CHAPTER 8

STATISTICS OF TIME AND FREQUENCY DATA ANALYSIS*●

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Contents

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●Parts of this Chapter have been modified from "Characterization of frequency stability" published in *IEEE Trans. on Instrum. and Meas.*, May 1971 (see reference [1]), publication herein is at concurrence of the IEEE.

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"As a concept for ordering and analyzing real events with variable amounts of information 'time' is much more complex than a simple clock-measure."

Patrick Meredith,
Study of Time, p. 84

The fluctuations about the average of a time or frequency signal characterize its quality. The statistical characteristics of the random fluctuations are needed in any clear statement regarding the stability of the signal as well as the credibility of the average value. We describe two methods of characterizing the random fluctuations of a signal, one in the frequency domain and one in the time domain.

In the frequency domain we characterize the fluctuations using the spectral density of the fluctuations of the time or of the fractional frequency. We use spectral densities which are one-sided and are on a per hertz basis. In the time domain the Allan variance is employed to characterize the time fluctuations and/or the fractional frequency fluctuations. The need to specify certain aspects of the measurement is emphasized, i.e., the sample time, the sample repetition rate, the measurement system bandwidth, and the number of samples in each variance. The relationships between these measures of stability are given.

Power law spectral densities can describe the commonly occurring kinds of random fluctuations which include white noise and flicker noise time or frequency fluctuations. For these commonly occurring noise processes some tables of convenient relationships between the frequency domain and the time domain measures are also given. In addition, for different, but specific, power law noise processes the dependence of the time domain stability measure on the number of samples or on the dead time of the measurement process is tabulated. The above stability measures have proven very useful, and examples of application are given, e.g., estimation of the time dispersion of a clock and specification of the detailed quality of a state-of-the-art frequency standard. Operational systems for measurement of frequency stability are described in detail sufficient for duplicating techniques and results.

Key words: Allan variance; frequency; frequency domain; frequency stability measurements; measurement system description; phase noise; sample variance; spectral density; stability definitions; terminology standards; time domain clock statistics; time/frequency statistics; variance.

8.1. INTRODUCTION

This chapter discusses some reasonable measures for time and/or frequency deviations. The subject is treated on both the theoretical and practical level. Section 8.2 presents some definitions and fundamentals to give understanding into what a deviation is. Sections 8.3 through 8.10 give a theoretical exposition into measures of frequency stability and contain major parts of the paper, "Characterization of frequency stability," by J. A. Barnes et al. [1].[1] Sections 8.11 through 8.15 show general applications and examples of stability measures in the laboratory, based on the paper cited above [1] and from the NBS Technical Note entitled "Frequency stability specification and measurement: High frequency and microwave signals" [2]. Standards of terminology and measurement techniques are recommended to facilitate conformity in reporting results from various laboratories with a commonality of reference. Some ten annexes are given. They include a glossary of terms, sample calculations and derivations, a frequency stability computer program, selected references and a bibliography applicable to frequency stability, and a reproduction of bias function tables. This latter is part of the NBS Technical Note, "Tables of bias functions, B_1 and B_2, for variances based on finite samples of processes with power law spectral densities" [3]. We have attempted to coordinate the chapter with sufficient independence of subject matter but allowing adequate material in a given area to satisfy both the theorist and the practical engineer.

8.2. DEFINITIONS AND FUNDAMENTALS

8.2.1. Aspects of Time

Time—as one of the four independent base units of measurement—has been adopted as a defined quantity. Currently, the unit of time is defined: "The second is the duration of 9 192 631 770 periods of the radiation corresponding to the transition between the two hyperfine levels of the ground state of the cesium-133 atom," as decreed by the General Conference on Weights and Measures [4] (see chap. 1, ann. 1.A.1). There are three fundamental aspects of time[2]: the first, covered by the definition above, is time interval which can be related to frequency—frequency being the inverse period of an oscillation. The second aspect is that of date or clock reading which often has been called epoch. We prefer the use of the word date because epoch has alternate meanings that could lead to confusion. Date or clock reading is simply the counting or accumulation—starting from some predetermined origin—of unit time intervals; e.g., "A special adjustment to the standard-frequency and time-signal emissions should be made at the end of 1971 so that the reading of the UTC scale will be 1 January 1972, 0h 0m 0s at the instant when the reading of Atomic Time (AT) indicated by the Bureau International de l'Heure (BIH) will be 1 January 1972, 0h 0m 10s" [5] (see chap. 1, ann. 1.B). The third aspect is simultaneity—the practical application of which is clock synchronization; i.e., two clocks have the same reading in some frame of reference.

8.2.2. Definitions: Stability, Reproducibility, Accuracy

Let us define *a clock* as a frequency standard coupled with a counter-divider. We next might question why a clock deviates from the ideal, where an ideal clock realizes the ultimate of the above three aspects of time. To further explain, the ideal clock's unit time interval would agree exactly with the definition, and at some particular moment the reading of the clock would agree exactly (would be synchronous) with some defined date or origin. Categorically there are two reasons for deviations. First, the clock may have *deterministic* biases; e.g., the frequency may drift with time, the calibration of the unit time interval or the frequency may have had an error—causing a systematic accumulation of time error, or the clock's initial reading (date) may have been set erroneously. This first category may be called nonrandom. Second, the clock will have *random* deviations caused by various kinds of random noise processes inherent in all clocks; e.g., the shot noise at the detector in an atomic beam frequency standard causes a random walk in the time deviations when this standard is integrated in a clock. Often, and meaningfully so, a random deviation in time or frequency is called a time fluctuation or a frequency fluctuation, respectively.

We will now define some terms to better describe nonrandom and random deviations in a time and/or frequency standard. In this regard we have drawn on the work of others [6] as well as collaborated with co-workers [7]; in addition a strong attempt was made to make the definitions consistent with current methods of characterizing frequency and time stability.

We consider *accuracy* as the degree of conformity of a measured and/or calculated value to some specified figure or definition. For example, the time accuracy is the degree of conformity of a clock's date with some defined date; and the time interval accuracy—which corresponds to the clock rate or frequency accuracy—is the conformity with the definition for the second as given in Section 8.2.1. Typically, the prime cause of inaccuracy results from nonrandom deviations, but naturally it involves random fluctuations as well.

[1] Figures in brackets indicate literature references at the end of this chapter.
[2] See Chapter 1 on "Basic Concepts of Precise Time and Frequency."

