Time and Frequency (Time-Domain) Characterization, Estimation, and Prediction of Precision Clocks and Oscillators

Invited Paper

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***Abstract*—A tutorial review of some time-domain methods of characterizing the performance of precision clocks and oscillators is presented. Characterizing both the systematic and random deviations is considered. The Allan variance and the modified Allan variance are defined, and methods of utilizing them are presented along with ranges and areas of applicability. The standard deviation is contrasted and shown not to be, in general, a good measure for precision clocks and oscillators. Once a proper characterization model has been developed, then optimum estimation and prediction techniques can be employed. Some important cases are illustrated. As precision clocks and oscillators become increasingly important in society, communication of their characteristics and specifications among the vendors, manufacturers, design engineers, managers, and metrologists of this equipment becomes increasingly important.**

Introduction

"WHAT THEN," asked St. Augustine, "is time? If no one asks me, I know what it is. If I wish to explain it to him who asks me, I do not know." Though Einstein and others have taught us a lot since St. Augustine, there are still many unanswered questions. In particular, can time be measured? It seems that it cannot; what is measured is the time *difference* between two clocks. The time of an event with reference to a particular clock can be measured. If time cannot be measured, is it physical, an abstraction, or is it an artifact?

We conceptualize some of the laws of physics with time as the independent variable. We attempt to approximate our conceptualized ideal time by inverting these laws so that time is the dependent variable. The fact is that time as we now generate it is dependent upon defined origins, a defined resonance in the cesium atom, interrogating electronics, induced biases, timescale algorithms, and random perturbations from the ideal. Hence, at a significant level, time—as man generates it by the best means available to him—is an artifact. Corollaries to this are that every clock disagrees with every other clock essentially always, and no clock keeps ideal or "true" time in an abstract sense except as we may choose to define it. Frequency or time interval, on the other hand, is fundamental to nature; hence the definition of the second can approach the ideal—down to some accuracy limit. Noise in nature is also fundamental. Characterizing the random variations of a clock opens the door to optimum estimation of environmental influences and to the design of optimum combining algorithms for the generation of uniform time and for providing a stable and accurate frequency reference.

Let us define $V(t)$ as the sine-wave voltage output of a precision oscillator:

$$V(t) = V_0 \sin \Phi(t) \tag{1}$$

where $\Phi(t)$ is the abstract but actual total time-dependent accumulated phase starting from some arbitrary origin $\Phi(t = 0) = 0$. We assume that the amplitude fluctuations are negligible around V_0. Cases exist in which this assumption is not valid, but we will not treat those in the context of this paper. This lack of treatment has no impact on the development or the conclusions in this paper. Since infinite bandwidth measurement equipment is not available to us, we cannot measure instantaneous frequency; therefore $\nu(t) = (1/2\pi)\, d\Phi/dt$ is not measurable. We can rewrite this equation with ν_0 being a constant nominal frequency and place all of the deviations in a residual phase $\phi(t)$:

$$V(t) = V_0 \sin (2\pi\nu_0 t + \phi(t)). \tag{2}$$ *

We then define a quantity $y(t) = (\nu(t) - \nu_0)/\nu_0$, which is dimensionless and which is the fractional or normalized frequency deviation of $\nu(t)$ from its nominal value. Integrating $y(t)$ yields the time deviation $x(t)$, which has the dimensions of time

$$x(t) = \int_0^t y(t')\, dt'. \tag{3}$$

From this, the time deviation of a clock can be written as a function of the phase deviation:

$$x(t) = \frac{\phi(t)}{2\pi\nu_0}. \tag{4}$$

Systematic Models for Clocks and Oscillators

The next question one may ask is why does a clock deviate from the ideal? We conceptualize two categories

