The Modified Allan Variance

The main property of the algorithm developed by J. J. Snyder is to increase the precision on the measure of periodic signal frequency, in presence of white phase noise [7]. It consists in dividing a time interval τ into n cycles of clock period τ_0 such as

$$\tau = n\tau_0. \tag{8}$$

Therefore, from a given observation time interval of duration 2τ, n overlapping time intervals of duration τ can be obtained, as depicted in Fig. 2. Another property of this algorithm is to reduce the total observation time by a factor $n/2$.

Following this way, Allan and Barnes have introduced the "modified Allan variance" [8] such as

$$\mathrm{Mod}\,\sigma_y^2(\tau) = \frac{1}{2\tau^2}\left\langle\left[\frac{1}{n}\sum_{i=1}^{n}\left\{\int_{t_0+(i+n)\tau_0}^{t_0+(i+2n)\tau_0} y(t)\, dt - \int_{t_0+i\tau_0}^{t_0+(i+n)\tau_0} y(t)\, dt\right\}\right]^2\right\rangle. \tag{9}$$

It can be easily seen from (9) that the calculation of each statistical sample involved in the definition of $\mathrm{Mod}\,\sigma_y^2(\tau)$ requires a signal observation of duration 3τ.

The impulse response $h_n(t)$ of the equivalent linear filter consists in finite sum of n shifted impulse responses $h_1(t)$. We have

$$h_n(t) = \frac{1}{n}\sum_{i=1}^{n} h_1(t - i\tau_0). \tag{10}$$

TABLE I
ANALYTICAL EXPRESSION FOR THE MODIFIED ALLAN VARIANCE WITHIN CONDITION $2\pi f_c \tau_0 >> 1$

NOISE TYPE	α	$\text{Mod}\ \sigma_y^2(\tau)$
WHITE P M	2	$\dfrac{3 h_2 f_c}{8 n\pi \tau^2}$
FLICKER P M	1	$\dfrac{h_1}{4\pi^2 n^2 \tau^2}\left[3n\ln(2\pi f_c \tau) + \sum_{k=1}^{n-1}(n-k)\left\{4\ln\left(\dfrac{n^2}{k^2}-1\right) - \ln\left(\dfrac{4n^2}{k^2}-1\right)\right\}\right]$
WHITE F M	0	$\dfrac{h_0}{2\tau} \times \dfrac{n^2+1}{2n^2}$
FLICKER F M	- 1	$\dfrac{2h_{-1}\ln 2}{n^2}\left[\dfrac{4n^2-3n+1}{2} + \dfrac{1}{n^2\ln 2} \times \sum_{k=1}^{n-1}(n-k) \times \left\{\dfrac{n}{2}\left[(k+2n)\ln(k+2n)-(k-2n)\ln(2n-k)\right] + \dfrac{1}{2}(k+n)(k-2n)\ln(k+n) + \dfrac{1}{2}(k-n)(k+2n)\ln\lvert k-n\rvert + 3k^2\ln k - k\left[(n+2k)\ln\left(k+\dfrac{n}{2}\right) - (n-2k)\ln\left\lvert k-\dfrac{n}{2}\right\rvert\right]\right\}\right]$
RANDOM WALK F M	- 2	$\dfrac{33}{40} + \dfrac{1}{8n^2} + \dfrac{1}{20n^4}$

*

In order to illustrate (10), variations with time of the shifted functions $h_1(t - i\tau_0)$ and of the impulse response $h_n(t)$ are represented in Fig. 3(a) and (b), respectively, for $n = 10$.

For $n = 1$, the Allan variance and the modified one are equal. We have

$$\text{Mod}\ \sigma_y^2(\tau) = \sigma_y^2(\tau). \tag{11}$$

One can express (9) in terms of the spectral density $S_y(f)$. We have

$$\text{Mod}\ \sigma_y^2(\tau) = \frac{2}{n^2\pi^2\tau^2} \cdot \left\{ n\int_0^\infty \frac{1}{f^2} S_y(f) \sin^4(\pi f n \tau_0)\, df + 2\sum_{k=1}^{n-1}(n-k)\int_0^\infty \frac{1}{f^2} S_y(f)\cos(2\pi f k \tau_0) \cdot \sin^4(\pi f n \tau_0)\, df \right\}. \tag{12}$$

It should be noted that the integrals involved in (12) are convergent for each noise component. The analytical expression for the modified Allan variance can therefore be deduced directly from this equation.