To determine the accuracy of a device or standard, it is necessary to evaluate every parameter which may cause a deviation from the definition. It is necessary to consider both (a) an estimate of the uncertainty in determining the magnitude of the deviation (intrinsic reproducibility of that parameter), and (b) the precision (see definition below of any measurement involved in the evaluation). If the parameters are independent in their effect in causing deviations from the definitions, then the uncertainties and precisions may be added as the square root of the sum of the squares to give an accuracy specification. It clearly should be stated also whether the accuracy has a one-, two-, or three-sigma (σ=standard deviation) confidence limit; naturally, the uncertainties and precisions should be consistent with the estimate of the accuracy specification.

By *stability* we mean the frequency domain and/or time domain behavior of a process. The proposed frequency domain measure of frequency stability is the spectral density of the fractional frequency fluctuations, $S_y(f)$. The proposed measure of stability in the time domain is the Allan variance (or its square root) [1, 8]. One commonly measures the instabilities caused by the random fluctuations in a device; obviously, though, the nonrandom deviations can contribute instabilities. Section 8.6 develops and defines stability measures for frequency and time.

Precision is the performance capability of a measurement process with contributions from both the measurement equipment and the devices or standards being measured. Precision is often the best consistently attainable indicator for the measurement process. Consider, too, the precision of a frequency standard measurement is limited by the instability of either the standard or the measurement equipment, whichever is worse. On the other hand, a time interval measuring instrument may have a precision of 0.1 ns inherent within the processing electronic counting and measuring devices; in contrast, it may have an accuracy of 1 ns, where the accuracy now requires also a knowledge of the actual delays involved in the instrument. The precision of a measurement for a standard with random uncorrelated fluctuations (usually idealized but an unrealistic assumption) in output is simply the standard deviation of the mean of these fluctuations; typically, a data set taken as a time series has some correlation. If such is the case, then a measure of the precision of a measurement could be the time domain stability ($\sigma_y(\tau)$ (defined in sec. 8.6.2)) at a sampling or averaging time, τ, such that the stability no longer improves with increasing τ. In such a specification the τ value or values need to be stated along with the number of samples, the measurement system bandwidth, and the dead time between data samples. The accuracy can never be better than the precision.

Reproducibility means the degree of agreement across a set of independent (in space and/or time) devices of the same design after exclusive evaluation of appropriate parameters in each device. An independent set of devices may consist of either an ensemble of devices or standards constructed according to the same procedure, or a single device having all parameters independently adjusted through reevaluation. For the single device, the degree of agreement is often called intrinsic reproducibility. Reproducibility is a relative measure in contrast to accuracy which is an absolute measure. The word accuracy is sometimes erroneously used in place of reproducibility, and it is clear that a device's accuracy can never be better than its reproducibility. One acceptable measure of reproducibility would be the following: given a mean value for each of a set of measurements across an ensemble or on a device as stated above, the reproducibility is the standard deviation of this set of measurements about the mean.

8.3. CHARACTERIZATION OF FREQUENCY STABILITY

The measurement of frequency and fluctuations in frequency has received such great attention for so many years that it is surprising that the concept of frequency stability does not have a universally accepted definition. At least part of the reason has been that some uses are most readily described in frequency domain and other uses in the time domain, as well as in combinations of the two. This situation is further complicated by the fact that only recently have noise models been presented which both adequately describe performance and allow a translation between the time and frequency domains. Indeed, only recently has it been recognized that there can be a wide discrepancy between commonly-used time domain measures themselves. Following the NASA–IEEE Symposium on Short-Term Stability in 1964 [9] and the Special Issue on Frequency Stability of the *Proc. IEEE* of February 1966 [10], it now seems reasonable to propose a definition of frequency stability. The paper by Barnes et al. [1] was presented as technical background for an eventual IEEE standard definition.

This section attempts to present (as concisely as practical) adequate, self-consistent definitions of frequency stability. Since more than one definition of frequency stability is presented, an important part of this section (perhaps the most important part) deals with translations among the suggested definitions of frequency stability. The applicability of these definitions to the more common noise models is demonstrated. Consistent with an attempt to be concise, the cited references have been selected for greatest value to the reader rather than for their exhaustive nature. Annex 8.G is a more comprehensive reference and bibliographic listing covering the subject of frequency stability.

Almost any signal generator is influenced to some extent by its environment. Thus, observed frequency instabilities may be traced, for example, to changes in ambient temperature, supply voltages, magnetic field, barometric pressure, humidity, physical vibration, or even output loading to mention the more obvious. While these environmental influences may be extremely inportant for many applications, the definition of frequency stability presented here is independent of these casual factors. In effect, we cannot hope to present an exhaustive list of environmental factors and a prescription for handling each, even though these environmental factors may be by far the most important in some cases. Given a particular signal generator in a particular environment, one can obtain its frequency stability with the measures presented herein, but one should not then expect the frequency stability always to be the same in a new environment.

It is natural to expect any definition of stability to involve various statistical considerations such as stationarity, ergodicity, average, variance, spectral density, etc. There often exist fundamental difficulties in rigorous attempts to bring these concepts into the laboratory. It is worth considering, specifically, the concept of stationarity since it is at the root of many statistical discussions.

A random process is mathematically defined as stationary if every translation of the time coordinate maps the ensemble onto itself. As a necessary condition, if one looks at the ensemble at one instant of time, t, the distribution in values within the ensemble is exactly the same as at any other instant of time, t'. This is not to imply that the elements of the ensemble are constant in time, but, as one element changes value with time, other elements of the ensemble assume the previous values. Looking at it in another way, by observing the ensemble at some instant of time, one can deduce no information as to when the particular instant was chosen. This same sort of invariance of the joint distribution holds for any set of time t_1, t_2, . . ., t_n and its translation $t_1+\tau$, $t_2+\tau$, . . ., $t_n+\tau$.

It is apparent that any ensemble that has both a finite past and future cannot be stationary, and this neatly excludes the real world and anything of practical interest. The concept of stationarity does violence to concepts of causality since we implicitly feel that current performance (i.e., the applicability of stationary statistics) cannot be logically dependent upon future events (i.e., if the process is terminated sometime in the distant future). Also, the verification of stationarity would involve hypothetical measurements which are not experimentally feasible, and therefore the concept of stationarity is not directly relevant to experimentation.