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* See Appendix Note # 11

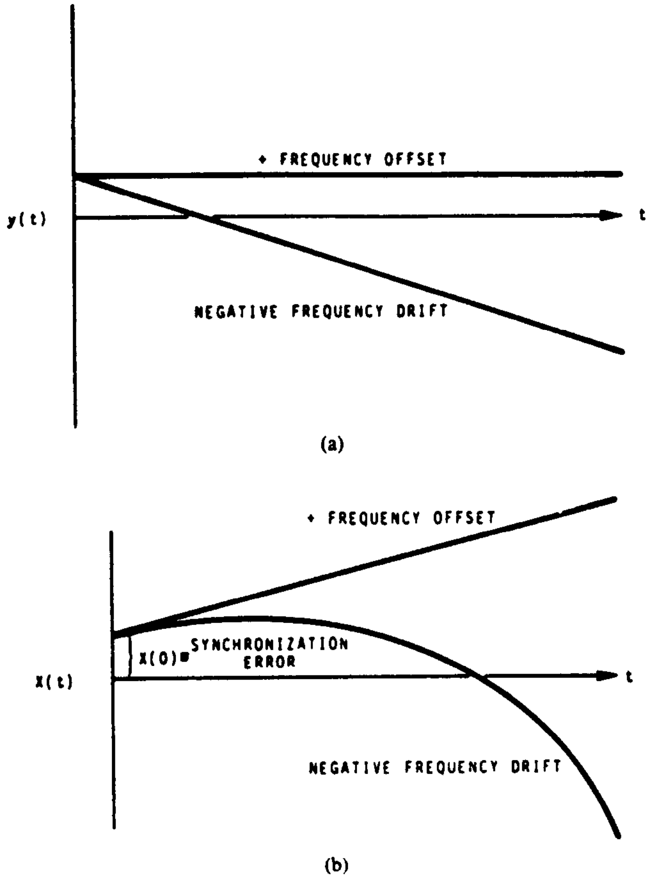

Fig. 1. Frequency $y(t)$ and time $x(t)$ deviations due to frequency offset and to frequency drift in clock. (a) Fractional frequency error versus time. (b) Time error versus time.

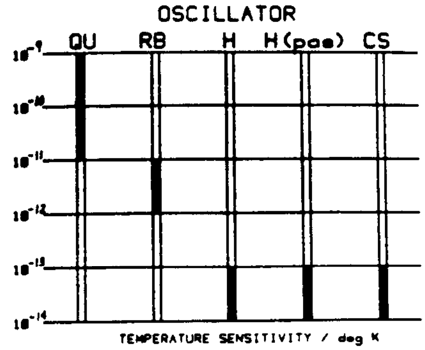

Fig. 2. Nominal values for temperature coefficient for frequency standards: QU = quartz crystal, RB = rubidium gas cell, H = active hydrogen maser, H(pas) = passive hydrogen maser, and CS = cesium beam.

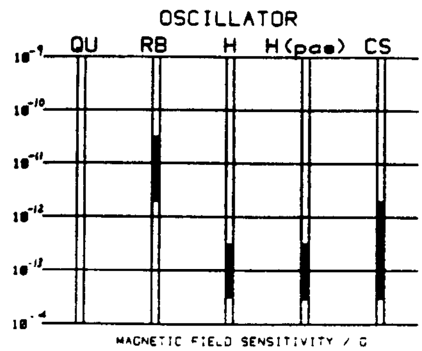

Fig. 3. Nominal values for magnetic field sensitivity for frequency standards: QU = quartz crystal, RB = rubidium gas cell, H = active hydrogen maser, H(pas) = passive hydrogen maser, and CS = cesium beam.

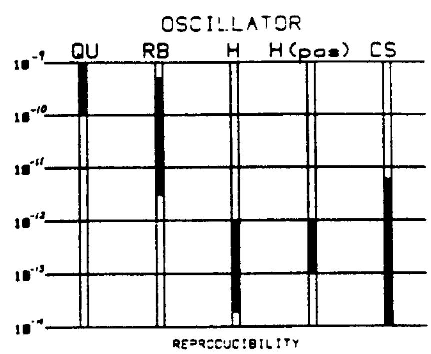

Fig. 4. Nominal capability of frequency standard to reproduce same frequency after period of time for standards: QU = quartz crystal, RB = rubidium gas cell, H = active hydrogen maser, H(pas) = passive hydrogen maser, and CS = cesium beam.

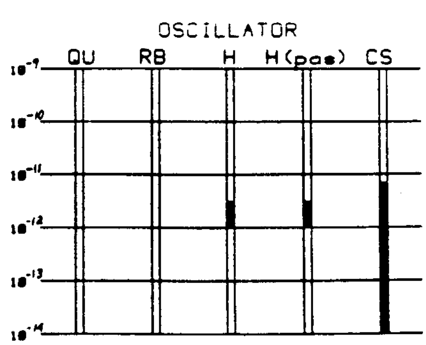

Fig. 5. Nominal capability for frequency standard to produce frequency determined by fundamental constants of nature for standards: QU = quartz crystal, RB = rubidium gas cell, H = active hydrogen maser, H(pas) = passive hydrogen maser, and CS = cesium beam.

of reasons, the first being systematics such as frequency drift (D), frequency offset (y_0), and time offset (x_0). In addition, there are systematic deviations that are often environmentally induced. The second category is the random deviations $\epsilon(t)$, which are usually not thought to be deterministic. In general, we may write

$$x(t) = x_0 + y_0 t + 1/2\, Dt^2 + \epsilon(t). \qquad (5)\ *$$

Though generally useful, the model in (5) does not apply in all cases; e.g., some oscillators have significant frequency-modulation sidebands, and in others the frequency drift D is not constant. In some clocks and oscillators, e.g., cesium-beam standards, setting $D = 0$ is usually a better model.

