# Stability of Frequency Locked Loops

F.L. Walls

National Institute of Standards and Technology,
Boulder, CO 80303, USA

## 1. Introduction

Passive frequency standards are characterized by the use of a reference resonance to stabilize the frequency of an external probe oscillator. A common configuration is shown in Fig. 1 [1-6]. The probe oscillator generally has phase modulation imposed on the carrier in order to interrogate the resonance with a minimum of offset. The resulting amplitude modulation is demodulated to yield an error curve that is essentially the derivative of the resonance. Although Fig. 1 shows a transmission system, similar schemes are sometimes possible in reflection [5]. The error signal from the demodulator is used to steer the probe signal toward the center of the resonance line [1-9]. For analysis times longer than one period of the modulation cycle, and under the condition that the probe oscillator wanders less than the half width of the error curve in the loop attack time, we can treat this curve as approximately static[1-4]. Near line center the loop error voltage $V_D$ at the synchronous detector, is approximately $V_D = k\ (\nu_o{}^o - \nu_R) + V_n$ where k is the slope of the error curve, $\nu_R$ is the resonance frequency of the reference, and $\nu_o{}^o$ is the open loop frequency of the probe oscillator and $V_n$ is the detector noise. If we now close the loop with gain $G(f)$, it can be shown that the spectral density of fractional frequency fluctuations $S_y(f)$ for the probe source becomes
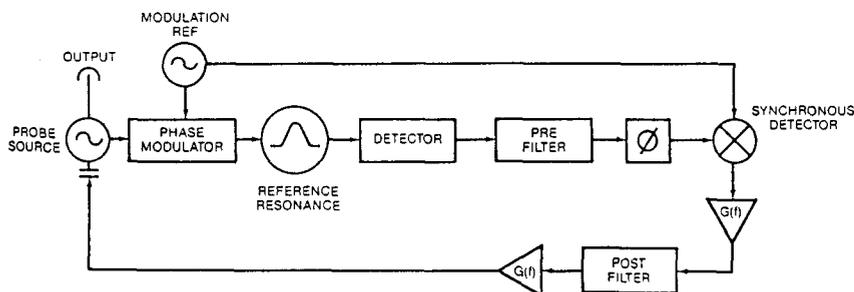


Figure 1. Generalized block diagram of a probe source locked to a reference resonance.

$$S_y(f) = \left(\frac{G(f)}{G(f)+1}\right)^2 \left(S_y{}^R + \frac{S_y{}^O(f)}{G^2(f)}\right) + S_y{}^N(f) \qquad (1)$$

where $S_y{}^O(f)$ is the open-loop spectral density of fractional frequency fluctuations of the probe source, $S_y{}^R(f)$ is that of the reference, and $S_y{}^N(f)$ is that of the detector and interrogation noise referred to the demodulator output. Cutler [10] has pointed out that noise in the local oscillator at the 2nd harmonic of the modulation frequency, which is usually ignored, causes a time varying frequency offset that is undistinguishable from reference noise. This sets the lower limit to the interrogation noise and often sets the lower limit of the noise performance of the local oscillator necessary not to degrade the overall performance. The magnitude of the 2nd harmonic noise modulation in radians/s is estimated in appendix B of [1] to be $k_3 = 2\pi\nu(S_y(\Omega/\pi))^{1/2}$, where $\Omega/(2\pi)$ is the modulation frequency. This leads to an interrogation noise term which is of order [1,4,10] $S_y{}^N(f) = 1/(16\pi) \, S_y{}^O(\Omega/\pi)$. For large values of $G(f)$, $S_y(f)$ of the probe source reflects that of the reference plus the added noise of the detection system [1-6].

The primary goal of this paper is to investigate the effect of various realistic forms of $G(f)$ on the spectral density of frequency and fractional-frequency stability. It will be shown that mod $\sigma_y(\tau)$ [11,12] is better suited than the traditional two-sample or Allan Variance $\sigma_y(\tau)$ [8], for evaluating the locked performance when the frequency stability of the reference is much greater than that of the local oscillator [7,8,11,12].