Actually the utility of statistics is in the formation of idealized models which *reasonably* describe significant observables of real systems. One may, for example, consider a hypothetical ensemble of noises with certain properties (such as stationarity) as a model for a particular real device. If a model is to be acceptable, it should have at least two properties: First the model should be tractable; that is, one should be able to easily arrive at estimates for the elements of the model; and, second, the model should be consistent with *observables* derived from the real device which it is simulating.

Notice that one does not need to know that the device was selected from a stationary ensemble, but only that the observables derived from the device are consistent with, say, elements of a hypothetically stationary ensemble. Notice also that the actual model used may depend upon how clever the experimenter-theorist is in generating models. It is worth noting, however, that while some texts on statistics give "tests for stationarity," these "tests" are almost always inadequate. Typically, these "tests" determine only if there is a substantial fraction of the noise power in Fourier frequencies whose periods are of the same order as the data length or longer. While this may be very important, it is *not* logically essential to the concept of stationarity. If a non-stationary model actually becomes common, it will almost surely result from usefulness or convenience, and not because the process is "actually non-stationary." Indeed, the phrase "actually non-stationary" appears to have no meaning in an operational sense. In short, stationarity (or non-stationarity) is a property of models and not a property of data [11].

Fortunately, many statistical models exist which adequately describe most present-day signal generators; many of these models are considered below. It is obvious that one cannot guarantee that *all* signal generators are adequately described by these models, but it is felt that they are adequate for the description of most signal generators presently encountered.

8.4. REQUIREMENTS FOR A MEASURE OF FREQUENCY STABILITY

To be useful, a measure of frequency stability must allow one to predict performance of signal generators used in a wide variety of situations as well as allow one to make meaningful relative comparisons among signal generators. One must be able to predict performance in devices which may most easily be described in the time domain, the frequency domain, or in a combination of the two. This prediction of performance may involve actual distribution functions, and thus second moment measures (such as power spectra and variances) are not totally adequate.

Two common types of equipment used to evaluate the performance of a frequency source are (analog) spectrum analyzers (frequency domain) and digital, electronic counters (time domain). On occasion the digital counter data are converted to power spectra by computers. One must realize that any piece of equipment simultaneously has certain aspects most easily described in the time domain and other aspects most easily described in the frequency domain. For example, an electronic counter has a high frequency limitation, and experimental spectra are determined with finite time averages.

Research has established that ordinary oscillators demonstrate noise which appears to be a superposition of causally generated signals and random, nondeterministic noises. The random noises include thermal noise, shot noise, noises of undetermined origin (such as flicker noise), and integrals of these noises. One might well expect, that for the more general cases, it would be necessary to use a nonstationary model (not stationary even in the wide sense, i.e., the covariance sense). Non-stationarity would, however, introduce significant difficulties in the passage between the frequency and time domains. It is interesting to note that, so far, experimenters have seldom found a non (covariance) stationary model useful in describing actual oscillators. In what follows, an attempt has been made to separate general statements (which hold for any noise or perturbation) from those which apply only to specific models. It is important that these distinctions be kept in mind.

8.5. CONCEPTS OF FREQUENCY STABILITY

To discuss the concept of frequency stability immediately implies that frequency can change with time and thus one is not considering Fourier frequencies (at least at this point). The conventional definition of instantaneous (angular) frequency is the time rate of change of phase; that is,

$$2\pi\nu(t) \equiv \frac{d\Phi(t)}{dt} \equiv \dot{\Phi}(t), \tag{8.1}$$

where $\Phi(t)$ is the instantaneous phase of the oscillator. By our convention, time dependent frequencies of oscillators are denoted by $\nu(t)$ (cycle frequency, hertz), and Fourier frequencies are denoted by ω (angular frequency, radians per second) or f (cycle frequency, hertz) where

$$\omega \equiv 2\pi f. \tag{8.2}$$

In order for eq (8.1) to have meaning, the phase $\Phi(t)$ must be a well-defined function. This restriction immediately eliminates some "nonsinusoidal" signals such as a pure, random, uncorrelated ("white") noise. For most real signal generators, the concept of phase is reasonably amenable to an operational definition and this restriction is not serious.

Of great importance to this presentation is the concept of spectral density, $S_g(f)$. The notation, $S_g(f)$, represents the one-sided spectral density of the (pure real) function, $g(t)$, on a per hertz basis; that is, the total "power" or mean square value of $g(t)$ is given by

$$\int_0^\infty S_g(f)df. \tag{8.3}$$

Since the spectral density is such an important concept to what follows, it is worthwhile to present some important references on spectrum estimation. Many workers have estimated spectra from data records, but worthy of special note are [12–15].

8.6. THE DEFINITION OF MEASURES OF FREQUENCY STABILITY

(Second Moment Type)

In a general sense, consider a signal generator whose instantaneous output voltage, $V(t)$, may be written as

$$V(t) = [V_0 + \epsilon(t)] \sin [2\pi\nu_0 t + \varphi(t)], \tag{8.4}$$

where V_0 and ν_0 are the nominal amplitude and frequency respectively of the output, and it is assumed that

$$\left|\frac{\epsilon(t)}{V_0}\right| \ll 1, \tag{8.5}$$

and

$$\left|\frac{\dot{\varphi}(t)}{2\pi\nu_0}\right| \ll 1. \tag{8.6}$$

for substantially all time, t. Making use of eqs (8.1) and (8.4) one sees that

$$\Phi(t) = 2\pi\nu_0 t + \varphi(t), \tag{8.7}$$

and

$$\nu(t) = \nu_0 + \frac{1}{2\pi}\dot{\varphi}(t). \tag{8.8}$$

Equations (8.5) and (8.6) are essential in order that $\varphi(t)$ may be defined conveniently and unambiguously (see sec. 8.9).

Since eq (8.6) must be valid even to speak of an instantaneous frequency, there is no real need to distinguish *stability* measures from *in*stability measures. That is, any fractional frequency stability measure will be far from unity, and the chance of confusion is slight. It is true that in a very strict sense people usually measure *in*stability and speak of stability. Because the chances of confusion are so slight, we choose to continue the custom of measuring "instability" and speaking of stability (a number always much less than unity).