Note that the quadratic D term occurs because $x(t)$ is the integral of $y(t)$, the fractional frequency, and is often the predominant cause of time deviation. In Fig. 1 we have simulated two systematic-error cases: a clock with frequency offset, and a clock with negative frequency drift. Figs. 2–6 summarize some of the important systematic influences on precision clocks and oscillators. In addition to Figs. 1–6, important systematic deviations may include modulation sidebands, e.g., 60 Hz, 120 Hz, daily, and annual dependences, which can be manifestations of environmental effects such as deviations induced by vibrations, shock, radiation, humidity, and temperature.

* See Appendix Note # 12

TABLE I
APPLICABLE OSCILLATORS AND RANGE OF APPLICABILITY

Typical Noise Types α	Name	Cs	H-Active	H-Passive	Qu	Rb
2	white-noise PM		≤ 100 s		≤ 1 ms	
1	flicker-noise PM				≤ 1 s	
0	white-noise FM	≥ 10 s	$100\ \text{s} \leq \tau \geq 10^4$ s	≥ 1 s		≥ 1 s
−1	flicker-noise FM	$\geq$ days	$\geq 10^4$ s	$\geq$ days	≥ 1 s	$\geq 10^4$
−2	random-walk FM	$\geq$ weeks	$\geq$ weeks	$\geq$ weeks	$\geq$ h	$\geq$ days

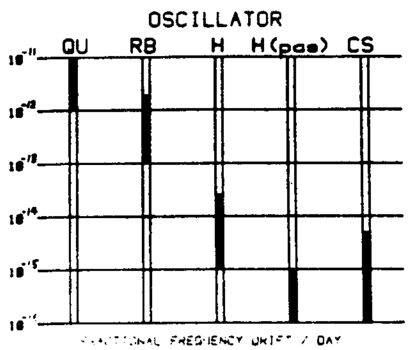

Fig. 6. Nominal values (ignoring sign) for frequency drift for frequency standards: QU = quartz crystal, RB = rubidium gas cell, H = active hydrogen maser, H(pas) = passive hydrogen maser, and CS = cesium beam.

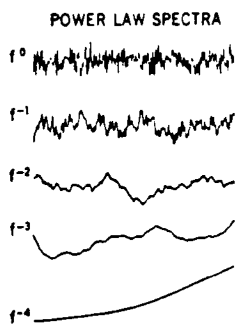

Fig. 7. Simulated random processes commonly occurring in output signal of atomic clocks. Power law spectra $S(f)$ are proportional to ω to some exponent, where f is Fourier frequency $((w = 2\pi f)$ and $S_y(f) = w^2 S_x(f))$. *

RANDOM MODELS FOR CLOCKS AND OSCILLATORS

The random-frequency deviations of precision clocks and oscillators can often be characterized by power-law spectra $S_y(f) \sim f^{\alpha}$, where f is the Fourier frequency and α typically takes on integer values, i.e., −2, −1, 0, 1, 2 [1]–[4]. Fig. 7 shows noise samples corresponding to these different power law spectra, and Table I shows the nominal range of applicability of these power-law models.

TIME-DOMAIN SIGNAL CHARACTERIZATION

Given a discrete set of time deviations x_i taken in sequence for the measurable time difference between a pair of clocks or between a clock and some primary reference, and given that the nominal spacing between adjacent time difference measurements is τ_0 (see Fig. 8 for an example), then the average fractional frequency for the ith measurement interval is

$$\bar{y}_i^{\tau_0} = \frac{x_{i+1} - x_i}{\tau_0} \tag{6}$$

where $-\tau_0$ over y_i denotes the average over an interval τ_0.

We can thus construct a set of discrete frequency values from such a time-difference data set. If the standard deviation is calculated for this set of values, one can show that for some kinds of power-law spectra encountered in precision oscillators the standard deviation is divergent [1], [2], [5], i.e., it does not converge to a well-defined value and is a function of data length. Hence the standard deviation is seldom useful and can be misleading in characterizing clocks. An IEEE subcommittee has recommended $S_y(f)$ in the frequency domain and a measure $\sigma_y^2(\tau)$ in the time domain [1]. $S_y(f)$ is the one-sided spectral density of y as a function of Fourier frequency f. The latter is often called the Allan variance or two-sample variance. The convergence of $\sigma_y(\tau)$ has been verified [1]–[4] for the power law spectra of interest in precision clocks and oscillators. The measure $\sigma_y^2(\tau)$ is defined as [1]

$$\sigma_y^2(\tau) = \frac{1}{2}\langle(\Delta\bar{y}^{\tau})^2\rangle \tag{7}$$

where $\Delta\bar{y}^{\tau}$ is the difference between adjacent fractional frequency measurements, each sampled over an interval τ, and the brackets $\langle\ \rangle$ indicate an infinite time average or expectation value. A pictorial description is shown in Fig. 9 for a finite data set. A data set of the order of 100 points is more than adequate for convergence of $\sigma_y(\tau)$, though of course the confidence of the estimate will typically improve as the data length increases [6].