In the following, it is assumed that the condition $2\pi f_c \tau_0 \gg 1$ is fulfilled. This means that the hardware bandwidth of the measurement system must be much larger than the reference clock frequency.

We have calculated the modified Allan variance for each noise component. Results are reported in Table I. It appears that the analytical expression for Mod $\sigma_y^2(\tau)$ is relatively simple except for flicker phase and flicker frequency noises where it is given as a finite sum of functions depending on n. In order to compare the Allan variance with the modified one, we

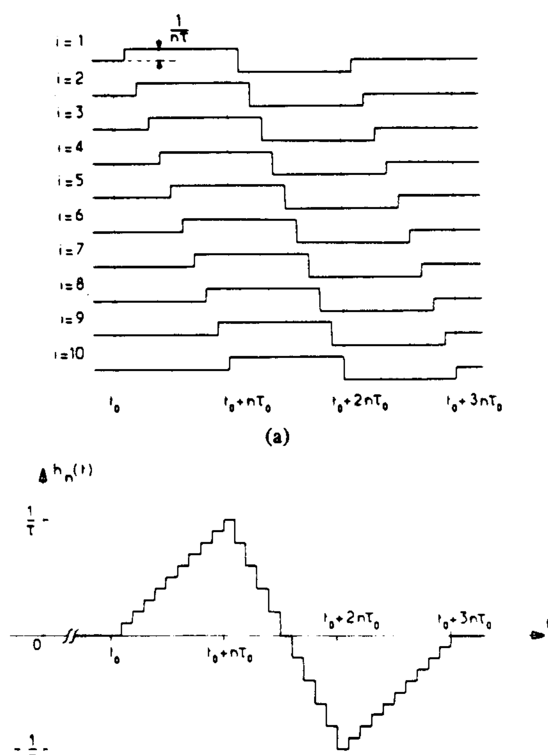

Fig. 3. (a) The impulse response $h_n(t)$, associated with the modified Allan variance calculation, represented as a sum of n shifted impulse response $h_1(t)$. It is assumed $n = 10$. (b) Variations with time of the impulse response $h_n(t)$, in the special case where $n = 10$.

* See Appendix Note # 35

TABLE II
ANALYTICAL EXPRESSIONS AND ASYMPTOTICAL VALUS FOR $R(n)$
(Results Are Valid within Condition $2\pi f_c\tau_0 >> 1$)

α	$R(n)$	$\lim_{n\to\infty} R(n)$
2	$\frac{1}{n}$	0
1	$\frac{1}{n^2}\left[n + \frac{1}{3\ln(2\pi f_c n\tau_0)} \times \sum_{k=1}^{n-1}(n-k)\left\{4\ln\left(\frac{n^2}{k^2}-1\right) - \ln\left(\frac{4n^2}{k^2}-1\right)\right\}\right]$	0
0	$\frac{n^2+1}{2n^2}$	0.5
-1	$\frac{1}{n^2}\left[\frac{4n^2-3n+1}{2} + \frac{1}{n^2\ln n^2} \times \sum_{k=1}^{n-1}(n-k) \times \left\{\frac{n}{2}\left[(k+2n)\ln(k+2n)-(k-2n)\ln(2n-k)\right] + \frac{1}{2}(k+n)(k-2n)\ln(k+n) + \frac{1}{2}(k-n)(k+2n)\ln\lvert k-n\rvert + 3k^2\ln k - k\left[(n+2k)\ln\left(k+\frac{n}{2}\right) - (n-2k)\ln\lvert k-\frac{n}{2}\rvert\right]\right\}\right]$	0.787
-2	$\frac{33}{40} + \frac{1}{8n^2} + \frac{1}{20n^4}$	0.825

*

consider the ratio $R(n)$ defined in [8] as

$$R(n) = \text{Mod}\,\sigma_y^2(\tau)/\sigma_y^2(\tau). \tag{13}$$

The analytical expressions for $R(n)$, deduced from Table I, are reported in Table II. One can see that $R(n)$ does not depend on the product $f_c\tau_0$, except for flicker phase noise modulation. The asymptotic values of $R(n)$ are also listed in Table II.