Of significant interest to many people is the radio frequency (RF) spectral density, $S_V(f)$. This is of direct concern in spectroscopy and radar. However, this is *not* a good primary measure of frequency stability for two reasons: First, fluctuations in the amplitude, $\epsilon(t)$, contribute directly to $S_V(f)$; and second, for many cases when $\epsilon(t)$ is insignificant, the RF spectrum, $S_V(f)$, is not uniquely related to the frequency fluctuations [16].

8.6.1. First Definition of the Measure of Frequency Stability—Frequency Domain

By definition, let

$$y(t) \equiv \frac{\varphi(t)}{2\pi\nu_0}, \tag{8.9}$$

where $\varphi(t)$ and ν_0 are as in eq (8.4). Thus $y(t)$ is the instantaneous fractional frequency deviation from the nominal frequency ν_0. A proposed definition of frequency stability is the spectral density $S_y(f)$ of the instantaneous fractional frequency fluctuations $y(t)$. The function $S_y(f)$ has the dimensions of Hz^{-1}.

One can show [17] that if $S_\varphi(f)$ is the spectral density of the phase fluctuations, then

$$S_y(f) = \left(\frac{1}{2\pi\nu_0}\right)^2 S_{\dot\varphi}(f),$$

$$= \left(\frac{1}{\nu_0}\right)^2 f^2 S_\varphi(f). \tag{8.10}$$

Thus, a knowledge of the spectral density of the phase fluctuations, $S_\varphi(f)$, characterizes the spectral density of the frequency fluctuations, $S_y(f)$—the first definition of frequency stability. Of course, $S_y(f)$ cannot be *perfectly* measured—this is the case for any physical quantity; useful estimates of $S_y(f)$ are, however, easily obtainable.

8.6.2. Second Definition of the Measure of Frequency Stability—Time Domain

The second definition is based on the sample variance of the fractional frequency fluctuations. This measure of frequency stability uses $\bar{y}_k$, defined as follows:

$$\bar{y}_k \equiv \frac{1}{\tau}\int_{t_k}^{t_k+\tau} y(t)dt = \frac{\varphi(t_k+\tau)-\varphi(t_k)}{2\pi\nu_0\tau}, \tag{8.11}$$

where $t_{k+1} = t_k + T$; $k = 0, 1, 2, \ldots$; T is the repetition interval for measurements of duration τ; and t_0 is arbitrary. Conventional frequency counters measure the number of cycles in a period τ; that is, they measure $\nu_0\tau(1+\bar{y}_k)$. When τ is one second they count the number $\nu_0(1+\bar{y}_k)$. The second measure of frequency stability, then, is defined in analogy to the sample variance by the relation

$$\langle\sigma_y^2(N, T, \tau)\rangle \equiv \left\langle \frac{1}{N-1}\sum_{n=1}^{N}\left(\bar{y}_n - \frac{1}{N}\sum_{k=1}^{N}\bar{y}_k\right)^2\right\rangle, \tag{8.12}$$

where $\langle g\rangle$ denotes the infinite time average of g; that is, the average of the set of all values which g has over running time, t. This measure of frequency stability is called the Allan variance of y. It is dimensionless since y is dimensionless.

In many situations it would be wrong to assume that eq (8.12) converges to a meaningful limit as $N\to\infty$. First, one cannot practically let N approach infinity, and, second, it is known that some actual noise processes contain substantial fractions of the total noise power in the Fourier frequency range below one cycle per year. It is important to specify a particular N and T to improve comparability of data. For the preferred definition we recommend choosing $N=2$ and $T=\tau$ (i.e., no dead time between measurements). Writing $\langle\sigma_y^2(N=2, T=\tau, \tau)\rangle$ as $\sigma_y^2(\tau)$, for this particular Allan variance [8], the proposed measure of frequency stability in the time domain may be written as

$$\sigma_y^2(\tau) = \left\langle\frac{(\bar{y}_{k+1}-\bar{y}_k)^2}{2}\right\rangle. \tag{8.13}$$

Of course, the experimental estimate of $\sigma_y^2(\tau)$ must be obtained from finite samples of data, and one can never obtain perfect confidence in the estimate—the true time average is not realizable in a real situation. One estimates $\sigma_y^2(\tau)$ from a finite number (say, $M-1$, M being the number of samples of $\bar{y}_k$) of values of $\sigma_y^2(2, \tau, \tau)$ and averages to obtain an estimate of $\sigma_y^2(\tau)$ as follows:

$$\sigma_y^2(\tau) \approx \frac{1}{2(M-1)}\sum_{k=1}^{M-1}(\bar{y}_{k+1}-\bar{y}_k)^2. \tag{8.13a}$$

It can be shown that the time average of $\sigma_y^2(2, \tau, \tau)$ is convergent (i.e., as $M\to\infty$) even for noise processes that do not have convergent $\langle\sigma_y^2(N, \tau, \tau)\rangle$ as $N\to\infty$ (see [1] app. I). Therefore, $\sigma_y^2(\tau)$ has greater utility as an idealization than does $\langle\sigma_y^2(\infty, \tau, \tau)\rangle$ even though both involve assumptions of infinite averages. In effect, increasing N causes $\sigma_y^2(N, T, \tau)$ to be more sensitive to the low frequency components of $S_y(f)$. In practice, one must distinguish between an experimental estimate of a quantity (say, of $\sigma_y^2(\tau)$) and its idealized value. It is reasonable to believe that extensions to the concept of statistical ("quality") control [18] may prove useful here. One should, of course, specify the actual number, M, of independent data samples used for an estimate of $\sigma_y^2(\tau)$. Confidence on the estimate has been calculated as a function of M in ref. [19].

In summary, therefore, $S_y(f)$ is the proposed measure of (instantaneous) frequency stability in the (Fourier) frequency domain and $\sigma_y^2(\tau)$ is the proposed measure of frequency stability in the time domain.

8.6.3. Distributions

This chapter does not attempt to specify a preferred probability distribution measure for frequency fluctuations. Whereas $(\bar{y}_{k+1}-\bar{y}_k)$ could be specified as the argument of a distribution function, we prefer to wait until further experience has demonstrated the need for some such specification.

The amplitude probability distribution of the random noise portions of frequency fluctuations is found usually to be Gaussian. See [19] for two examples, one for white noise of phase and one for flicker noise of frequency.

8.6.4. Treatment of Systematic Variations

a. *General.* The definition of frequency stability $\sigma_y^2(\tau)$ in the time domain is useful for many situations. However, some oscillators exhibit an aging or almost linear drift of frequency with time. For some applications, this trend may be calculated and removed [8] before estimating $\sigma_y^2(\tau)$.