Given a discrete set of stored evenly spaced data, the value of τ can be varied in the software [7]. If τ_0 is the minimum data spacing for the original stored data set $\bar{y}_i^{\tau_0}$, then one can change the sampling time to $\tau = n\tau_0$ by averaging n adjacent values of $\bar{y}_i^{-\tau_0}$ to obtain a new fractional ** frequency estimate $\bar{y}_i^{\tau}$, with sample time τ as input to (7). Note this is different from averaging adjacent values of x. Hence in a very convenient way one can calculate $\sigma_y(\tau)$ as a function of τ, which will be shown to be very useful. For a finite data set of M values of $\bar{y}_i^{\tau_0}$, (7) for general τ becomes (see Fig. 8 for an example computation of $\sigma_y(\tau$

* See Appendix Note # 13

** See Appendix Note # 14

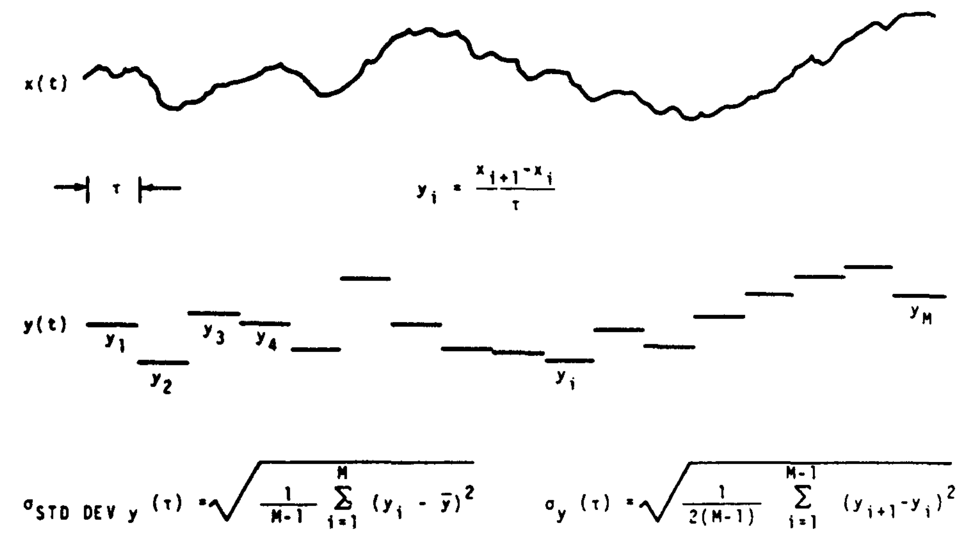

Fig. 8. Simulated time deviation plot $x(t)$ with indicated sample time τ over which each adjacent fractional frequency y_i is measured. Equations are for standard deviation and for estimate of $\sigma_y(\tau)$ for finite data set of M frequency measurements. Often standard deviation diverges as data length increases when measuring long-term frequency stability of precision oscillators, whereas $\sigma_y(\tau)$ converges.

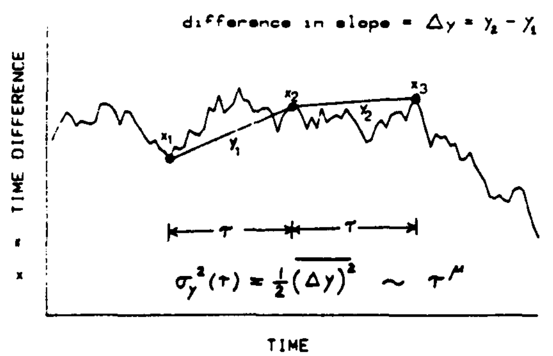

Fig. 9. Pictorial of computation of Allan variance. Simulated time variations plotted are random walk. At set sample time τ, $\Delta y = (x_3 - 2x_2 - x_1)/\tau$ is computed. With time of measurement of x_3 τ ahead of x_2 and that of x_2 τ ahead of x_1, all possible values of Δy are computed. Each Δy is squared and average squared value determined, $\overline{(\Delta y)^2}$; taking 1/2 of this yields two-sample or Allan variance for that value of τ. Value of τ can then be changed either in hardware or software to determine Allan variance for another value of τ.

