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Characterization and measurement of time and frequency stability *

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The roles which spectral density of fractional frequency fluctuations, two-sample variance, and power spectra play in different parts of the electromagnetic spectrum are introduced. Their relationship is discussed. Data acquisition in the frequency and the time domain is considered, and examples are given throughout the spectrum. Recently proposed methods for the characterization of a single high-quality frequency source are briefly described. Possible difficulties and limitations in the interpretation of measurement results are specified, mostly in the presence of a dead time between measurements. The link between past developments in the field, such as two-sample variance and spectral analysis from time domain measurement, and recently introduced structure functions is emphasized.

1. INTRODUCTION

Progress in the characterization of time and frequency stability has been initiated owing to the work of the various authors of papers delivered at the IEEE-NASA Symposium on Short Term Frequency Stability [1964] and of articles published in a special issue of the *Proceedings of the IEEE* [1966]. Presently widespread definitions of frequency stability have been given by *Barnes et al.* [1971]. Many of the most important articles on the subject of time and frequency have been gathered in the *NBS Monograph 140* [1974]. Since that time, many papers have been published which outline different aspects of the field. Owing to the extent of the subject, they will be only partly reviewed here. We will emphasize recently proposed principles of measurements and recent developments in the time domain characterization of frequency stability. The subject of time prediction and modeling as well as its use for estimation of the spectrum of frequency fluctuations [*Percival*, 1978] are beyond the scope of this paper. Recent reviews which outline several different aspects of the field of time and frequency characterization have been published [*Barnes*, 1976; *Winkler*, 1976; *Barnes*, 1977; *Rutman*, 1978; *Kartaschoff*, 1978].

2. DEFINITIONS: MODEL OF FREQUENCY FLUCTUATIONS

The instantaneous output voltage of a frequency generator can be written as

$$v(t) = [V_0 + \Delta V(t)] \cos [2\pi\nu_0 t + \varphi(t)] \quad (1)$$

where V_0 and ν_0 are constants which represent the nominal amplitude and frequency, respectively. $\Delta V(t)$ and $\varphi(t)$ denote time-dependent voltage and phase variations.

Fractional amplitude fluctuations are defined by

$$\varepsilon(t) = \Delta V(t)/V_0 \quad (2)$$

A power spectral density of fractional amplitude fluctuations $S_\varepsilon(f)$ can be introduced if amplitude fluctuations are random and stationary in the wide sense. Usually, for high-quality frequency sources, one has

$$|\varepsilon(t)| \ll 1 \quad (3)$$

and amplitude fluctuations are neglected. However, it is known that amplitude fluctuations can be converted into phase fluctuations in electronic circuits used for frequency metrology [*Barillet and Audoin*, 1976; *Bava et al.*, 1977*a*] and that they may perturb measurement of phase fluctuations [*Brendel et al.*, 1977]. It is then likely that amplitude fluctuations will become the subject of more detailed analysis in the future.

According to the conventional definition of instantaneous frequency we have

$$\nu(t) = \nu_0 + (1/2\pi)\dot{\varphi}(t) \quad (4)$$

In a stable frequency generator the condition

$$|\dot{\varphi}(t)|/2\pi\nu_0 \ll 1 \quad (5)$$

is generally satisfied.

We will use the following notations [*Barnes et*

* See Appendix Note # 24

al., 1971]:

$$x(t) = \frac{\varphi(t)}{2\pi\nu_0} \quad \text{and} \quad y(t) = \frac{\dot{\varphi}(t)}{2\pi\nu_0} \tag{6}$$

where $x(t)$ and $y(t)$ are the fractional phase and frequency fluctuations, respectively. The quantity $x(t)$ represents the fluctuation in the time defined by the generator considered as a clock.

At first, we will make the following assumptions:

1. The quantities $x(t)$ and $y(t)$ are random functions of time with zero mean values, which implies that systematic trends are removed [*Barnes et al.*, 1971]. They might be due to ageing or to imperfect decoupling from environmental changes such as temperature, pressure, acceleration, or voltage. Characterization of drifts will be considered in section 8.

2. The statistical properties of the stable frequency generators are described by a model which is stationary of order 2. This point has been fully discussed in the literature [*Barnes et al.*, 1971; *Boileau and Picinbono*, 1976; *Barnes*, 1976]. This assumption allows one to derive useful results and to define simple data processing for the characterization of frequency stability.

Actual experimental practice shows that, besides long-term frequency drifts, the frequency of a high-quality frequency source can be perturbed by a superposition of independent noise processes, which can be adequately represented by random fluctuations having the following one-sided power spectral density of fractional frequency fluctuations:

$$S_y(f) = \sum_{\alpha=-2}^{2} h_\alpha f^\alpha \tag{7}$$

$S_y(f)$ is depicted in Figure 1. Its dimensions are Hz^{-1}. Lower values of α may be present in the spectral density of frequency fluctuations. They have not been clearly identified yet because of experimental difficulties related to very long term data acquisition and to control of experimental conditions for long times. Moreover, the related noise processes may be difficult to distinguish from systematic drifts.

Finite duration of measurements introduces a low-frequency cutoff which prevents one from obtaining information at Fourier frequencies smaller

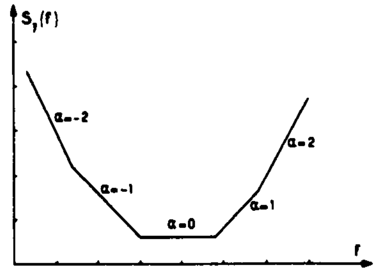

Fig. 1. Asymptotic log-log plot os $S_y(f)$ for commonly encountered noise processes.

than $1/\theta$, approximately, where θ is the total duration of the measurement [*Cutler and Searle*, 1966]. Alternatively, this made it possible to invoke physical arguments to remove some possible mathematical difficulties related to the divergence of $S_y(f)$ as $f \rightarrow 0$ for $\alpha < 0$.

Furthermore, high pass filtering is always present in the measuring instruments or in the frequency generator to be characterized. It insures convergence conditions at the higher-frequency side of the power spectra for $\alpha > 0$.

The spectral density of fractional phase fluctuations is also often considered. From (6), one can write, at least formally,

$$S_x(f) = (1/4\pi^2 f^2) S_y(f) \tag{8}$$

The dimensions of $S_x(f)$ are $\mathrm{s}^2\ \mathrm{Hz}^{-1}$. Similarly, the spectral density of phase fluctuations $\varphi(t)$ is such as

$$S_\varphi(f) = (2\pi\nu_0)^2 S_x(f) \tag{9}$$

It is expressed in $(\mathrm{rad})^2\ \mathrm{Hz}^{-1}$.

The quantity $\mathscr{L}(f)$ [*Halford et al.*, 1973] is sometimes considered to characterize phase fluctuations. If phase fluctuations at frequencies $>f$ are small compared with 1 rad, one has

$$\mathscr{L}(f) = \tfrac{1}{2} S_\varphi(f) \tag{10}$$

where $S_\varphi(f)$ is the spectral density of phase fluctuations of the frequency generator considered. The definition of $\mathscr{L}(f)$ implies a connection with the radio frequency spectrum, and its use is not recommended.

Since the class of noise processes for which $y(t)$ is stationary is broader than that for which the

TABLE 1. Designation of noise processes

α	Designation	Class of Stationarity
2	white noise of phase	stationary phase fluctuations
1	flicker noise of phase	stationary first-order phase increments
0	white noise of frequency	stationary first-order phase increments
−1	flicker noise of frequency	stationary second-order phase increments
−2	random walk of frequency	stationary second-order phase increments

phase is stationary [*Boileau and Picinbono*, 1976], $S_y(f)$ should preferably be used in mathematical analysis. However, it is true that many experimental setups transduce $\varphi(t)$ into voltage fluctuations and allow one to experimentally determine an estimate of its power spectral density $S_\varphi(f)$.

Table 1 shows the designations of the noise processes considered. It also indicates the class of stationarity to which they pertain, as will be justified later on.

$S_y(f)$ is one of the recommended definitions of frequency stability [*Barnes et al.*, 1971]. It gives the widest information on frequency deviations $y(t)$ within the limits stated previously.

3. NOISE PROCESSES IN FREQUENCY GENERATORS

The white phase noise ($\alpha = 2$) predominates for f large enough. It is the result of the additive thermal (for the lower part of the electromagnetic spectrum, including microwaves) or quantum (for optical frequencies) noise which is unavoidably superimposed on the signal generated in the oscillator [*Cutler and Searle*, 1966]. It leads to a one-sided spectral density $S_y(f)$ of the form $FkTf^2/\nu_0^2P$ or $Fh\nu_0 f^2/\nu_0^2 P$, depending on the frequency range, where k is Boltzmann's constant, h is Planck's constant, T is the absolute temperature, F is the noise figure of the components under consideration, and P is the power delivered by atoms.

The flicker phase noise ($\alpha = 1$) is generated mainly in transistors, where this noise modulates the current [*Halford et al.*, 1968; *Healey*, 1972]. The theory of this noise is not yet very well understood. Diffusion processes across junctions of semiconductor devices may produce this noise.