## 2. The Effect of Different Forms of Servo Gain

Figure 2 shows the effect of locking a probe source with $S_y{}^O(f) = 2 \times 10^{-28}/f^2 + 1 \times 10^{-24}/f + 2 \times 10^{-30}f^2$ (which roughly corresponds to that of a low-noise 5 MHz quartz oscillator) to a reference resonance with $S_y{}^R(f)$ of $2 \times 10^{-30}$ and $_{Sy}{}^N(f) = 5.6 \times 10^{-28}$. $S_y{}^N(f)$ is the estimated interrogation noise for a modulation frequency of 47 Hz [1,4,10]. Curves A, B, and C show the effect of using a first-, second-, or third- order loop each having a loop bandwidth of approximately 0.1 Hz or an attack time of 1.6 s[1]. The solid Curves A, B, and C of Fig. 3 show $\sigma_y(\tau)$ calculated for curves A, B, and C from Fig. 2 with a noise bandwidth of 3 Hz. The improvement in stability of the local oscillator due to the servo scales roughly as $\tau/\tau_O$ where $\tau_O$ is the attack time and $\tau$ is the measurement time. Considerable insight into the effect of various gain stages on the frequency stability can be obtained by considering

$$\frac{\Delta\nu}{\nu_0} \sim \sqrt{\overline{\Delta\varphi^2/(2\pi\nu_R\tau)^2}} \qquad (2)$$

as a measure of fractional time or frequency stability in the region of measurement times from about 1 to $10^5$ s where the effects of low frequency
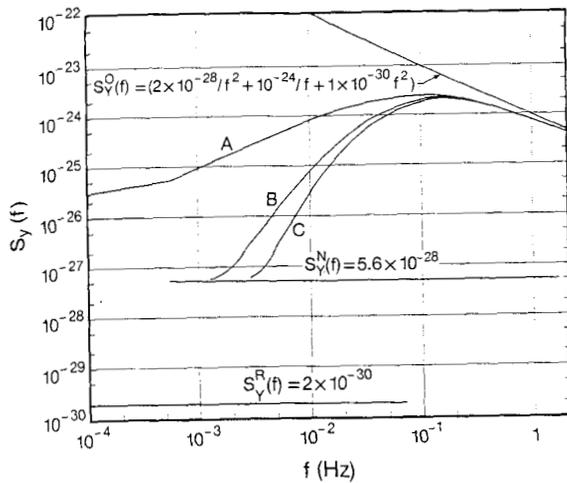
146

Figure 2. $S_y(f)$ of the probe source locked to a reference resonance. For A $G(f) = (1/(10f))$, for B $G(f) = (1/(10f))(1 + 1/(40f))$, for C $G(f) = 1/(10f))(1 + 1/(40f))(1 + 1/(160f))$.
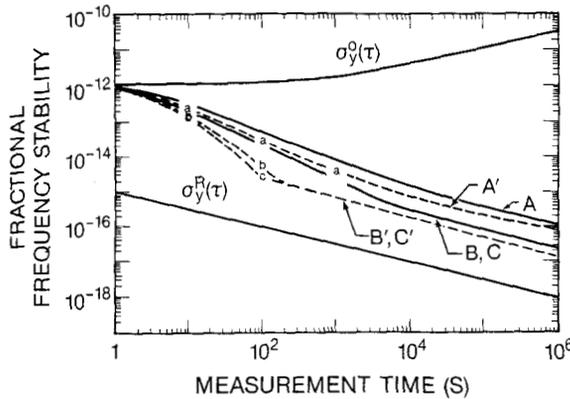
In Figure 2: $S_Y^O(f) = (2 \times 10^{-28}/f^2 + 10^{-24}/f + 1 \times 10^{-30} f^2)$; $S_Y^N(f) = 5.6 \times 10^{-28}$; $S_Y^R(f) = 2 \times 10^{-30}$



Figure 3. The solid curves show $\sigma_y(\tau)$ and the dashed curves show mod $\sigma_y(\tau)$ versus $\tau$ for curves A, B, and C of Fig. 2. The a, b, and c points show estimations of mod $\sigma_y(\tau)$ using Eqs. 2 and 3 with $f_h = f_{ho}$. Also shown is $\sigma_y^O(\tau)$ and $\sigma_y^R(\tau)$.

divergence can often be ignored [7,8]. The squared phase deviation of the zero crossings is given approximately by[8]

$$\Delta\varphi^2 = \nu_R^2 \int_{1/2\pi^2\tau}^{f_h} (S_y(f))/f^2 \, df \qquad (3)$$

for a sample time $\tau$ and noise bandwidth $f_h$. For a noise spectrum which varies as $S_y(f) = K'f^n$ where $n > 2$, the integral is dominated by the high frequency bandwidth even for very long measurement times. The fractional-frequency (or time) stability given by Eq. 2 decreases as $\tau^{-1}$ just as does $\sigma_y(\tau)$, due to the increase in measurement time and not to a decrease in the value of $\Delta\varphi^2$. The integral in Eq. 3 can be integrated by parts for the various segments of $S_y(f)$. This makes it easier to evaluate and optimize the performance of the overall system than by using a process based on $\sigma_y(\tau)$.