Fig. 4 depicts the variations of $R(n)$ with n. It shows that, for large values of n, white phase and flicker phase noise modulations have different dependences. As outlined in [8], this gives a means to easily distinguish these two noise processes, in the time domain. For large n, and for $\alpha = 0, -1, -2$, $R(n)$ remains a constant. Consequently the Allan variance can be deduced from the modified one, for these noise processes.

A comparison with results of [8] shows a good agreement for $\alpha = 2$, 0 and -2. But, for $\alpha = 1$ and -1, our expressions for the modified Allan variance and ratio $R(n)$ disagree, especially for flicker phase noise modulation. This discrepancy might be due to the fact that in [8], Mod $\sigma_y^2(\tau)$ is expressed in terms of the autocorrelation function of phase fluctuations which is not defined for $\alpha = 1$.

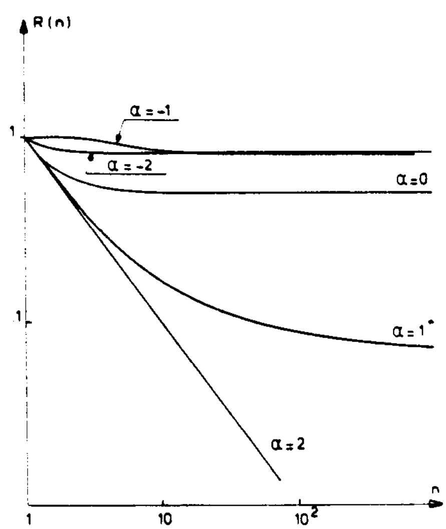

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Fig. 4. Variations with n of the ratio $R(n)$, for fractional frequency fluctuations with power law spectrum $S_y(f) = h_\alpha \cdot (1/1 + (f/f_c)^2)f^\alpha$, within condition $2\pi f_c\tau_0 >> 1$. (*For $\alpha = +1$, $R(n)$ is a function of f_c and τ_0. The reported variations are for $2\pi f_c\tau_0 = 10^4$.)

IV. Uncertainty on the Estimate of the Modified Allan Variance

Equation (9) shows that the definition of the modified Allan variance theoretically implies an infinite set of time intervals.

Practically, one can only estimate this quantity from a finite set of m successive cycles similar to the one depicted in Fig. 3(b).

Let Mod $\hat{\sigma}_y^2(\tau)$ be the estimated modified Allan variance (EMAV) such as

$$\text{Mod}\,\hat{\sigma}_y^2(\tau) = \frac{1}{2\tau^2 n^2} \times \frac{1}{m}\sum_{k=1}^{m}\left\{\sum_{i=1}^{n} A_{i,k}\right\}^2 \tag{14}$$

where

$$A_{i,k} = \int_{t_0+(i+n)\tau_0+(k-1)3n\tau_0}^{t_0+(i+2n)\tau_0+(k-1)3n\tau_0} y(t)\,dt - \int_{t_0+i\tau_0+(k-1)3n\tau_0}^{t_0+(i+n)\tau_0+(k-1)3n\tau_0} y(t)\,dt. \tag{15}$$

* See Appendix Note # 35

The EMAV is a random function of m. Its calculation requires an observation time of duration $3m\tau$.