$= \tau_0$), i.e., $n = 1$)

$$\sigma_y^2(\tau) = \frac{1}{2(M - 2n + 1)} \sum_{k=1}^{M-2n+1} (\bar{y}_{k+n}^{\tau} - \bar{y}_k^{\tau})^2 \quad (8)$$

where $\bar{y}_{k+n}^{\tau}$ and $\bar{y}_k^{\tau}$ are still adjacent fractional frequencies (i.e., no dead time exists between the measurements), each averaged over $\tau = n\tau_0$, and

$$\bar{y}_k^{\tau} = \frac{1}{n} \sum_{i=k}^{k+n-1} \bar{y}_i^{\tau_0} = \frac{x_{k-n} - x_k}{\tau}. \quad (9)*$$

Alternately, one may write (see Fig. 9 for an example)

$$\sigma_y^2(\tau) = \frac{1}{2\tau^2(M - 2n + 1)} \sum_{i=1}^{M-2n+1} (x_{i+2n} - 2x_{i+n} + x_i)^2 \quad (10)$$

* See Appendix Note # 15

where x_i is taken from the set of $M + 1 = N$ discrete time deviation measurements between a pair of clocks or oscillators, $i = 1$ to $M + 1$:

$$x_{k+1} = \tau_0 \sum_{i=1}^{k} \bar{y}_i^{\tau_0} + x_1. \quad (11)$$

Equation (8) is obtained from a first difference on frequency, and (10) from the second difference on the time; they are mathematically identical, yielding the option of using frequency or time (phase) data.

For power-law spectra the following proportionality applies: $\sigma_y^2(\tau) \sim \tau^{\mu}$, where μ is typically constant for a particular value of α. A simple and elegant relationship exists between the spectral density exponent α (in the relationship $S_y(f) \sim f^{\alpha}$) and μ, i.e., $\mu = -\alpha - 1$ ($-3 < \alpha \leq 1$) and $\mu = -2$ ($\alpha \geq 1$) [8]. For example, for a significant range of τ values, $\sigma_y(\tau) \sim \tau^{\mu/2}$ is proportional to $\tau^{-(1/2)}$ for cesium, rubidium, and passive hydrogen maser frequency standards. Therefore μ has the value of -1, and hence α has the value of 0 (white-noise frequency modulation). This is the classical noise exhibited by an important set of atomic clocks for τ's beyond a few seconds. In this case, $\sigma_y(\tau_0)$ is equal to the standard deviation. Fortunately, for most cases with precision clocks and oscillators where $\tau \geq 1$ s, the simple relationship $\mu = -\alpha - 1$ is applicable. It is convenient to plot $\log \sigma_y(\tau)$ versus $\log \tau$ to estimate the value of μ and to let $n = 2^l$, $l = 0, 1, 2, \cdots$ ($\tau = n\tau_0$).

An ambiguity exists at $\mu = -2$; one cannot conveniently tell whether the noise process is flicker-noise phase modulation (PM), $\alpha = +1$, or white-noise PM, $\alpha = +2$. This ambiguity can be resolved by realizing that for these cases $\sigma_y(\tau)$ depends on the measurement bandwidth [2], [3]. One can construct a variable software bandwidth f_s by realizing the following [9], [10]. In any measurement system a hardware bandwidth f_h exists through which we

TABLE II[a]

Typical Noise Types α	Name	$\sigma_y^2(\tau) =$	Classical Standard Deviation of x	Classical Standard Deviation of y
2	white-noise PM	$a_2\tau^{-3}$ [b]	$\tau \cdot \sigma_y(\tau)/\sqrt{3}$ (constant)	$\sigma_y(\tau)\sqrt{2(N+1)/3N}$
1	flicker-noise PM	$a_1\tau^{-2}$	$\sim\tau \cdot \sigma_y(\tau)\sqrt{\ln M/\ln 2}$	$\sim\sigma_y(\tau)\sqrt{2(N+1/3N}$
0	white-noise FM	$a_0\tau^{-1}$	$\tau_0 \cdot \sigma_y(\tau_0)\sqrt{(M+1)/6}$	$\sigma_y(\tau_0)$
−1	flicker-noise FM	$a_{-1}\tau^0$	undefined	$\sigma_y(\tau)\sqrt{N\ln N/(2(N-1)\ln 2)}$
−2	random-walk FM	$a_{-2}\tau$	undefined	$\sigma_y(\tau)\sqrt{N/2}$

[a] Note τ is a general averaging time and τ_0 is the initial averaging time ($\tau = n\tau_0$, where n is an integer). Also note that the last four entries in the fourth column and the last two entries in the fifth column go to infinity as M or N go to infinity. M is the initial number of frequency difference measurements and N the number of phase or time difference measurements $N = M + 1$. If the spectral density is given by $S_y(f) = h_\alpha f^\alpha$, then

$$a_2 = \left(\frac{1}{2\pi}\right)^2 3f_h\tau_0 h_2{}^{b}$$

$$a_1 = \left(\frac{1}{2\pi}\right)^2 (1.038 + 3\log_e(2\pi f_h\tau)h_1)$$

$$a_0 = \left(\frac{1}{2}\right)h_0$$

$$a_{-1} = 2\log_e(2)h_{-1}$$

$$a_{-2} = \frac{1}{6}(2\pi)^2 h_{-2}.$$

[b] Note this equality assumes use of modified $\sigma_y^2(\tau) = \bar{\sigma}_y^2(\tau)$.