In general, a systematic trend is perfectly deterministic (i.e., predictable in detail) while the noise is nondeterministic (i.e., predictable only in a statistical sense). Consider a function, $g(t)$, which may be written in the form

$$g(t)=c(t)+n(t), \tag{8.14}$$

where $c(t)$ is some deterministic function of time and $n(t)$, the noise, is a nondeterministic function of time. We will define $c(t)$ to be the *systematic trend* to the function $g(t)$. A problem of significance here is the determination of when and in what sense $c(t)$ is measurable.

b. *Specific Case—Linear Drift.* As an example, if we consider a typical quartz crystal oscillator whose fractional frequency deviation is $y(t)$, we may let

$$g(t)=\frac{d}{dt}\,y(t). \tag{8.15}$$

Let $c(t)$ be the drift rate of the oscillator (e.g., 10^{-10} per day) and $n(t)$ is related to the frequency "noise" of the oscillator by a time derivative. One sees that the time average of $g(t)$ becomes

$$\frac{1}{T}\int_{t_0}^{t_0+T} g(t)dt=c_1+\frac{1}{T}\int_{t_0}^{t_0+T} n(t)dt, \tag{8.16}$$

where $c(t)=c_1$ is assumed to be the constant drift rate of the oscillator. In order for c_1 to be an observable, it is natural to expect the average of the noise term to vanish, that is, converge to zero.

It is instructive to assume [8, 20] that in addition to a linear drift the oscillator is perturbed by a flicker noise frequency modulation (FM), i.e.,

$$S_y(f)=\begin{cases} h_{-1}f^{-1}, & 0<f\leq f_h \\ 0, & f>f_h, \end{cases} \tag{8.17}$$

where h_{-1} is a constant (see sec. 8.7.1(b)) and thus,

$$S_n(f)=\begin{cases} (2\pi)^2 h_{-1}f, & 0\leq f\leq f_h \\ 0, & f>f_h, \end{cases} \tag{8.18}$$

for the oscillator we are considering. With these assumptions, it is seen that

$$\lim_{T\to\infty}\frac{1}{T}\int_{t_0}^{t_0+T} n(t)dt=\kappa(0)=0, \tag{8.19}$$

and that

$$\lim_{T\to 1}\left\{\text{Variance}\left[\frac{1}{T}\int_{t_k}^{t_k+T} n(t)dt\right]\right\}=0, \tag{8.20}$$

where $\kappa(f)$ is the Fourier transform of $n(t)$. Since $S_n(0)=0$, $\kappa(0)$ must also vanish both in probability and in mean square. Thus, not only does $n(t)$ average to zero. but arbitrarily good confidence in the result may be obtained by longer averages.

Having shown that one can reliably estimate the drift rate, c_1, of this (common) oscillator, it is instructive to attempt to fit a straight line to the frequency aging. That is, let

$$g(t)=y(t), \tag{8.21}$$

and, thus,

$$g(t)=c_0+c_1(t-t_0)+n'(t), \tag{8.22}$$

where c_0 is the frequency intercept at $t=t_0$ and c_1 is the drift rate previously determined. A problem arises here because

$$S_{n'}(f)=S_y(f), \tag{8.23}$$

and

$$\lim_{T\to 1}\left\{\text{Variance}\left[\frac{1}{T}\int_{t_k}^{t_k+T} n'(t)dt\right]\right\}=\infty, \tag{8.24}$$

for the noise model we have assumed. This follows from the fact that the (infinite N) variance of a flicker noise process is infinite [17, 8, 20]. Thus, c_0 cannot be measured with any realistic precision—at least, in an absolute sense.

We may interpret these results as follows: After experimenting with the oscillator for a period of time one can fit an empirical equation to $y(t)$ of the form

$$y(t)=c_0+tc_1+n'(t), \tag{8.25}$$

where $n'(t)$ is nondeterministic. At some later time it is possible to reevaluate the coefficients c_0 and c_1. According to what has been said, the drift rate c_1 should be reproducible to within the confidence estimates of the experiment regardless of when it is reevaluated. For c_0, however, this is not true. In fact, the more one attempts to evaluate c_0, the larger are the fluctuations in the result.

Depending on the spectral density of the noise term, it may be possible to predict future measurements of c_0 and to place realistic confidence limits on the prediction [21]. For the case considered here, however, these confidence limits increase when the prediction interval is increased. Thus, in a certain sense, c_0 is "measurable" but it is not in statistical control (to use the language of the quality control engineer [18]).

8.7. TRANSLATIONS AMONG FREQUENCY STABILITY MEASURES

8.7.1. Frequency Domain to Time Domain

a. *General.* It is of value to define $r = T/\tau$; that is, r is the ratio of the time interval between successive measurements to the duration of the averaging period. Cutler has shown that ([1] app. I)

$$\langle \sigma_y^2(N, T, \tau) \rangle = \frac{N}{(N-1)} \int_0^\infty df S_y(f) \frac{[\sin^2(\pi f \tau)]}{(\pi f \tau)^2} \left\{ 1 - \frac{\sin^2(\pi r f N \tau)}{N^2 \sin^2(\pi r f \tau)} \right\}. \quad (8.26)$$

Equation (8.26) in principle allows one to calculate the time domain stability $\langle \sigma_y^2(N, T, \tau) \rangle$ from the frequency domain stability $S_y(f)$.

b. *Specific model.* A model which has been found useful [17–22] consists of a set of five independent noise processes, $z_n(t)$, $n = -2, -1, 0, 1, 2$, such that

$$y(t) = z_{-2}(t) + z_{-1}(t) + z_0(t) + z_1(t) + z_2(t), \quad (8.27)$$

and the spectral density of z_n is given by

$$S_{z_n}(f) = \begin{cases} h_n f^n, & 0 \le f \le f_h \\ 0, & f > f_h, \end{cases} \quad n = -2, -1, 0, 1, 2, \quad (8.28)$$

where the h_n are constants. Thus, $S_y(f)$ becomes

$$S_y(f) = h_{-2} f^{-2} + h_{-1} f^{-1} + h_0 + h_1 f + h_2 f^2, \quad (8.29)$$

for $0 \le f \le f_h$ and $S_y(f)$ is assumed to be negligible beyond this range. In effect, each z_n contributes to both $S_y(f)$ and $\langle \sigma_y^2(N, T, \tau) \rangle$ independently of the other z_n. The contributions of the z_n to $\langle \sigma_y^2(N, T, \tau) \rangle$ are tabulated in Appendix II of reference [1] (see also table 8.1 in sec. 8.12).