We consider ϵ, the fractional deviation of the EMAV relative to the modified Allan variance defined as follows:

$$\epsilon = \frac{\text{Mod } \hat{\sigma}_y^2(\tau) - \text{Mod } \sigma_y^2(\tau)}{\text{Mod } \sigma_y^2(\tau)}. \tag{16}$$

The standard deviation $\sigma(\epsilon)$ of ϵ defines the relative uncertainty on the measurement of the modified Allan variance, due to the finite number of averaging cycles. We have

$$\sigma(\epsilon) = \frac{1}{\text{Mod } \sigma_y^2(\tau)} \{\sigma^2 [\text{Mod } \hat{\sigma}_y^2(\tau)]\}^{1/2} \tag{17}$$

where $\sigma^2 [\text{Mod } \hat{\sigma}_y^2(\tau)]$ denotes the true variance of the EMAV such as

$$\sigma^2 [\text{Mod } \hat{\sigma}_y^2(\tau)] = \langle [\text{Mod } \hat{\sigma}_y^2(\tau)]^2 \rangle - [\text{Mod } \sigma_y^2(\tau)]^2. \tag{18}$$

We assume that the fluctuations $y(t)$ are normally distributed [10]. One can therefore express $\langle [\text{Mod } \hat{\sigma}_y^2(\tau)]^2 \rangle$ as

$$\begin{aligned} m^2 \langle [\text{Mod } \hat{\sigma}_y^2(\tau)]^2 \rangle &= (m^2 + 2m) [\text{Mod } \sigma_y^2(\tau)]^2 \\ &\quad + 4 \sum_{p=1}^{m-1} (m - p) \left\{ 2 \sum_{i=1}^{n-1} (n - i) I_n + n I_0 \right\}^2 \end{aligned} \tag{19}$$

where I_n are integrals which depend on n and on the noise process. We have

$$\begin{aligned} 8\pi^2\tau^2 n^2 I_n = \int_0^\infty \frac{S_y(f)}{f^2} \cos 6\pi n p f \tau_0 & \\ \times \{6 \cos 2\pi f \tau_0 i - 4 \cos 2\pi f \tau_0 (i + n) & \\ - 4 \cos 2\pi f \tau_0 (i - n) + \cos 2\pi f \tau_0 (i + 2n) & \\ + \cos 2\pi f \tau_0 (i - 2n)\} \, df. & \end{aligned} \tag{20}$$

For each noise component, the expression for $\sigma(\epsilon)$ can be deduced from the calculation of integrals involved in (20). These expressions are generally lengthy and complicated except for white phase and white frequency noise modulations, where integrals I_n equal zero. We have limited the present analysis to these two noise components. We get for $\sigma(\epsilon)$

$$\sigma(\epsilon) = \frac{2}{m}, \quad \text{for } \alpha = 2 \text{ and } 0. \tag{21}$$

We now compare (21) with previously published results related to the estimate of the Allan variance [5]. For a given time observation of duration $3m\tau$, it can be easily deduced from [5] that the relative uncertainty on the estimate of the Allan variance varies asymptotically as $1.14\, m^{-1/2}$ and $1.0\, m^{-1/2}$ for $\alpha = 2$ and 0, respectively. For these two noise components, the uncertainty on the EMAV is larger than the uncertainty on the estimated Allan variance, but of the same order of magnitude.

V. Conclusion

We have calculated the analytical expression for the modified Allan variance for each component of the model usually considered to characterize random frequency fluctuations in precision oscillators. These expressions have been compared with previously published results and the link between the Allan variance and the modified Allan variance has been specified.

The uncertainty on the estimate of the modified Allan variance has been studied and numerical values have been reported for white phase and white frequency noise modulations.

In conclusion, the modified Allan variance appears to be well suited for removing the ambiguity between white and flicker phase noise modulation. Nevertheless, the calculation of the modified Allan variance requires signal processing which is complicated, compared to the Allan variance. In the presence of white or flicker phase noise, the Allan variance cannot be easily deduced from the modified Allan variance. Furthermore, for a given source exhibiting different noise components, the determination of the Allan variance from the modified one is difficult to perform. For most of time-domain measurements, the use of the Allan variance is preferred.

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