measure the phase difference or the time difference between a pair of oscillators or clocks and we define $\tau_h = 1/f_h$. In other words, τ_h is the sample time period through which the time or phase date are observed or averaged. Averaging n time or phase readings increases the sample time window to $n\tau_h = \tau_s$. Let $\tau_s = 1/f_s$; then $f_s = f_h/n$, i.e., the software bandwidth is narrowed to f_s. In other words, $f_s = f_h/n$ decreases as we average more values; i.e., increase n ($\tau = n\tau_0$). One can therefore construct a second difference composed of time deviations so-averaged and then define a modified $\sigma_y^2(\tau) = \bar{\sigma}_y^2(\tau)$ that will remove the ambiguity through bandwidth variation:

$$\bar{\sigma}_y^2(\tau) \simeq \frac{1}{2\tau^2 n^2(N-3n+1)} \sum_{j=1}^{N-3n+1}\left(\sum_{i=j}^{n+j-1}(x_{i+2n} - 2x_{i+n} + x_i)\right)^2 \quad * \tag{12}$$

where $N = M + 1$, the number of time-deviation measurements available from the data set. Now if $\bar{\sigma}_y^2(\tau) \sim \tau^{\mu'}$, then $\mu' = -\alpha - 1$ ($1 \le \alpha \le 3$) [10], [11]. Thus $\bar{\sigma}_y(\tau)$ is typically employed as a subroutine to remove the ambiguity if $\sigma_y(\tau) \sim \tau^{-1}$. This is because the $\mu' = -\alpha - 1$ relationship is valid as an asymptotic limit for large n and $\alpha < 1$ and is not valid in general; however, there is evidence that $\bar{\sigma}_y^2(\tau)$ may be a better measure [12]. Specifically, for $\alpha = 2$ and 1, $\mu'/2$ equals $-3/2$ and -1, respectively, providing a clean differentiation between white-noise PM and flicker-noise PM.

If three or more independent oscillators or clocks are available along with time (phase) or frequency measurements between them, then it is possible to estimate a variance for each oscillator or clock. Often there is a reference to which the rest are periodically measured at a sampling rate $1/\tau_0$. If at each measurement the time or frequency differences between the clocks are measured at nominally the same time, then the time difference or frequency difference can usually be estimated or calculated between every possible pair in the set of oscillators or clocks. Given a series of measurements, variances s_{ij}^2 can be calculated on the time or frequency data between all pairs. It has been shown [13] that the individual clock variances can be estimated using the following equations:

$$\sigma_i^2 \doteq \frac{1}{m-2}\left(\sum_{j=1}^{m} s_{ij}^2 - B\right)$$

where

$$B = \frac{1}{m-1}\sum_{i<j}^{m} s_{ij}^2, \tag{13}$$

m is the number of clocks available in the set, and $s_{ii}^2 = 0$. If the variance measures used are $\sigma_y^2(\tau)$ or $\bar{\sigma}_y^2(\tau)$, then (13) can be used to estimate the individual variances as a function of τ.

Table II illustrates why one should not use the standard deviation to characterize clocks. For the different kinds of noise processes we list the standard deviation of the time deviations and of the fractional frequency deviations as a function of $\sigma_y(\tau)$. The divergent nature of the standard deviation is apparent. Even for classical white-noise FM the standard deviation of the time diverges as the square root of the data length, i.e., the number of samples N [2].

* See Appendix Note # 16

TABLE III

α	Typical Noise Types Name	Optimum Prediction $x(\tau_p)$ rms[a]	Time Error: Asymptotic Form
2	white-noise PM	$\tau_p \cdot \sigma_y(\tau_p)/\sqrt{3}$	constant
1	flicker-noise PM	$\sim \tau_p \cdot \sigma_y(\tau_p)\sqrt{\ln \tau_p / 2 \ln \tau_0}$	$\sqrt{\ln \tau_p}$
0	white-noise FM	$\tau_p \cdot \sigma_y(\tau_p)$	$\tau_p^{1/2}$
−1	flicker-noise FM	$\tau_p \cdot \sigma_y(\tau_p)/\sqrt{\ln 2}$	τ_p
−2	random-walk FM	$\tau_p \cdot \sigma_y(\tau_p)$	$\tau_p^{3/2}$

[a] τ_p is the prediction interval.