Any electronic device has a finite bandwidth and this certainly applies to frequency measuring equipment also. For fractional frequency fluctuations, $y(t)$, whose spectral density varies as

$$S_y(f) \sim f^\alpha, \ \alpha \ge -1, \quad (8.30)$$

for the higher Fourier components, one sees (from ref. [1], app. I) that $\langle \sigma_y^2(N, T, \tau) \rangle$ may depend on the exact shape of the frequency cutoff. This is true because a substantial fraction of the noise "power" may be in these higher Fourier components. As a simplifying assumption, this chapter assumes a sharp cutoff in noise "power" at the frequency f_h for the noise models. It is apparent from the tables in reference [1] (app. II) that the time domain measure of frequency stability may depend on f_h in a very important way, and, in some practical cases, the actual shape of the frequency cutoff may be very important [17]. On the other hand, there are many practical measurements where the value of f_h has little or no effect. Good practice, however, dictates that the system noise bandwidth, f_h, should be specified with any results.

In actual practice, the model of eqs (8.27), (8.28), and (8.29) seems to fit almost all real frequency sources. Typically, only two or three of the h-coefficients are actually significant for a real device and the others can be neglected. Because of its applicability, this model is used in much of what follows. Since the z_n are assumed to be independent noises, it is normally sufficient to compute the effects for a general z_n and recognize that the superposition can be accomplished by simple additions for their contributions to $S_y(f)$ or $\langle \sigma_y^2(N, T, \tau) \rangle$.

8.7.2. Time Domain to Frequency Domain

a. *General.* For general $\langle \sigma_y^2(N, T, \tau) \rangle$ no simple prescription is available for translation into the frequency domain. For this reason, one might perfer $S_y(f)$ as a general measure of frequency stability. This is especially true for theoretical work.

b. *Specific model.* Equations (8.27), (8.28), and (8.29) form a realistic model which fits the random, nondeterministic noises found on most signal generators. Obviously, if this is a good model, then the tables in reference [1] (app. II) or table 8.1 may be used in reverse to translate into the frequency domain.

Allan [8] and Vessot [22] showed that if

$$S_y(f) = \begin{cases} h_\alpha f^\alpha, & 0 \le f \le f_h \\ 0, & f > f_h, \end{cases} \quad (8.31)$$

where α is a constant, then

$$\langle \sigma_y^2(N, T, \tau) \rangle \sim |\tau|^\mu, \ 2\pi\tau f_h >> 1, \quad (8.32)$$

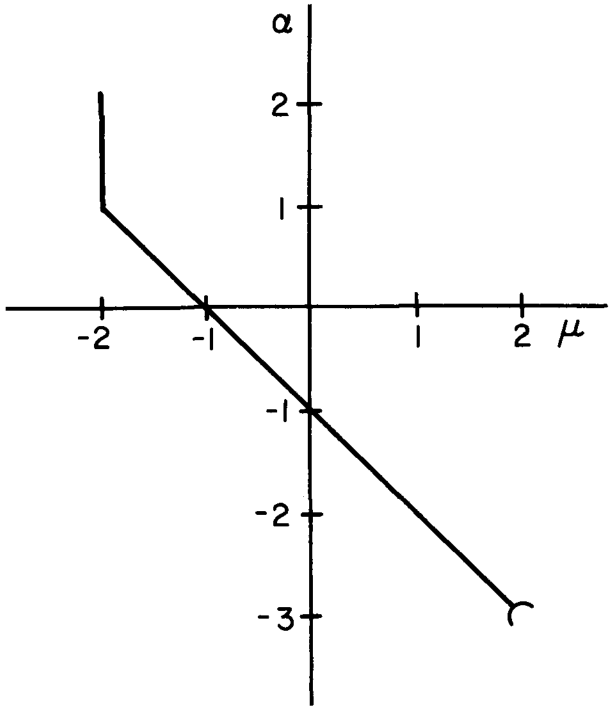


FIGURE 8.1. Mapping of exponents $\mu - \alpha$ (see eq 8.37).

for N and $r \equiv \frac{T}{\tau}$ held constant, and the constant μ is related to α by the mapping shown[3] in figure 8.1. If eqs (8.31) and (8.32) hold over a reasonable range for a signal generator, then eq (8.31) can be substituted into eq (8.26) and evaluated to determine the constant h_α from measurements of $\langle \sigma_y^2(N, T, \tau) \rangle$. It should be noted that the model of eqs (8.31) and (8.32) may be easily extended to a superposition of similar noises as in eq (8.29).

8.7.3. Translations Among the Time Domain Measures

a. *General.* Since $\langle \sigma_y^2(N, T, \tau) \rangle$ is a function of N, T, and τ (for some types of noise f_h is also important), it is very desirable to be able to translate among different sets of N, T, and τ (f_h held constant); this is, however, not possible in general.

b. *Specific model.* It is useful to restrict consideration to a case described by eqs (8.31) and (8.32). Superpositions of independent noises with different power-law types of spectral densities (i.e., different α's) can also be treated by this technique, e.g., eq (8.29). One may define two "bias functions," B_1 and B_2 by the relations [3]:

$$B_1(N, r, \mu) \equiv \frac{\langle \sigma_y^2(N, T, \tau) \rangle}{\langle \sigma_y^2(2, T, \tau) \rangle}, \tag{8.33}$$

and

$$B_2(r, \mu) \equiv \frac{\langle \sigma_y^2(2, T, \tau) \rangle}{\langle \sigma_y^2(2, \tau, \tau) \rangle}, \tag{8.34}$$

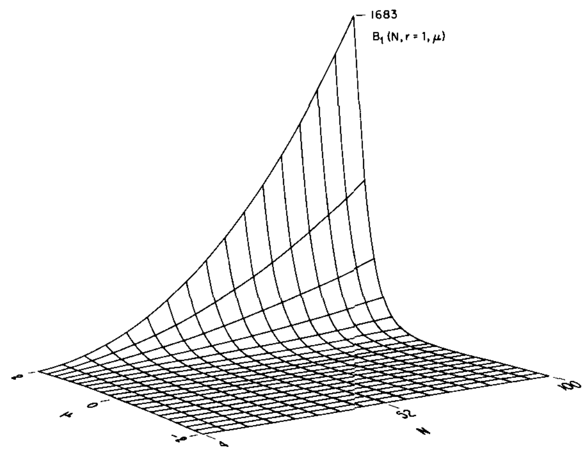


FIGURE 8.2. The bias function, $B_1(N, r = 1, \mu)$.