White noise of frequency ($\alpha = 0$) is present in oscillators. It is the result of the noise perturbation in the generation of the oscillation which is due to white noise within the bandwidth of the frequency-determining element of the oscillator [*Blaquière*, 1953*a*, *b*]. It is often masked by other types of noise but has been observed in lasers [*Siegman and Arrathoon*, 1968] and more recently in masers [*Vessot et al.*, 1977]. The one-sided spectral density of fractional frequency fluctuations is then kT/PQ^2 of $h\nu_0/PQ^2$ depending on the frequency range, as stated above. Q is the quality factor of the frequency-determining element.

White noise of frequency is typical of passive frequency standards such as cesium beam tube and rubidium cell devices as well as stabilized lasers. It is related to the shot noise in the detection of the resonance to which an oscillator is slaved [*Cutler and Searle*, 1966].

Flicker noise of frequency and the random walk frequency noise for which $\alpha = -1$ and -2, respectively, are sources of limitation in the long-term frequency stability of frequency sources. They are observed in active devices as well as passive ones. For instance, flicker and random walk frequency noises have been observed in quartz crystal resonators [*Wainwright et al.*, 1974] and rubidium masers [*Vanier et al.*, 1977]. The origin is not well understood yet. It might be connected, in the first case, with fluctuations in the phonon energy density [*Musha*, 1975].

Figures 3 and 5 show, for the purpose of illustration, $S_y(f)$ for a hydrogen maser for $10^{-5} \lesssim f \lesssim 3$ Hz [*Vessot et al.*, 1977] and for an iodine-stabilized He-Ne laser for $10^{-2} \lesssim f \lesssim 100$ Hz [*Cérez et al.*, 1978]. In both cases, $S_y(f)$ is derived from the results of time domain frequency measurements

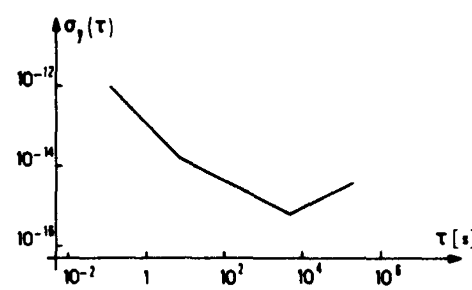

Fig. 2. Frequency stability, characterized by the root mean square of the two-sample variance of fractional frequency fluctuation, of a hydrogen maser. The part of the graph with a slope of $-\frac{1}{2}$ is typical of white noise of frequency.

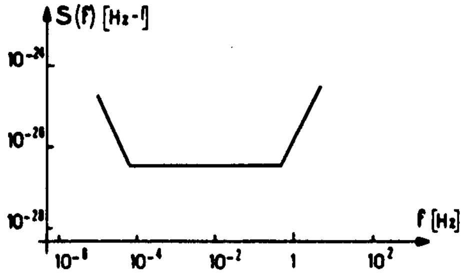

Fig. 3. Spectral density of fractional frequency fluctuations of a hydrogen maser. The parts of the graph with slopes 0 and 2 coincide with theoretical expectations.

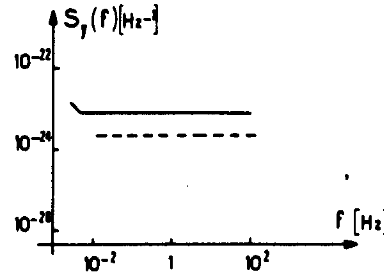

Fig. 5. Spectral density of fractional frequency fluctuations of a He-Ne iodine-stabilized laser. The solid line represents the spectral density of fractional frequency fluctuations corresponding to experimental results, and the dotted line represents the expected value of $S_y(f)$.

(see section 6.4), as shown in Figures 2 and 4 but is very close to theoretical limits specified above.

It is worth pointing out here that in systems where a frequency source is frequency slaved to a frequency reference or phase locked to another frequency generator the different kinds of noise involved are filtered in the system [*Cutler and Searle,* 1966; C. Audoin, unpublished manuscript, 1976]. In these cases, at the output of the system, one can find noise contributions pertaining to the model (7) but appearing on the Fourier frequency scale in an order different than that shown in Figure 1. This is depicted in Figures 6 and 7 for the case of a cesium beam frequency standard consisting of a good quartz crystal oscillator which is frequency controlled by a cesium beam tube resonator.

The model for the frequency fluctuations is more useful if the noise processes can be assumed to be gaussian ones (in particular, momenta of all orders can then be expressed with the help of momenta of second order). The deviation of the frequency being the result of a number of elementary perturbations, this assumption seems a reasonable one. Furthermore, the normal distribution of $\bar{y}$, the mean value of frequency fluctuations averaged over time interval τ as defined in (16), has been experimentally checked for $\alpha = 2, 1, 0,$ and -1 [*Lesage and Audoin,* 1973, 1977]. This is shown in Figure 8 for white noise of frequency, for instance.

4. MEASUREMENTS IN THE FREQUENCY DOMAIN *

Measurement of power spectral density of frequency and phase fluctuations can be performed in the frequency domain for Fourier frequencies greater than a few 10^{-3} Hz owing to the availability of good low-frequency spectrum analyzers.

4.1. *Use of a frequency discriminator*

Frequency discriminators are of current use to characterize radio frequency and microwave generators. A resonant device such as a tuned circuit or a microwave cavity acts as a transducer which

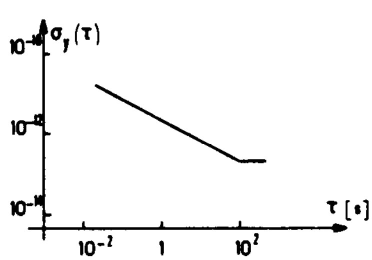

Fig. 4. Frequency stability, characterized by the root mean square of the two-sample variance of fractional frequency fluctuations, of a He-Ne iodine-stabilized laser.

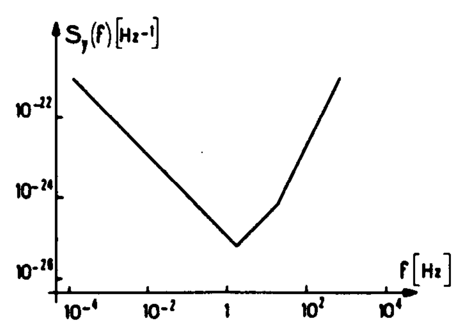

Fig. 6. Spectral density of fractional frequency fluctuations of a good quartz crystal oscillator.

* See Appendix Note # 6

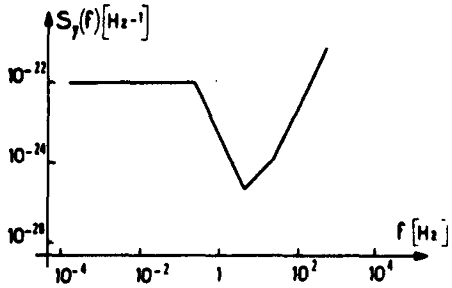

Fig. 7. Spectral density of fractional frequency fluctuations of the same quartz crystal oscillator as in Figure 6 but frequency controlled by a Cs beam tube resonance.

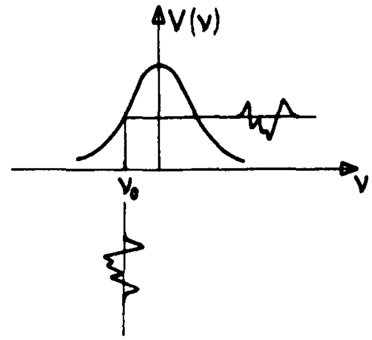

Fig. 9. Principle of frequency to voltage transfer in a frequency discriminator.

transforms frequency to voltage fluctuations. This method can be applied to optical frequency sources too, as shown in Figures 9 and 10. Here the source is, for instance, a CW dye laser, and the frequency selective device is a Fabry-Perot etalon. The second light pass allows one to compensate for the effects of amplitude fluctuations and to adjust to a null the mean value of the output voltage. The slope of this frequency discriminator equals 1 V MHz^{-1}, typically, with a good Fabry-Perot etalon in the visible.

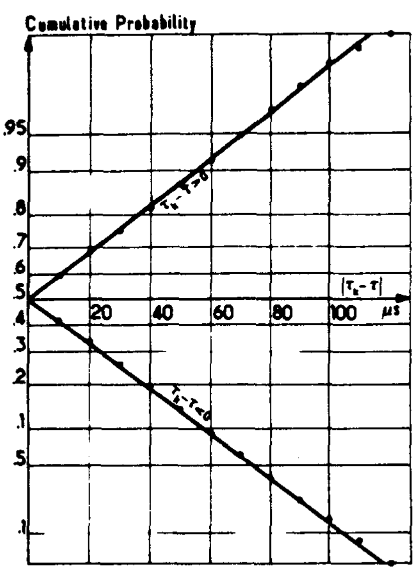

Fig. 8. Distribution of counting time results for white frequency noise (cesium beam frequency standards, τ = 10 s) in Galtonian coordinates. Circles represent the cumulative probability corresponding to $|\tau_k - \tau|$ with $\tau = \langle\tau_k\rangle$. Solid lines correspond to the normal distribution of the same width.

4.2. *Use of a phase detector*

This technique is well suited for the study of frequency sources in the radio frequency domain 0.2 MHz $< \nu_0 <$ 500 MHz, in a range where very low noise balanced-diode mixers which utilize Schottky barrier diodes are available. This technique has mainly been promoted by the National Bureau of Standards [*Shoaf*, 1971; *Walls and Stein*, 1977].