Time and Frequency Estimation and Prediction

Using $\sigma_y(\tau)$, $\bar{\sigma}_y(\tau)$, $S_y(f)$, or $S_\phi(f)$, one can characterize typical power law processes. Once characterized, this opens the opportunity for determining optimum estimates of values by employing the statistical theorem that the optimum estimate of a white-noise process is the simple mean.

For example, consider the very common and very important case of white-noise FM typically found on the signals from cesium standards, rubidium standards, and passive hydrogen masers. The optimum estimate of the frequency is the simple mean frequency, which is equivalent to $(x_N - x_1)/M\tau_0$. It is still all too common within our discipline to see our colleagues erroneously determining the frequency for these kinds of oscillators by calculating the slope from a linear least-squares fit to the time deviations and quoting the standard deviation around that fit as a measure of the clock performance. There are three problems in proceeding this way. First, the frequency estimate is not optimum in a mean-square-error sense. It is equivalent to throwing away about 20 percent of the data and thereby increasing the cost in the case of a calibration. Second, the standard deviation diverges as the square root of the data length. Third, the standard deviation is significantly dependent on the filter form, e.g., linear least squares, as well as the clock deviations. On the other hand, such a filter is sometimes useful for assessing outliers. The optimum "end-point" method outlined earlier has the risk that if either of the points is abnormal, (i.e., the model fails), the result will of course be adversely effected. Therefore such a filter is useful to assess whether there are outliers—paying special attention to the end points. Also, if the measurement noise exceeds the combined noise in the clocks, then the end points will be adversely affected. The key message is that the end-point method for estimating frequency is only optimum if the noise is pure white FM, which is easy to determine from a log $\sigma_y(\tau)$ versus log τ plot.

There are other useful, and maybe not so obvious, optimum estimators appropriate for time-difference data sets.

1) Given white-noise PM, the best time estimate is the simple mean of the time deviations; the frequency estimate then is the slope from a linear least-squares fit to the time deviations, and the frequency drift D is determined from a quadratic least-squares fit to the time deviations per (1).
2) Given white-noise FM, the optimum estimate of the time is the last value; the optimum-frequency estimate is outlined in the previous paragraph, and the optimum-frequency-drift estimate is derived from a linear least-squares fit to the frequency.
3) Given random-walk FM, the current optimum time estimate is the last value plus the last slope (clock rate) times the time since the last value; the optimum-frequency estimate is obtained from the last slope of the time deviations; and the optimum-frequency-drift estimate is calculated from the mean second difference of the time deviations. Caution needs to be exercised here, for typically there will be higher frequency component noise in a real data stream, such as white-noise FM, along with random-walk FM, and this can significantly contaminate the drift estimate from a mean second difference. If random-walk FM is the predominant long-term power-law process, which is often the case, then the effect of high-frequency noise can be reduced by calculating the second difference from the first, middle, and end-time deviation points of the data set.

The flicker-noise cases are significantly more complicated, though filters can be designed to approximate optimum estimation [14]–[16]. As the data length increases without limit, time is not defined for flicker-noise PM, and frequency is not defined for flicker-noise FM. This has some philosophical implications for the definitions of time and frequency, unless some low-frequency cutoff limits exist. If significant frequency drift exists in the data, it should be optimally subtracted from the data or it will bias the long-term values of $\sigma_y(\tau)$:

$$\sigma_y(\tau) = \frac{D\tau}{\sqrt{2}}. \tag{14}$$

Once the power-law spectra are deduced for a pair of oscillators or clocks, then one can also develop an optimum predictor. Table III gives both the optimum prediction uncertainty values for the various relevant pure power-law spectra as well as their asymptotic forms. Special forecasting techniques must be used for optimal prediction when combinations of these processes are present [17]. To illustrate how these concepts relate to real devices, Fig. 10 shows a $\sigma_y(\tau)$ diagram for some interesting

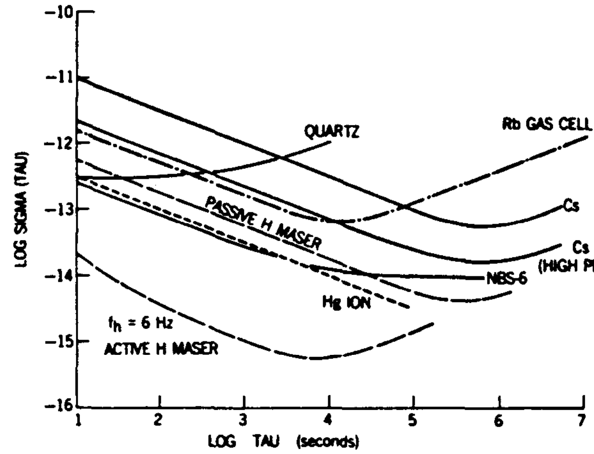

Fig. 10. Square root of Allan variance for variety of state-of-the-art precision oscillators including NBS-6, NBS primary frequency standard.