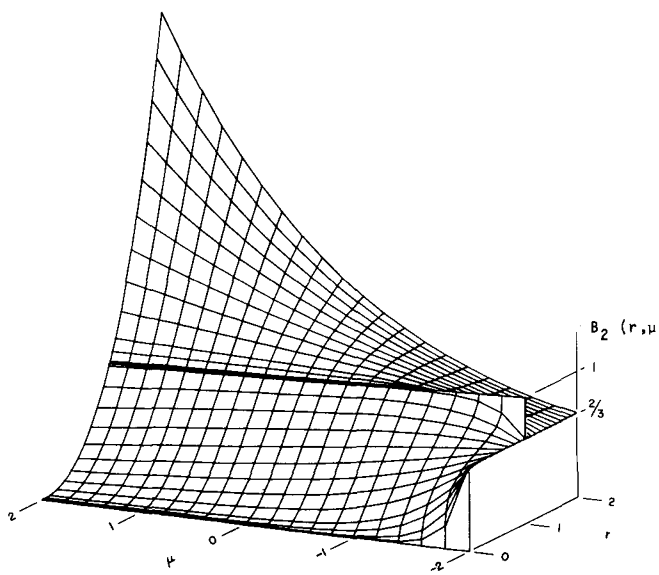


FIGURE 8.3. The bias function, $B_2(r, \mu)$.

[3] It should be noted that in Allan [8] the exponent, α, corresponds to the spectrum of *phase* fluctuations while variances are taken over average frequency fluctuations. In the present paper, α is identical to the exponent $\alpha + 2$ in reference [8].

where $r \equiv T/\tau$ and μ is related to α by the mapping of figure 8.1. In words, B_1 is the ratio of the average variance for N samples to the average variance for 2 samples (everything else held constant); while B_2 is the ratio of the average variance with dead time between measurements ($r \neq 1$) to that of no dead time ($r=1$, $N=2$, and τ held constant). These functions are tabulated in reference [3], and reproduced in Annex 8.J. Figures 8.2 and 8.3 show a computer plot of $B_1(N, r=1, \mu)$ and $B_2(r, \mu)$.

Suppose one has an experimental estimate of $\langle \sigma_y^2(N_1, T_1, \tau_1) \rangle$ and its spectral type is known—that is, eqs (8.31) and (8.32) form a good model and μ is known. Suppose also that one wishes to know the variance at some other set of measurement parameters, N_2, T_2, τ_2. An unbiased estimate of $\langle \sigma_y^2(N_2, T_2, \tau_2) \rangle$ may be calculated by the equation:

$$\langle \sigma_y^2(N_2, T_2, \tau_2) \rangle = \left(\frac{\tau_2}{\tau_1}\right)^{\mu} \left[\frac{B_1(N_2, r_2, \mu) B_2(r_2, \mu)}{B_1(N_1, r_1, \mu) B_2(r_1, \mu)}\right] \langle \sigma_y^2(N_1, T_1, \tau_1) \rangle, \quad (8.35)$$

where $r_1 \equiv T_1/\tau_1$ and $r_2 \equiv T_2/\tau_2$.

c. *General-Bias Functions.* While it is true that the concept of the bias functions, B_1 and B_2, could be extended to other processes besides those with the power-law types of spectral densities, this generalization has not been done. Indeed, spectra of the form given in eq (8.31) (or superpositions of such spectra as in eq (8.29)) seem to be the most common types of non-deterministic noises encountered in signal generators and associated equipment. For other types of fluctuations (such as causally generated perturbations), translations must be handled on an individual basis.

8.8. EXAMPLES OF APPLICATIONS OF PREVIOUSLY DEVELOPED MEASURES

8.8.1. Applications of Stability Measures

Obviously, if one of the stability measures is exactly the important parameter in the use of a signal generator, the stability measure's application is trivial. Some nontrivial applications arise when one is interested in a different parameter, such as in the use of an oscillator in Doppler radar measurements or in clocks.

a. Doppler Radar

(1) *General.* From its transmitted signal, a Doppler radar receives from a moving target a frequency-shifted return signal in the presence of other large signals. These large signals can include clutter (ground return) and transmitter leakage into the receiver (spillover). Instabilities of radar signals result in noise energy on the clutter return, on spillover, and on local oscillators in the equipment. The limitations of subclutter visibility (SCV) rejections due to the radar signals themselves are related to the RF spectral density, $S_V(f)$. The quantity typically referred to is the carrier-to-noise ratio and can be mathematically approximated by the quantity

$$\frac{S_V(f)}{\int_0^\infty S_V(f')df'}.$$

(This quantity is actually the noise-to-carrier ratio, but using the reciprocal is more convenient in what follows, and there is little chance for confusion.) The effects of coherence of target return and other radar parameters are amply considered in the literature [23–26].

(2) *Special Case.* Because FM effects generally predominate over AM effects, this carrier-to-noise ratio is approximately given by [16]

$$\frac{S_V(f)}{\int_0^\infty S_V(f')df'} \approx \frac{1}{2} S_\varphi(|f-\nu_0|), \quad (8.36)$$

for many signal sources provided $|f-\nu_0|$ is sufficiently greater than zero. Thus, if $f-\nu_0$ is a frequency separation from the carrier, the carrier-to-noise ratio at that point is approximately

$$\frac{1}{2} S_\varphi(|f-\nu_0|) = \frac{1}{2}\left(\frac{\nu_0}{f-\nu_0}\right)^2 S_y(|f-\nu_0|). \quad (8.37)$$

b. Clock Errors

(1) *General.* A clock is a device which counts the cycles of a periodic phenomenon. Thus, the reading error $x(t)$ of a clock run from the signal given by eq (8.4) is

$$x(t) = \frac{\varphi(t)}{2\pi\nu_0}, \quad (8.38)$$

and the dimensions of $x(t)$ are seconds. If this clock is a secondary standard, one could have a past history of $x(t)$, the time error relative to the standard clock. It often occurs that one is interested in predicting the clock error $x(t)$ for some future date, say $t_0+\tau$, where t_0 is the present date. Obviously, this is a problem in pure prediction and can be handled by conventional methods [13].