Figure 11 shows the principle of the determination of the phase fluctuations in frequency multipliers, for instance. The two frequency multipliers are driven by the same source. A phase shifter is adjusted in order to satisfy the quadrature condition. One then has

$$v(t) = D\,[\varphi_1(t) - \varphi_2(t)] \qquad (11)\ *$$

where D is a constant and φ_1 and φ_2 are the phase fluctuations introduced in the devices under test. It is assumed that the mixer is properly used to allow a balance of the phase and amplitude fluctuations of the frequency source.

This technique is often used to characterize phase fluctuations of two separate frequency sources of the same frequency. The quadrature condition is

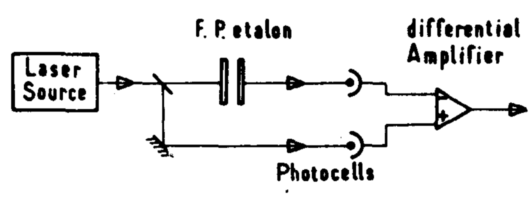

Fig. 10. Principle of frequency noise analysis of a dye laser.

* See Appendix Note # 25

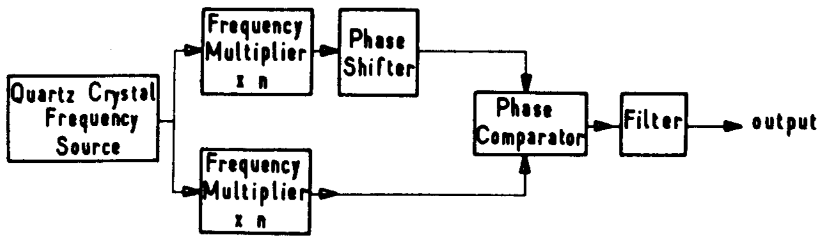

Fig. 11. Principle of measurement of phase noise with a high-quality balanced mixer used as a phase comparator.

insured by phase locking the reference oscillator, number 2 of Figure 12, to the oscillator under test. Fluctuations of the output voltage $u(t)$ at frequencies f larger than the frequency cutoff of the phase loop are proportional to phase fluctuations of oscillator number 1. On the contrary, components of $u(t)$ at frequencies smaller than the above frequency cutoff are representative of frequency fluctuations of oscillator number 1.

The requirement of having a reference oscillator of the same quality as the oscillator to be tested may be inconvenient. It has recently been shown that the phase noise of a single oscillator can be measured by using the mixer technique, but with a delay line [*Lance et al.*, 1977]. Figure 13 shows a schematic of the setup. The signal from the frequency source is split into two channels. The reference channel includes a phase shifter for the purpose of adjustment. It feeds one of the mixer inputs. The other channel delays the signal before it is applied to the second mixer input. It can be seen that the power spectra density of the mixer output is proportional to $(2\pi f \tau_d)^2 S_\varphi(f)$, where τ_d is the delay. The sensitivity of this technique is then reduced for low Fourier frequencies. However, some signal to noise enhancement can be achieved in a more elaborate configuration with two differential delay line systems in which cross-spectrum analysis is performed on the signal output from the two delay line systems [*Lance et al.*, 1978].

Another method has been proposed to determine the power spectrum of fractional frequency fluctuations of a single high-quality frequency source [*Groslambert*, 1977]. It is shown in Figure 14. Two auxiliary oscillators, which do not need to be of the same quality as the oscillator under test, are used. They are phase locked to the frequency generator to be characterized. The control voltages $v_1(t)$ and $v_2(t)$ are appropriately filtered in order to obtain, at their outputs, a voltage $v_1'(t) = K_1(\dot{\varphi}_1 - \dot{\varphi}_0)$ and $v_2'(t) = K_2(\dot{\varphi}_2 - \dot{\varphi}_0)$, respectively, where K_1 and K_2 are constants and the subscripts 0, 1, and 2 refer to the oscillator under test, oscillator number 1, and oscillator number 2, respectively. It can be shown that the cross-correlation function of v_1' and v_2' is proportional to the autocorrelation function of the frequency fluctuations of the oscillator under test. Its spectral density of fractional frequency fluctuations can then be obtained via Fourier transform.

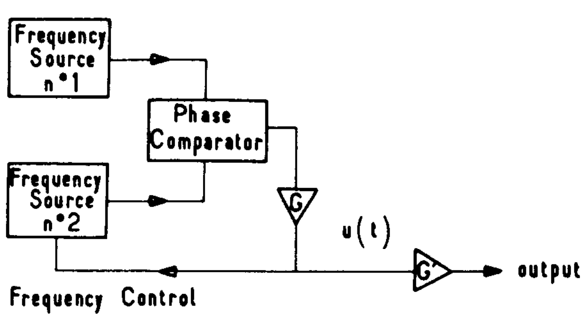

Fig. 12. Principle of phase noise measurement of oscillators. A phase lock loop insures the phase quadrature of the two phase-compared signals.

4.3. *Precision of measurement in the frequency domain*

A spectrum analyzer includes a filter of bandwidth Δf, centered at frequency f, a nonlinear device which measures the power in the filtered signal, and a low pass filter which integrates the output signal for the time T. The integration time T is not infinite, and the filter bandwidth Δf is not extremely narrow. Only an estimate $\hat{S}(f, \Delta f, T)$ of $S(f)$ can then be obtained. Well-known results show that the precision p in the measurement of the power spectral density of a gaussian process is given by

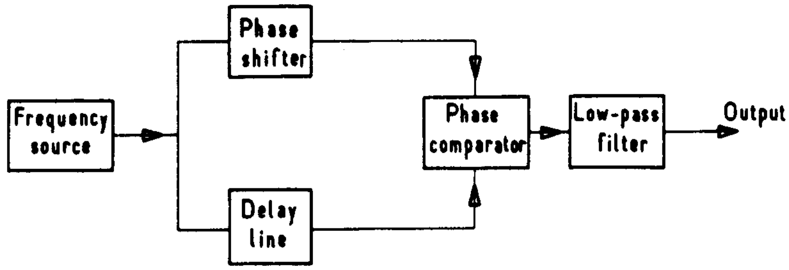

Fig. 13. Principle of phase noise measurement of a single-frequency source with a delay line.

$$p \simeq (T \cdot \Delta\nu)^{-1/2} \qquad (12)*$$

No attempt has been made, to our knowledge, to specify p more accurately for the different noise processes which can be encountered in frequency metrology of stable sources.

5. MEASUREMENTS IN THE TIME DOMAIN

Time and frequency counting techniques are well known [*Cutler and Searle,* 1966]. They are the easiest to implement to provide information on the low-frequency content ($f \leq 1$ Hz) of the power spectra of fractional frequency fluctuations.

5.1. *The beat frequency method*

A beat note at frequency ν_1 is obtained from two frequency sources under test, with frequencies ν_0 and ν_0', respectively, such that $\nu_0 \simeq \nu_0'$. If $\Delta\nu_0$ and $\Delta\nu_0'$ denote the frequency fluctuations of the two sources and $\Delta\nu_1$ the frequency fluctuations of the beat note, one has

$$y_1 = \frac{\Delta\nu_1}{\nu_1} = \frac{\nu_0}{\nu_1}\left|\frac{\Delta\nu_0}{\nu_0} - \frac{\Delta\nu_0'}{\nu_0'}\right| \qquad (13)$$

The fractional frequency fluctuations of the beat note are then proportional to those of the frequencies ν_0 and ν_0' but multiplied by the factor ν_0/ν_1 which is much larger than unity.

With stable generators at frequencies lower than approximately 100 GHz the frequency fluctuations are small enough that the beat note can be at low frequency. The counter is then used as a period meter, and a high precision in the measurement is achievable.

Optical frequency standards show larger frequency fluctuations in absolute value. For instance, a laser stabilized at 500 THz (λ = 0.6 μm) with a fractional frequency stability of 1×10^{-13} exhibits frequency fluctuations of 50 Hz. They can be easily measured if the beat note is at 50 MHz, say, when the counter is used as a frequency meter. In the case of iodine-stabilized He-Ne lasers the beat note is easily obtained by locking the two lasers to different hyperfine components of the considered iodine transition. Otherwise, the frequency offset technique is used [*Barger and Hall,* 1969].

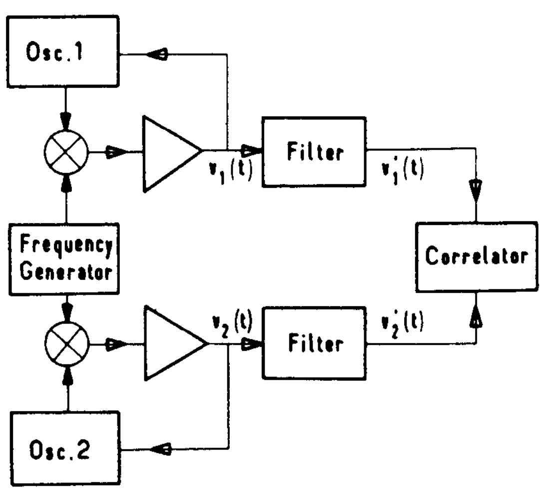

Fig. 14. Principle of phase noise measurement of a single high-quality frequency source with a correlator.