147

The contribution of the high frequency noise can be greatly reduced by phase averaging the data points[9]. The data at measurement time $\tau = n\tau_0$ (where $\tau_0$ is the data interval) is obtained by averaging the n adjacent phase points. This is equivalent to using mod $\sigma_y(\tau)$ to analyze the data[9,11,12]. The standard expression for mod $\sigma_y(\tau)$ contains an enormous number of terms and is quite laborious to compute[11-12]. Gary [13] has pointed out that mod $\sigma_y(\tau)$ can be reduced to

$$\text{mod } \sigma_y^2(\tau) = \frac{2}{n^4\pi^2\tau_0^2} \int_0^{f_h} \frac{S_y(f)\sin^6(\pi\tau_0 nf)}{f^2\sin^2(\pi\tau_0 f)} \, df. \tag{4}$$

which is much more manageable. Never the less it still requires numerical calculations to determine which segment of the phase noise dominates the integral. We can estimate mod $\sigma_y(\tau)$ from Eqs. 2 and 3 by using $f_h = f_{ho}/n$, where $f_{ho}$ is the hardware bandwidth of the measurement system and $\tau = n\tau_0$ is the measurement time. The integral for $\Delta\varphi^2$ in Eq. 3 is now substantially reduced for measurement times large enough that $f_{ho}/n$ is less than the bandwidth of the servo system. This is in contrast to the original calculation for Eq. 3 where the integral must always increase with $\tau$. This integral can also be divided into parts and integrated analytically. The fractional frequency stability computed for A, B, and C of Fig. 2 using mod $\sigma_y(\tau)$ are shown as the dashed curves A', B', and C' of Fig. 3. The points labeled a, b, c show the results of estimating mod $\sigma_y(\tau)$ using Eqs. 2 and 3 with $f_h = f_{ho}/n$. The agreement with the dashed curves is very good. The calculations for $\sigma_y(\tau)$ and mod $\sigma_y(\tau)$ are virtually independent of using an upper cutoff frequency for the integration or a simple low pass filter of the same bandwidth. The use of mod $\sigma_y(\tau)$ reduces by a factor of about 100 the time necessary to reach the performance of the reference plus the interrogation noise, which in this case limits the performance for times longer than 100 s.

## 3. Discussion

The primary utility of these results is the insight into the origin of the major contributions to $\sigma_y(\tau)$ and mod $\sigma_y(\tau)$ as a function of the servo gain and the analysis bandwidth, and the effect of narrowing the bandwidth with longer measurement times. The specific examples illustrate that it is generally necessary to use a second-order loop to lock the probe source to the reference resonance, but that a third-order loop offers little additional improvement when drift in the probe is not serious. In addition, if the reference resonance is substantially more stable than the probe source, it can be very useful to use mod $\sigma_y(\tau)$ to analyze the output of the probe source. We have introduced a simple measure of frequency stability that is easy to use for optimizing the servo and the analysis system. With the techniques introduced here it is possible to

realize fractional frequency stabilities and time prediction of the local oscillator which are characteristic of the reference resonance in a way that is orders of magnitude faster than those using more traditional approaches.

## 4. Acknowledgements

## 5. References

1.  F. L. Walls and S. R. Stein, NBS Tech Note 692, U.S. Government Printing Office, Washington, D.C., SD Cat. # C1346:692
2.  C. Audoin and V. Candelier, IEEE Trans. on UFFC, UFFC-34, 573-581 (1987)
3.  A. DeMarchi, G. D. Rovera, and A. Premoli, IEEE Trans. on UFFC, UFFC-34, 582-591 (1987)
4.  F. W. Walls, IEEE Trans. on UFFC, UFFC-34, 592-597 (1987)
5.  S. R. Stein, Proc. of 29th Symposium on Frequency Control, 321-327 (1975)
6.  J. Vanier, M. Tetu, and L. G. Berniez, IEEE Trans. on I&M IM-28, 188-193
7.  D. W. Allan, Proc. IEEE, 54, 221-230 (1966)
8.  J. A. Barnes et al., IEEE Trans. I&M, IM-20, 120 (1971)
9.  J. J. Snyder, Appl. Opt., 19, 1223-1225 (1980)
10. L. S. Cutler, Hewlett Packard, Private communication, (Sept. 1988).
11. D. W. Allan and J. A. Barnes, Proc. 35th Annu. Symp. Frequency Control, 470-474 (1981)
12. P. Lesage and T. Ayi, IEEE Trans. I&M, IM-33, 332-336 (1984)
13. J. Gary, National Bureau of Standards, Boulder, CO, private communication.