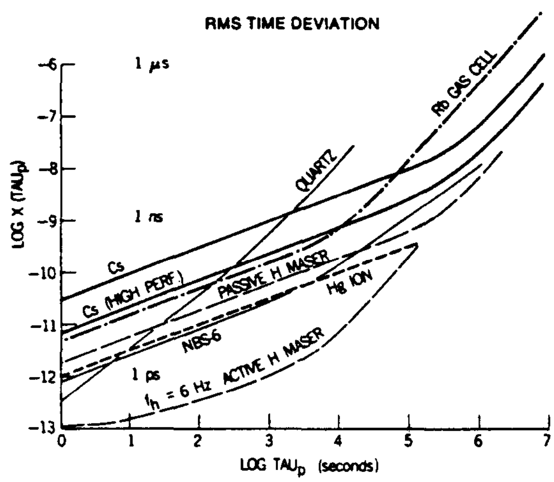

Fig. 11. From frequency stability characterization shown in Fig. 10, optimum prediction algorithms to minimize time error can be obtained. Based on optimum prediction procedures rms time prediction error for prediction interval τ_p can be calculated for each oscillator shown in Fig. 10 and corresponding values are plotted in Fig. 11.

state-of-the-art oscillators, and Fig. 11 shows the rms time prediction errors for the same set of oscillators.

Conclusion

In conclusion, it is clear that classical statistics do not allow characterization of common kinds of random signal variations found in precision oscillators. The two-sample Allan variance provides a valuable and convergent measure of the power-law spectral-density models useful in characterizing random deviations for most oscillators and clocks. Once characterized, we can calculate optimum time and frequency estimates as well as predicted values. Characterizing the random variations also provides near-optimum estimation of systematic effects, which often cause the predominant time and frequency deviations. For example, if we wanted to optimally determine the static temperature dependence with the temperature set at two different values, we would stabilize the oscillator at one temperature and measure the frequency against a reference for a time τ_m, corresponding to the τ for the nominal minimum $\sigma_y(\tau)$ value. We would then change the temperature to the other value and repeat the measurement with the same criteria and note the $\Delta \bar{y}^{\tau_m}$ between the two optimally determined frequency values. If these two steps are repeated several times, an arbitrarily good precision for the temperature coefficient is achieved if it is linear. The uncertainty is approximately given by $\sigma_y(\tau_m)/\sqrt{P}$, where P is the number of $\Delta \bar{y}^{\tau_m}$ values obtained from switching back and forth. Knowing the characteristics of both the random and the systematic deviations of precision clocks and oscillators clearly is useful to the designer, the manufacturer, the planner, and the user as well as the vendor of these devices.

The aforementioned procedures usually work well if the clocks or oscillators are in a reasonable environment. If the environment is adverse, other procedures and analysis methods may have to be employed. As a general rule it is often useful to analyze the data in the frequency domain as well as the time domain. The frequency domain is especially useful if there are bright lines, i.e., sidebands to the carrier frequency. The effect of a modulation sideband f_m on $\sigma_y(\tau)$ can be calculated, and is given by [18]

$$\sigma_y(\tau) = \frac{x_{pp}}{\tau} \sin^2 (\pi f_m \tau) \qquad (15)$$

where x_{pp} is equal to the peak-to-peak or twice the amplitude of the time-deviation modulation.

If one is trying to estimate the power-law spectral behavior between a pair of oscillators or clocks using $\sigma_y(\tau)$, it is apparent from (15) that if significant modulation sidebands are present on the signal, these can seriously contaminate that estimate. However, if a $\sigma_y(\tau)$ plot displays a character as given by (15), then the amplitude and frequency of that modulation sideband can be estimated from this time-domain analysis technique. In practice, this approach is often used, but these modulation sidebands can be more efficiently estimated in the frequency domain. If the measurement sampling rate $1/\tau_0$ is set equal to f_m, then the modulation sideband is aliased away and has no effect on $\sigma_y(\tau)$.

The best rule in all analysis is to use common sense. Very often the most revealing information may be in a plot of the raw time (phase) difference or frequency-difference residuals after some trend has been removed. Such a plot is usually the first thing to look at when characterizing clocks and oscillators. Caution here is also important as a pure random walk on the time residuals (white FM) may be visually interpreted as having frequency steps. This is especially true for flicker FM as often seen in quartz-crystal oscillators. Following the time-residual plot with a $\sigma_y(\tau)$ analysis often answers the question as to whether or not such steps are statistically significant.

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David W. Allan for a photograph and biography, please see page 571 of this Transactions.