5.2. *The time difference method*

The time difference method [*Allan and Daams,* 1975] must be used with time standards which deliver pulses as time scale marks. Distant time comparison and synchronization by TV pulses, light pulses, Loran-C pulses, for instance, pertain to this category. It provides information on the relative phases of the two clocks under test.

Time interval measurements being very precise

* See Appendix Note # 26

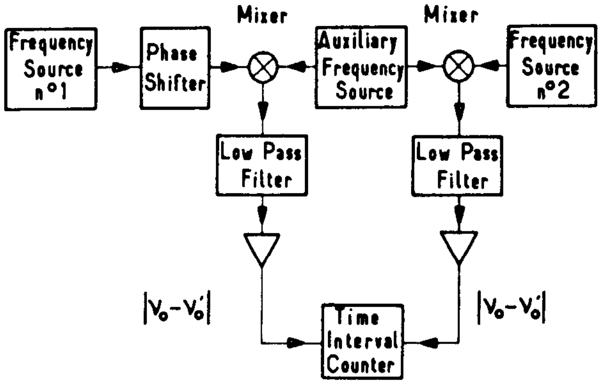

Fig. 15. Principle of the time difference method.

(a precision of 10–100 ns is typical, depending on the class of the counter), they are also used with c.w. frequency generators, as shown in Figure 15. This method is then well suited to the case in which the frequency sources under test have the same nominal frequency ν_0, e.g., atomic frequency standards. An auxiliary frequency source, such as a frequency synthesizer, with frequency ν_0' allows one to obtain, at the output of the mixers, two beat notes at the desired frequency $\nu_1 = |\nu_0 - \nu_0'|$. After amplification the zero crossing of one of the beat notes starts the time interval counter, and the zero crossing of the other beat note stops it. One has

$$x_1 = (\nu_0/\nu_1)(x_0 - x_0') \tag{14}$$

where x_1 is the fractional phase fluctuation of the beat note and x_0 and x_0' that of the two frequency standards. For instance, with ν_0 = 5 MHz, ν_1 = 0.5 Hz, and a precision in the time interval measurement of 0.1 μs a precision of 10^{-14} s at the nominal frequency ν_0 is achieved.

6. CHARACTERIZATION OF FREQUENCY STABILITY IN THE TIME DOMAIN

6.1. *Significance of experimental data*

It is well established that measurement in the time domain with an electronic counter samples phase increments and gives $\Delta_\tau\varphi(t_k)$ defined as

$$\Delta_\tau\varphi(t_k) = \varphi(t_k + \tau) - \varphi(t_k) \tag{15}$$

The phase increment $\Delta_\tau\varphi(t_k)$ is related to $\overline{y}_k$, the average over time interval $[t_k, t_k + \tau]$ of fractional frequency fluctuations. We have

$$\overline{y}_k = \frac{1}{\tau}\int_{t_k}^{t_k+\tau} y(t')\,dt' = \frac{1}{2\pi\nu_1\tau}\Delta_\tau\varphi(t_k) \tag{16}$$

where ν_1 is the mean frequency of the processed signal.

Samples of $\overline{y}_k$ can be combined in many different ways. Some of those which have been considered will be reviewed here. On the other hand, the number of samples is finite, and the question arises as to the related uncertainty in the characterization of frequency stability and of the best use of the data.

6.2. *N-sample variance*

The sequence of measurement is as shown in Figure 16. The mean duration of each measurement is τ, and T is the time interval between the beginnings of them.

In statistical estimation it is common to consider sample variance [*Papoulis*, 1965]. The N-sample variance of y_k is defined as

$$\sigma_y^2(N, T, \tau) = \frac{1}{N-1}\sum_{k=0}^{N-1}\left(\overline{y}_k - \frac{1}{N}\sum_{k=0}^{N-1}\overline{y}_k\right)^2 \tag{17}$$

where the factor $N/(N - 1)$ removes bias in the estimation.

The dependence of the expectation value of the N-sample variance on the number N of samples, the sample time τ, and the power spectral density has been considered by *Allan* [1966]. We will only consider special cases in the following.

It can be shown that computation of the average of the N-sample variance introduces a filtering of the power spectral density $S_y(f)$ [*Barnes et al.*, 1971]. We have

$$\langle\sigma_y^2(N, T, \tau)\rangle = \int_0^\infty S_y(f)|H(f)|^2\,df \tag{18}$$

$H(f)$ is the transfer function of a linear filter which has the following expression:

$$|H(f)|^2 = \frac{N}{N-1}\left[\frac{\sin\pi f\tau}{\pi f\tau}\right]^2\left\{1 - \left[\frac{\sin\pi fNT}{N\sin\pi fT}\right]^2\right\} \tag{19}$$

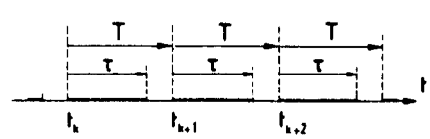

Fig. 16. Sequence of time domain measurement.

For f such that

$$\pi f T \ll 1 \tag{20}$$

we have

$$|H(f)|^2 \simeq \frac{N(N+1)}{3} (\pi f T)^2 \tag{21}$$

Equation (21) shows that for finite N the integral in (18) will converge at the lower limit for $\alpha = -1$ and $\alpha = -2$, as well as for $\alpha = 2$, 1, and 0. One sees that very low frequency components of $S_y(f)$ are best eliminated for small values of N.

6.3. *Variance of time-averaged frequency fluctuations*

When N goes to infinity, $\langle \sigma_y^2(\infty, \tau, \tau) \rangle$ becomes $\sigma^2(\bar{y}_k)$, the variance of time-averaged frequency fluctuations or of the first difference of phase fluctuations [*Cutler and Searle,* 1966] as given by

$$\sigma^2(\bar{y}_k) = \langle (\bar{y}_k)^2 \rangle \tag{22}$$

where angle brackets denote mathematical expectation.

In the presence of a single-pole low-pass filter with cutoff frequency f_c we have the following relation between $\sigma_y^2(\bar{y}_k)$ and $S_y(f)$:

$$\sigma^2(\bar{y}_k) = \int_0^\infty S_y(f) \frac{1}{1 + (f^2/f_c^2)} |H_1(f)|^2 df \tag{23}$$

with

$$|H_1(f)|^2 = \left[\frac{\sin \pi f \tau}{\pi f \tau} \right]^2 \tag{24}$$

Equation (24) shows that $\sigma^2(\bar{y}_k)$ converges for $\alpha = 2$, 1, 0 but diverges for flicker noise of frequency ($\alpha = -1$) and random walk of frequency ($\alpha = -2$). The variance $\sigma^2(\bar{y}_k)$ is no longer used to characterize frequency stability. However, it is useful to relate the RF power spectral density to $S_y(f)$ [*Rutman,* 1974*a*].

6.4. *Two-sample variance*

For the special case $N = 2$, (17) gives

$$\langle \sigma_y^2(2, T, \tau) \rangle = \tfrac{1}{2} \langle (\bar{y}_{k+1} - \bar{y}_k)^2 \rangle \tag{25}$$

and we have

$$\langle \sigma_y^2(2, T, \tau) \rangle = \int_0^\infty S_y(f) \frac{1}{1 + (f/f_c)^2} |H_2(f)|^2 df \tag{26}$$

with

$$|H_2(f)|^2 = 2 \left[\frac{\sin \pi f \tau}{\pi f \tau} \right]^2 (\sin \pi f T)^2 \tag{27}$$

For small f, such that $\pi f T \ll 1$, $|H_2(f)|^2$ varies as f^2. The integral in (26) is thus defined for flicker noise of frequency ($\alpha = -1$) and random walk of frequency ($\alpha = -2$), as well as for $\alpha = 2$, 1, and 0. It is easy to show that the quantity ($\bar{y}_{k+1} - \bar{y}_k$) represents a second-order difference of phase fluctuations. It follows that second-order phase increments are stationary for $\alpha = -1$ and -2 (as specified in Table 1).

6.4.1. *Two-sample variance without dead time.* The two-sample variance (Allan variance) without dead time, for $T = \tau$, is now generally accepted as the measure of frequency stability in the time domain. One sets

$$\sigma_y^2(\tau) = \langle \sigma_y^2(2, \tau, \tau) \rangle \tag{28}$$

Table 2 gives asymptotic expressions of $\sigma_y^2(\tau)$ in the cases $2\pi f_c \tau \gg 1$ and $2\pi f_c \tau \ll 1$. Expressions of $\sigma_y^2(\tau)$ in the presence of a sharp high-frequency cutoff f_h have been given by *Barnes et al.* [1971] for the case $2\pi f_h \tau \gg 1$.

One sees in Table 2 that $\sigma_y^2(\tau)$ has a characteristic τ dependence for each type of noise considered,

TABLE 2. Asymptotic expressions of the two-sample variance for the noise processes considered *

	$\sigma_y^2(\tau)$	
$S_y(f)$	$2\pi f_c \tau \gg 1$	$2\pi f_c \tau \ll 1$
$h_2 f^2$	$\frac{3 h_2 f_c}{8\pi\tau^2}$	$\frac{h_2 f_c^2}{2\tau}$
$h_1 f$	$\frac{3 h_1 \ln(2\pi f_c \tau)}{4\pi^2\tau^2}$	$2 h_1 f_c^2 \ln 2$
h_0	$\frac{h_0}{2\tau}$	$\frac{2}{3}\pi^2 h_0 f_c^2 \tau$
$h_{-1} f^{-1}$	$2 h_{-1} \ln 2$	$2\pi^2 h_{-1} f_c^2 \tau^2$
$h_{-2} f^{-2}$	$\frac{2\pi^2}{3} h_{-2} \tau$	$\pi^3 h_{-2} f_c \tau^2$

* See Appendix Note # 27

such as $\sigma_y^2(\tau) = k/\tau^{\mu}$. This is specified as follows for $2\pi f_c \tau \gg 1$:

α	μ
2	2
1	2
0	1
−1	0
−2	−1

For $\alpha = -1$ the $\sigma_y(\tau)$ graph is a horizontal line, which justifies the designation 'flicker floor' for that part of the graph.

Usually, under experimental conditions the relation $2\pi f_c \tau \gg 1$ is satisfied. The noise processes which perturb the oscillation can then be identified from a $\sigma_y(\tau)$ graph if it is assumed that the aforementioned model of frequency fluctuations is valid. Table 2 shows in which cases $\sigma_y^2(\tau)$ depends on the frequency cutoff. The latter must then be specified.

The case $2\pi f_c \tau \ll 1$ is useful for the analysis of the effect of frequency of phase servocontrol loops where the frequency fluctuations of the frequency reference are low-pass filtered. Bias functions have been given to relate (1) the two-sample variance with and without dead time and (2) the two-sample variance to the N-sample variance [*Barnes et al.*, 1971].

6.4.2. *Two-sample variance with dead time.* * General expressions for the N-sample variance with dead time have been given [*Barnes et al.*, 1971] for useful values of α if the condition $2\pi f_h \tau \gg 1$ is satisfied. The case of the two-sample variance with dead time has not been emphasized enough yet. Table 3 compares the two-sample variance with and without dead time when the condition $2\pi f_c \tau \gg 1$ is fulfilled. The condition of negligible dead time is then $2\pi f_c(T - \tau) \ll 1$. Results for $2\pi f_c \tau \ll 1$ are also available (P. Lesage and C. Audoin, private communication, 1978). Table 3 shows that in the presence of dead time, i.e., $2\pi f_c(T - \tau) \gg 1$, the expression for the two-sample variance is noticeably modified for $\alpha = -1$ and -2.

TABLE 3. Comparison of the two-sample variance with and without dead time for $2\pi f_c \tau \gg 1$

	$\sigma_y^2(2, T, \tau)$	
$S_y(f)$	$2\pi f_c(T-\tau) \ll 1$	$2\pi f_c(T-\tau) \gg 1$
$h_2 f^2$	$\dfrac{3h_2 f_c}{8\pi\tau^2}$	$\dfrac{h_2 f_c}{4\pi\tau^2}$
$h_1 f$	$\dfrac{3h_1 \ln(2\pi f_c \tau)}{4\pi^2\tau^2}$	$\dfrac{h_1 \ln(2\pi f_c \tau)}{2\pi^2\tau^2}$
h_0	$h_0/2\tau$	$h_0/2\tau$
$h_{-1} f^{-1}$	$2h_{-1} \ln 2$	$h_{-1} \ln(T/\tau); T \gg \tau$
$h_{-2} f^{-2}$	$\dfrac{2\pi^2}{3} h_{-2}\tau$	$\pi^2 h_{-2} T$

The case of the flicker noise of frequency is particularly interesting. Figure 17 shows the variation of the two-sample variance with dead time as a function of $\tau/(T - \tau)$. The flicker floor does not exist anymore if the value of this ratio is modified when the sampling time τ is changed. The identification of the noise process which perturbs the oscillation might then be wrong if the effect of dead time is not taken into account. **

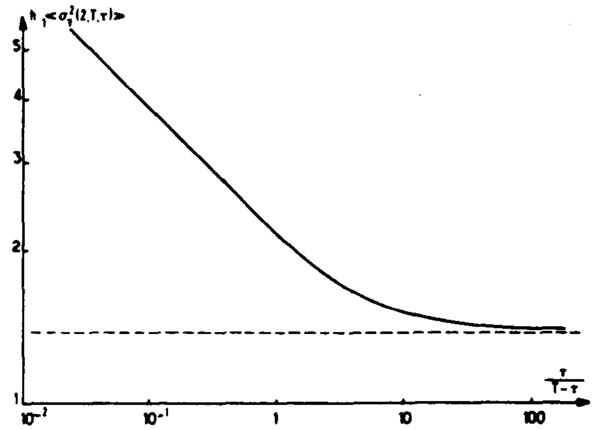

Fig. 17. The solid line represents the variation of $h_{-1}\langle\sigma_y^2(2, T, \tau)\rangle$ versus $\tau/(T - \tau)$ for the flicker noise of frequency. The dotted line represents the asymptotic value for $(T - \tau) \ll \tau$.

6.5. *Precision in the estimation of the two-sample variance*

Measurements are always of finite duration, and therefore the number of available values of y_k is finite. We are then faced with the problem of the precision in the estimation of the time domain frequency stability measurement. This is an important one because successive characterizations of the frequency stability of a given device allow one to get information on the stationarity of the processes involved in the perturbation of its frequency but within the limits of the precision of the characterization. Precision in the estimation of the frequency stability of individual oscillators of a set of p frequency generators ($p > 2$) [*Gray and Allan,*

* See Appendix Note # 28

** See Appendix Note # 29

1974] critically depends on the precision of the $p(p-1)/2$ frequency comparisons which can be performed by arranging oscillators in pairs. Furthermore, the uncertainty in the determination of $\sigma_y(\tau)$ translates directly into the uncertainty in determining the h_α coefficients if the frequency generator is perturbed by noise processes modeled by (7).

The precision in the estimation of time domain measurements of frequency stability has been considered by several authors [*Tausworthe*, 1972; *Lesage and Audoin*, 1973; *Yoshimura*, 1978]. It has been determined for most of the experimental situations which can be encountered in the two-sample variance characterization of frequency stability, with or without dead time (P. Lesage and C. Audoin, private communication, 1978).

Calculation of the expectation value of the two-sample variance according to (25) requires an infinite number of data. But, in practice, only m counting results are available, and one calculates the estimated average of the two-sample variance as follows:

$$\hat{\sigma}_y^2(2, T, \tau, m) = \frac{1}{2(m-1)} \sum_{k=1}^{m-1} (\bar{y}_{k+1} - \bar{y}_k)^2 \qquad (29)$$

One can easily show that the expectation value of $\hat{\sigma}_y^2(2, T, \tau, m)$ equals the averaged two-sample variance with dead time. Thus the finite number of measurements does not introduce bias in the estimation of the two-sample variance.

The estimated averaged two-sample variance (EATSV) being a random function of m, we need to characterize the uncertainty in the estimation. We thus introduce the variance of the EATSV, according to the common understanding of a variance. We set

$$\sigma^2[\hat{\sigma}_y^2(2, T, \tau, m)] = \langle [\hat{\sigma}_y^2(2, T, \tau, m) - \langle \sigma_y^2(2, T, \tau)\rangle]^2 \rangle \qquad (30)$$

With the expression (29) of the EATSV we get

$$\sigma^2[\hat{\sigma}_y^2(2, T, \tau, m)] = \left[\frac{1}{2(m-1)}\right]^2 \left\langle \sum_{i=1}^{m-1} \sum_{j=1}^{m-1} \beta_i \beta_j \right\rangle \qquad (31)$$

with

$$\beta_i = (\bar{y}_{i+1} - \bar{y}_i)^2 - 2\langle \sigma_y^2(2, T, \tau)\rangle \qquad (32)$$

The classical law of large numbers [*Papoulis*, 1965] which states that the true variance of a sum of $(m-1)$ uncorrelated random variables decreases as $1/(m-1)$, even for small values of $(m-1)$, does not apply here. We are considering the quantities β_i which are correlated because two adjacent differences $(\bar{y}_{i+1} - \bar{y}_i)$ and $(\bar{y}_{i+2} - \bar{y}_{i+1})$ are obviously not independent.

Equation (31) can also be written as

$$\sigma^2[\hat{\sigma}_y^2(2, T, \tau, m)] = \frac{1}{m-1}\left[\frac{\Gamma_0}{4} + \frac{1}{2(m-1)} \sum_{k=1}^{m-2} (m-1-k)\Gamma_k \right] \qquad (33)$$

with

$$\Gamma_k = \langle \beta_i \beta_{i-k} \rangle \qquad (34)$$

Γ_k, which does not depend on m, represents the autocorrelation coefficient of β_i and β_{i-k}. Since the same data are used in two adjacent pairs, the autocorrelation coefficient Γ_1, and possibly others, differs from zero. Equation (33) then shows that the $1/(m-1)$ dependence also occurs for the random variables considered, but asymptotically for large enough values of $(m-1)$.

The variance of the EATSV can be related to $S_y(f)$ if it is assumed that the quantities $\bar{y}_i$ are normally distributed. This is a reasonable assumption, as shown in section 2.

It is useful to introduce $\Delta(m)$, the fractional deviation of $\hat{\sigma}_y^2(2, T, \tau, m)$ defined as

$$\Delta(m) = \frac{\hat{\sigma}_y^2(2, T, \tau, m) - \langle \sigma_y^2(2, T, \tau)\rangle}{\langle \sigma_y^2(2, T, \tau)\rangle} \qquad (35)$$

The standard deviation $\sigma[\Delta(m)]$ defines the precision in the estimation of the two-sample variance. Expressions for $\sigma[\Delta(m)]$ which are valid for $m > 2$ have been established for all possible values of $2\pi f_c \tau$ and $2\pi f_c(T-\tau)$ but will not be given here.

In practice, the time domain frequency stability of a frequency source is characterized by the standard deviation $[\hat{\sigma}_y^2(2, T, \tau, m)]^{1/2}$. We therefore consider δ defined as

$$\delta = \frac{[\hat{\sigma}_y^2(2, T, \tau, m)]^{1/2} - \langle \sigma_y^2(2, T, \tau)\rangle^{1/2}}{\langle \sigma_y^2(2, T, \tau)\rangle^{1/2}} \qquad (36)$$

$\sigma(\delta)$ specifies the precision in the estimation of the time domain frequency stability measurement

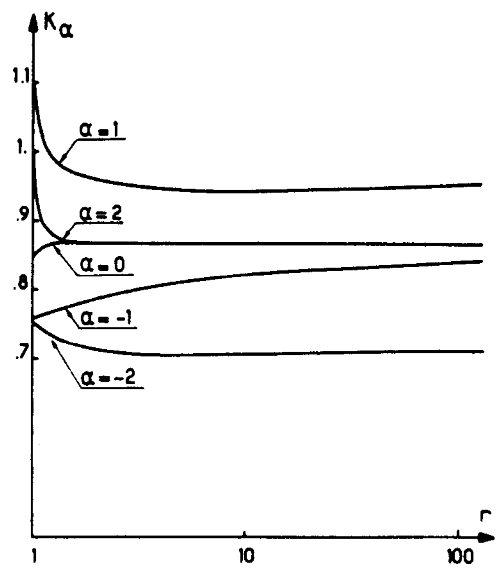

Fig. 18. Variation of K_α as a function of $r = T/\tau$ for the commonly encountered noise processes and for $2\pi f_c \tau = 10$.

from a limited number of data and allows one to draw error bars on a frequency stability graph.

For $2\pi f_c \tau \gg 1$ and $m \gg 1$ we have

$$\sigma(\delta) \simeq K_\alpha m^{-1/2} \qquad (37)$$

The values of K_α are given as follows subject to the condition that the dead time is negligible, i.e., $2\pi f_c(T - \tau) \ll 1$:

α	K_α
2	0.99
1	0.99
0	0.87
−1	0.77
−2	0.75

In the presence of dead time the values of K_α depend on the noise process considered as well as the values of $2\pi f_c \tau$ and $r = T/\tau$. Figure 18, valid for $2\pi f_c \tau = 10$ shows that the dependence of K_α with dead time is especially pronounced in the vicinity of $r = 1$ for $\alpha = 1$ and 2.

7. CHARACTERIZATION OF FREQUENCY STABILITY VIA FILTERING OF PHASE OR FREQUENCY NOISE

Equations (27) and (28) show that the definition of the time domain measurement of frequency stability $\sigma_y^2(\tau)$ involves a filtering of $S_y(f)$ in a linear filter. Figure 19 shows the impulse response of this filter, which represents the sequence of measure-

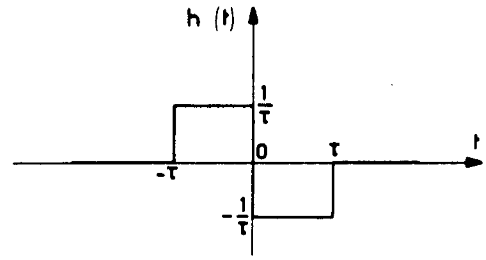

Fig. 19. Impulse response of a linear filter which represents computation of two-sample variance.

ment for $T = \tau$, and Figure 20 depicts the related transfer function. One can also consider the effect of filtering a voltage proportional to $y(t)$ or $x(t)$ in a physically realizable analog filter.

A high pass filter of cutoff frequency $1/\pi\tau$ has been considered [*Rutman*, 1974*b*; *Rutman and Sauvage*, 1974]. Its input receives a voltage proportional to $x(t)$. It is provided by a mixer used as a phase comparator. The rms value of the filtered signal is measured. When the frequency cutoff is changed, this rms value shows the same μ versus α dependence as shown in section 6.4.1. More interesting is a bandpass filter centered at the variable frequency $f = 1/2\tau$ but with a fixed value of its quality factor. It allows one to distinguish white and flicker noise of phase, as it gives $\mu = 3$ for $\alpha = 2$; the μ versus α dependence being otherwise unchanged for $\alpha = 1, 0, -1$, and -2.

Similarly, a frequency discriminator, giving an output proportional to $y(t)$, followed by two cascaded resistance-capacitance (RC) filters and a rms voltmeter allows one to obtain a useful approximation of the two-sample variance. The filters insure low-pass and high-pass filtering with $RC = \tau/2$ [*Wiley*, 1977].

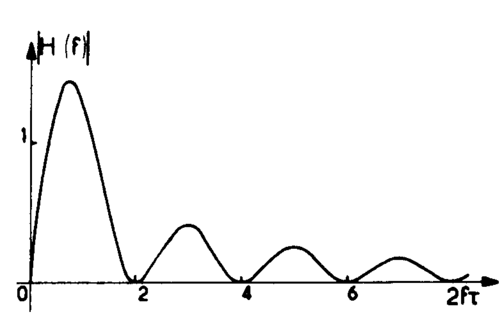

Fig. 20. Transfer function of the linear filter with impulse response shown in Figure 19.

8. SPECTRAL ANALYSIS INFERRED FROM TIME DOMAIN MEASUREMENTS

The methods of time domain characterization of frequency stability reviewed above allow one to identify noise process if they are described by

$$S_y(f) = \sum_{\alpha=-2}^{2} h_\alpha f^\alpha \tag{38}$$

This may not be the case. Furthermore, it is of interest to determine the power spectral density of fractional frequency fluctuations for Fourier frequencies lower than 1 Hz. In this region, time domain measurements are the most convenient, and the question arises as to their best use for spectral analysis.

8.1. *Selective numerical filtering*

Equations (24) and (27) show that calculation of the variance of the second difference of phase fluctuations (the two-sample variance) involves a more selective filtering than calculation of the variance of the first difference of phase fluctuations. One can then consider higher-order differences [*Barnes*, 1966; *Lesage and Audoin*, 1975*a*, *b*]. The nth-order difference of phase fluctuations is denoted as ${}^{n}\Delta_{T,\tau}\varphi(t_k)$, where τ and T have the same meaning as in section 6.1. and 6.2. This nth difference is defined by the following recursive equation:

$${}^{n}\Delta_{T,\tau}\varphi(t_k) = {}^{(n-1)}\Delta_{T,\tau}\varphi(t_k + T) - {}^{(n-1)}\Delta_{T,\tau}\varphi(t_k) \tag{39}$$

which introduces binomial coefficients C^{i}_{n-1}. We have

$${}^{n}\Delta_{T,\tau}\varphi(t_k) = \sum_{i=0}^{n-1} (-1)^i C^{i}_{n-1} \{\varphi[t_k + (n-1-i)T + \tau] - \varphi[t_k + (n-1-i)T]\} \tag{40}$$

The transfer function $H_n(f, T, \tau)$ of the linear filter which represents the calculation of the variance of the nth difference of phase fluctuations is given by

$$|H_n(f, T, \tau)| = \frac{2^{n-1}}{\pi f \tau} (\sin \pi f T)^{n-1} \sin \pi f \tau \tag{41}$$

It should be pointed out that for $f\tau \ll 1$, one has

$$|H_n(f, T, \tau)| \simeq (2\pi f T)^{n-1} \qquad f\tau \ll 1 \tag{42}$$

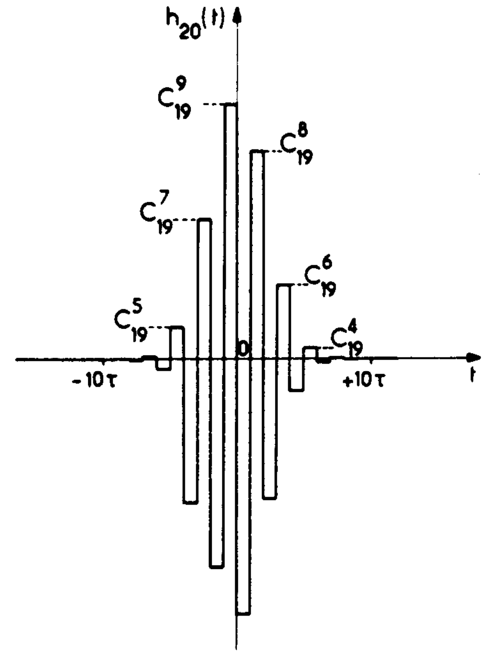

Fig. 21. Impulse response of a linear filter which represents computation of the variance of the 20th difference of phase fluctuations. C^i_j represents binomial coefficients.

Figure 21 shows the impulse response of the linear filter which represents the calculation of the variance of the 20th difference of phase fluctuations, and Figure 22 shows the related transfer function. A selective filtering is then involved around frequency $1/2\tau$.

Such a variance is also known as a modified Hadamard variance [*Baugh*, 1971]. The spurious responses at frequencies $(2l + 1)/2\tau$, where l is an integer, can be eliminated by a proper weighting

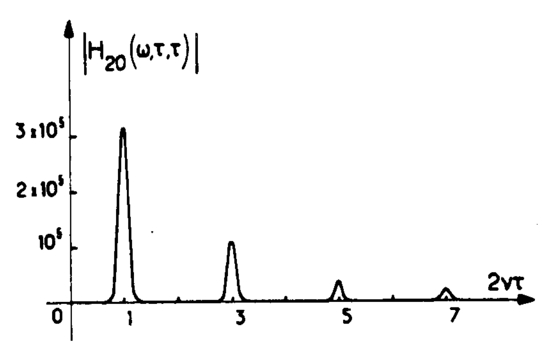

Fig. 22. Transfer function of the linear filter considered in Figure 21.

of measurement results or by filtering with the help of an analog filter [*Groslambert*, 1976].

Such a technique of linear filtering has been used to show that good quartz crystal oscillators exhibit flicker noise of frequency for Fourier frequencies as low as 10^{-3} Hz [*Lesage and Audoin*, 1975*b*]. Furthermore, it is well suited to the design of automated measurement setups [*Peregrino and Ricci*, 1976; *Groslambert*, 1977].

The best use of experimental time domain data for selective filtering has been considered by *Boileau* [1976].

8.2. *High-pass filtering*

If frequency fluctuations $y(t)$ are filtered in an ideal high-pass filter with transfer function $G_0(f_1, f)$ such that

$$G_0(f_1, f) = \begin{cases} 0 & f < f_1 \\ 1 & f > f_1 \end{cases} \tag{43}$$

its output $z(t)$ is such that

$$\sigma_z^2 = \int_0^\infty G_0(f, f_1) S_y(f)\, df = \int_{f_1}^\infty S_y(f)\, df \tag{44}$$

Equation (44) shows that the derivative of σ_z^2 is $-S_y(f)$, and spectral analysis, and therefore characterization of frequency stability, are possible, in principle, by high-pass filtering.

Possible realization of the high-pass filter by techniques of digital data processing have been specified, such as the method of finite-time variance and the method of finite-time frequency control. Processing of finite-time data is aimed to properly deal with the nonintegrable singularity of the power spectral density at $\nu = 0$ [*Boileau*, 1975; *Boileau and Picinbono*, 1976]. The method is well suited to the analysis of drifts or slow frequency changes. Practical use of this method has not been reported yet.

8.3. *Use of the sample spectral density*

It has been shown in section 8.1. that spectral analysis from the Hadamard variance or its modified forms requires a series of measurements at time interval τ in order to specify the spectral density at frequency $1/2\tau$. Another point of view has been considered [*Boileau and Lecourtier*, 1977]. From a set of measurements of y_k, sampled at frequency $1/\tau$, it allows one to obtain an estimation of the spectral density for discrete values of the Fourier frequency.

9. STRUCTURE FUNCTIONS OF OSCILLATOR FRACTIONAL PHASE AND FREQUENCY FLUCTUATIONS

Interest in the variance of nth-order difference of phase fluctuations was recognized early in the field of time keeping (see for instance, *Barnes* [1966]). This can be easily understood from (43), which shows that an efficient filtering of low-frequency components of frequency fluctuations is then introduced. It allows one to deal properly with frequency drifts, which will now be considered, and poles of $S_y(f)$ of order $2(n-1)$ at the origin. It is equivalent to saying that the nth difference of phase fluctuations allows one to consider random processes with stationary nth-order phase increments.

This question has been formalized by *Lindsey and Chie* [1976, 1977], who introduce structure functions of oscillator phase fluctuations. The nth order structure function of phase fluctuations is nothing else but the variance of the nth difference of phase fluctuations, as considered in section 8. Then, by definition, the nth-order structure function of fractional phase fluctuations is given by

$$D_x^{(n)}(\tau) = E\{[{}^n\Delta_{\tau,\tau} x(t_k)]^2\} \tag{45}$$

where E{·} means expectation value. The fractional phase (or the clock reading) at time t_k is $x(t_k)$. We assume $T = \tau$.

Let us consider an oscillator, the phase $\varphi'(t)$ of which is of the following form except for an additive constant:

$$\varphi'(t) = \sum_{k=2}^{l} \Omega_{k-1} \frac{t^k}{k!} + \varphi(t) \tag{46}$$

where Ω_k is a random variable modeling the kth-order frequency drift and $\varphi(t)$ represents random phase fluctuations. We then have

$$x'(t) = \sum_{k=2}^{l} d_{k-1} \frac{t^k}{k!} + x(t) \tag{47}$$

and

$$y'(t) = \sum_{k=1}^{l-1} d_k \frac{t^k}{k!} + y(t) \tag{48}$$

where $d_k = \Omega_k / 2\pi\nu_0$ is the normalized drift coefficient and $x(t)$ and $y(t)$ have the same meaning as in the preceding sections. The notations x' and y' refer to an oscillator with drift.

It can then be shown that we have

$$D_x^{(n)}(\tau) = \tau^{2n} E\,[d_{n-1}^2] + 2^{2n} \int_0^\infty S_y(f) \frac{\sin^{2n}(\pi f \tau)}{(2\pi f)^2}\, df \qquad l = n \tag{49}$$

and

$$D_x^{(n)}(\tau) = 2^{2n} \int_0^\infty S_y(f) \frac{\sin^{2n}(\pi f \tau)}{(2\pi f)^2}\, df \qquad l > n \tag{50}$$

If one applies (49) to the case of an oscillator without drift, one can easily show that the following equations are satisfied:

$$\sigma^2(\overline{y_k}) = (1/\tau^2) D_x^{(1)}(\tau) \tag{51}$$

and

$$\sigma_y^2(\tau) = (1/2\tau^2) D_x^{(2)}(\tau) \tag{52}$$

This is indeed not surprising because apart from more or less complicated mathematical formalism the definitions of the considered variances and structure functions are closely related, as has been emphasized here.

Relations between sample variance and structure functions have been given by *Lindsey and Chie* [1976], whereas the relation between structure functions and several different approaches of frequency stability characterization has been analyzed by *Rutman* [1977, 1978].

For the generally accepted noise model defined by (7) the τ dependence of higher-order structure functions is the same as the two-sample variance, as shown in Table 4 [*Lindsey and Chie*, 1978*b*]. As stated above, structure functions of order n allow one to consider spectral densities which vary as f^α at the origin with $\alpha \geq -2(n-1)$. For instance, for $n = 3$ it is possible to characterize frequency fluctuations of an oscillator with a power spectral density of fractional frequency fluctuations given by $S_y(f) = \Sigma_{\alpha=-4}^{2} h_\alpha f^\alpha$. This oscillator exhibits stationary third-order increments of phase fluctuations.

In the presence of a frequency drift described by a polynomial of degree $l-1$, structure functions of degree $n < l$ are meaningless: their computation yields a time-dependent result. For $n = l$ the lth structure function shows a long-term τ dependence proportional to τ^{2l}. This dependence disappears for $n > l$. Although a power spectral density of the form $f^{-(2l-1)}$ would also give the structure functions a variation of the form τ^{2l}, this variation does not depend on n, provided that the function is meaningful. It is then possible, at least in principle, to identify frequency drifts and to specify their order. This is illustrated in Figure 23 according to *Lindsey and Chie* [1978*b*]. However, there are not yet experimental proofs that such a characterization is achievable in practice.

10. POWER SPECTRAL DENSITY OF STABLE FREQUENCY SOURCES

The power emitted by a source of time-dependent voltage $v(t)$ given by (1) is $S_v(\nu)\,d\nu$ in the frequency range $[\nu, \nu + d\nu]$, where $S_v(\nu)$ is the power spectral density of the source. The dimensions of $S_v(\nu)$ are V^2 Hz^{-1}. The main interest of power spectral density, in frequency metrology, is related to high-order frequency multiplication. We will only introduce the subject by giving the relations between $S_v(\nu)$ and $S_\phi(f)$ and stating present problems in the field.

TABLE 4. Structure functions of orders 1, 2, 3, and 4 for fractional phase fluctuations of commonly encountered noise processes for $2\pi f_h \tau \gg 1$

$S_y(f)$	$D_x^{(1)}(\tau)$	$D_x^{(2)}(\tau)$	$D_x^{(3)}(\tau)$	$D_x^{(4)}(\tau)$
$h_2 f^2$	$\frac{1}{2\pi^2} h_2 f_h$	$\frac{3}{2\pi^2} h_2 f_h$	$\frac{5}{\pi^2} h_2 f_h$	$\frac{35}{2\pi^2} h_2 f_h$
$h_1 f$	$\frac{1}{2\pi^2} h_1 \ln(\pi f_h \tau)$	$\frac{3}{2\pi^2} h_1 \ln(\pi f_h \tau)$	$\frac{5}{\pi^2} h_1 \ln(\pi f_h \tau)$	$\frac{35}{2\pi^2} h_1 \ln(\pi f_h \tau)$
h_0	$\frac{1}{2} h_0 \tau$	$h_0 \tau$	$3 h_0 \tau$	$10 h_0 \tau$
$h_{-1} f^{-1}$		$4 h_{-1} \tau^2 \ln 2$	$6.75\, h_{-1} \tau^2$	$20.7\, h_{-1} \tau^2$
$h_{-2} f^{-2}$		$\frac{4}{3} \pi^2 h_{-2} \tau^3$	$2\pi^2 h_{-2} \tau^3$	$\frac{16}{3} \pi^2 h_{-2} \tau^3$

From *Lindsey and Chie* [1978*b*]

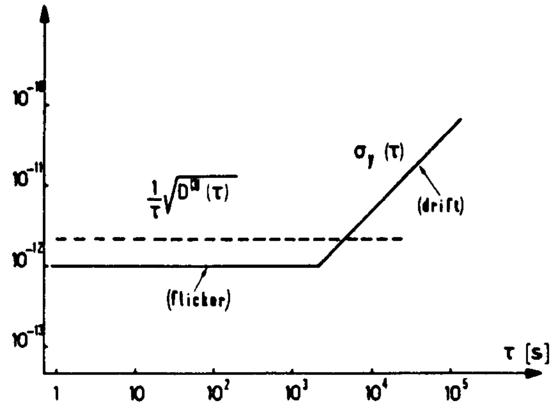

Fig. 23. (Solid line) Two-sample variance of an oscillator showing a linear frequency drift of 10^{-10} per day and flicker noise of frequency given by $S_y(f) = 7.2 \times 10^{-25} f^{-1}$. (Dotted line) The third difference of fractional phase fluctuation is independent of the drift (according to *Lindsey and Chie* [1978*b*].

Negligible amplitude noise and gaussian phase fluctuations being assumed, it is well known that the autocorrelation function of $v(t)$ is $R_v(\tau)$ given by

$$R_v(\tau) = \frac{V_0^2}{2} \cos 2\pi\nu_0\tau \exp\left[-\tfrac{1}{2}(2\pi\nu_0)^2\sigma^2(\overline{y_k})\right] \quad (53)$$

As $\sigma^2(\overline{y_k})$ is only defined for stationary phase fluctuations and for phase fluctuations with stationary first increments, the same is true for $R_v(\tau)$ and therefore $S_v(f)$.

10.1. *White noise of frequency*

This is the simplest to deal with. If the frequency of the source is perturbed by a broadband white noise of frequency, one has $S_y(f) = h_0$ and $\sigma^2(\overline{y_k}) = (h_0/2\tau)$. Whence

$$R_v(\tau) = \frac{V_0^2}{2} \cos 2\pi\nu_0\tau \exp\left(-\frac{|\tau|}{\tau_c}\right) \quad (54)$$

where τ_c is the coherence time of the signal. We have

$$\tau_c = (\pi^2 \nu_0^2 h_0)^{-1} \quad (55)$$

The one-sided power spectral density is then represented by a Lorentzian given by

$$S_v(\nu) = V_0^2 \frac{2\pi\Delta\nu}{(2\pi\Delta\nu)^2 + [2\pi(\nu - \nu_0)]^2} \quad (56)$$

where $\Delta\nu$ is the half width at half maximum of the power spectra defined as

$$2\Delta\nu = \pi\nu_0^2 h_0 \quad (57)$$

We obviously have $2\pi\Delta\nu \cdot \tau_c = 1$. If oscillators are considered, one has $h_0 = kT/PQ^2$ in the radiofrequency and microwave domain and $h_0 = h\nu_0/PQ^2$ in the optical frequency domain, where P is the power delivered by the oscillator and Q the quality factor of the frequency-determining element. Table 5 gives theoretical values of coherence time and linewidth of good oscillators. It is only intended to illustrate a comparison, often made, of the spectral purity of oscillators. It must be pointed out that, in practice, other noise processes exist which modify these results. Even any meaning of τ_c and $\Delta\nu$ is removed if $R_v(\tau)$ is not defined.

Multiplication of the frequency by n multiplies $\Delta\nu$ and divides τ_c by the factor n^2.

10.2. *White noise of phase*

Presently available good quartz oscillators are affected by white noise of phase. It is easy to show from the definition (16) of y_k that the following equation is satisfied:

$$\tfrac{1}{2}\left[2\pi\nu_0\tau\sigma(\overline{y_k})\right]^2 = R_\varphi(0) - R_\varphi(\tau) \quad (58)$$

where $R_\varphi(\tau)$ denotes the autocorrelation function of the stationary phase fluctuations $\varphi(t)$.

The expression of the one-sided $S_v(\nu)$ then follows [*Rutman*, 1974*a*; *Lindsey and Chie*, 1978*a*]:

$$S_v(\nu) = \frac{V_0^2}{2} e^{-R_\varphi(0)} [\delta(\nu - \nu_0)$$
$$+ S_\varphi(\nu - \nu_0) + \tfrac{1}{2} S_\varphi(\nu) * S_\varphi(\nu) + \cdots] \quad (59)$$

where the asterisk denotes convolution and the bracket contains an infinite set of multiple-convolution products of $S_\varphi(\nu)$ by itself. Such an equation is not easily tractable. It is the reason why the

TABLE 5. Theoretical values of correlation time and power spectrum linewidth for various oscillators

Oscillator	ν_0, Hz	h_0, Hz^{-1}	τ_c, s	$2\Delta\nu$, Hz
5-MHz quartz xtal	5×10^6	4×10^{-27}	10^{12}	3×10^{-13}
H maser	1.4×10^9	4×10^{-27}	10^7	3×10^{-8}
He-Ne laser	5×10^{14}	3×10^{-29}	10^{-2}	30

approximation of small-phase fluctuations is often made. If $R_\varphi(0) = \overline{\varphi^2} \ll 1$, one has

$$S_v(\nu) = \frac{V_0^2}{2} e^{-\overline{\varphi^2}} [\delta(\nu - \nu_0) + S_\varphi(\nu - \nu_0)] \qquad (60)$$

In this approximation the power spectrum consists of a carrier at frequency ν_0 around which the spectrum of the phase fluctuations is translated.

If the frequency of the signal is multiplied by n, the mean squared frequency fluctuation becomes $n^2\overline{\varphi^2}$. If $n^2\overline{\varphi^2} \ll 1$, the power spectral density is then given by

$$S_v(\nu) = \frac{V_0^2}{2} e^{-n\overline{\varphi^2}} [\delta(\nu - n\nu_0) + n^2 S_\varphi(\nu - \nu_0)] \qquad (61)$$

The power in the carrier decreases, and the power in the pedestal increases. The relative powers in the carrier P_c and in the pedestal P_p are then given by

$$P_c = e^{-\overline{\phi^2}} \qquad (62)$$

$$P_p = 1 - e^{-\overline{\phi^2}} \qquad (63)$$

respectively, where ϕ^2 represents the mean squared value of phase fluctuations at the signal frequency considered. It has been proved that (62) and (63) are valid, even if the condition $\overline{\phi^2} \ll 1$ is not satisfied (F. Clerc, private communication, 1977).

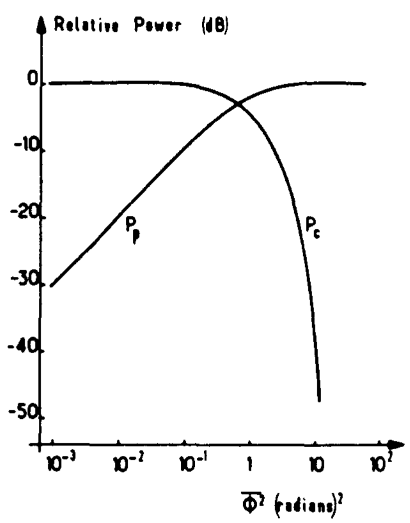

Fig. 24. Variation of the relative power in the carrier P_c and the pedestal P_p as a function of ϕ^2, the mean squared phase fluctuations.

Figure 24 shows the variation of P_c and P_p as a function of ϕ^2. One easily understands that the carrier may disappear if the multiplicative factor is high enough. This has been theoretically and experimentally investigated by *Walls and de Marchi* [1975], *Bava et al.* [1977*b*], and *Godone et al.* [1978]. A signal has been synthesized at 761 GHz, starting from a 5-MHz quartz oscillator, which verifies theoretical conclusions.

10.3. *Other noise processes*

Much work remains to be done to analyze properly the effect of noise such as the flicker noise of frequency or the random walk of frequency which contributes power very close to the carrier. The very interesting semiempirical approach by *Halford* [1971] has not yet been justified either theoretically or experimentally in a convincing manner.

11. CONCLUSION

Widely used theoretical and experimental methods for the characterization of frequency stability in the time and frequency domain have been outlined. Recently used or proposed experimental methods have been reviewed. The effect of dead time on the interpretation of time domain measurements, as well as on their precision has been emphasized. Recently introduced structure functions have been considered as well as their interest for the elimination of frequency drifts. The problems in the relation between the radiofrequency power spectral density and the power spectral density of phase fluctuations have been briefly summarized.

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