(2) *Special Case.* Although one could handle the prediction of clock errors by the rigorous methods of prediction theory, it is more common to use simpler prediction methods [20, 21]. In particular, one often predicts a clock error for the future by adding a correction to the present error; this correction is derived from the current rate of gain (or loss) of time. That is, the predicted error $\hat{x}(t_0+\tau)$ is related to the past history of $x(t)$ by the equation

$$\hat{x}(t_0+\tau)=x(t_0)+T\left[\frac{x(t_0)-x(t_0-T)}{T}\right]. \tag{8.39}$$

As a specific example, let $T=\tau$, then the mean-square error of prediction becomes

$$\langle[x(t_0+\tau)-\hat{x}(t_0+\tau)]^2\rangle=\langle[x(t_0+\tau) -2x(t_0)+x(t_0-\tau)]^2\rangle, \tag{8.40}$$

which, with the aid of eq (8.13), can be written in the form

$$\langle[x(t_0+\tau)-\hat{x}(t_0+\tau)]^2\rangle=2\tau^2\sigma_y^2(\tau). \tag{8.41}$$

8.9. MEASUREMENT TECHNIQUES FOR FREQUENCY STABILITY

8.9.1. Heterodyne Techniques (General)

It is possible for oscillators to be very stable and values of $\sigma_y(\tau)$ can be as small as 10^{-14} in some state of the art equipment. Thus, one often needs measuring techniques capable of resolving very small fluctuations in $y(t)$. One of the most common techniques is a heterodyne or beat frequency technique. In this method, the signal from the oscillator to be tested is mixed with a reference signal of almost the same frequency as the test oscillator; one is left then with a lower average frequency for analysis without reducing the frequency (or phase) fluctuations themselves. Following Vessot et al. [27], consider an ideal reference oscillator whose output signal is

$$V_r(t)=V_{0r}\sin 2\pi\nu_0 t, \tag{8.42}$$

and a second oscillator whose output voltage $V(t)$ is given by eq (8.4): $V(t)=[V_0+\epsilon(t)]\sin[2\pi\nu_0 t+\varphi(t)]$. Let these two signals be mixed in a product detector; that is, the output of the product detector $v(t)$ is equal to the product $\gamma V(t)\times V_r(t)$, where γ is a constant (see fig. 8.4).

Let $v(t)$, in turn, be processed by a sharp, low-pass filter with cutoff frequency f_h' such that

$$0<f_h<f_h'<\nu_0. \tag{8.43}$$

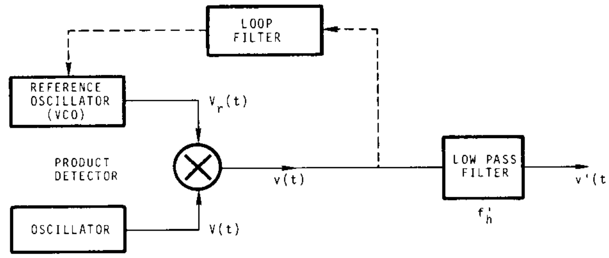


FIGURE 8.4. Example of heterodyne scheme.

One may write

$$\gamma V(t)\cdot V_r(t)=\gamma V_{0r}(V_0+\epsilon)[\sin 2\pi\nu_0 t][\sin(2\pi\nu_0 t+\varphi)] =v(t)=\gamma\frac{(V_{0r}V_0)}{2}\left(1+\frac{\epsilon}{V_0}\right)[\cos\varphi-\cos(4\pi\nu_0 t+\varphi)]. \tag{8.44}$$

Assume that $\cos[\varphi(t)]$ has essentially no power in Fourier frequencies f in the region $f\geq f_h'$. The effect of the low-pass filter then is to remove the second term on the extreme right of eq (8.44); that is,

$$v'(t)=\gamma\frac{V_{0r}V_0}{2}\left(1+\frac{\epsilon}{V_0}\right)\cos\varphi(t). \tag{8.45}$$

This separation of terms by the filter is correct only if $\left|\frac{\dot{\varphi}(t)}{2\pi\nu_0}\right|<<1$ for all t (see eq (8.6)).

The following two cases are of interest:

(a) *Case I*

The relative phase of the oscillators is adjusted so that $|\varphi(t)|<<1$ (in-phase condition) during the period of measurement. Under these conditions

$$v'(t)\approx\frac{\gamma}{2}V_{0r}V_0+\frac{\gamma}{2}V_{0r}\epsilon(t), \tag{8.46}$$

since $\cos\varphi(t)\approx 1$. That is to say one detects the amplitude noise $\epsilon(t)$ of the signal.

(b) *Case II*

The relative phase of the oscillators is adjusted to be in approximate quadrature; that is

$$\varphi'(t)=\varphi(t)+\frac{\pi}{2}, \tag{8.47}$$

where $|\varphi'(t)|<<1$. Under these conditions,

$$\cos\varphi(t)=\sin\varphi'(t)\approx\varphi'(t), \tag{8.48}$$

and

$$v'(t) = \frac{\gamma}{2} V_{0r}V_0\varphi'(t) + \frac{\gamma}{2} V_{0r}\varphi'(t)\epsilon(t). \quad (8.49)$$

If it is true that $\left|\frac{\epsilon(t)}{V_0}\right| << 1$ for all t (see eq (8.5)), then eq (8.49) becomes

$$v'(t) \approx \frac{\gamma}{2} V_{0r}V_0\varphi'(t); \quad (8.50)$$

that is, $v'(t)$ is proportional to the phase fluctuations. Thus, in order to observe $\varphi'(t)$ by this method, eqs (8.5) and (8.6) must be valid. For different average phase values, mixtures of amplitude and phase noise are observed.

To maintain the two signals in quadrature for long observational periods, one can use a voltage controlled oscillator (VCO) for a reference and feed back the phase error voltage (as defined in eq (8.50)) to control the frequency of the VCO [28]. In this condition of the phase-locked oscillator, the voltage $v'(t)$ is the analog of the *phase* fluctuations for Fourier frequencies above the loop cutoff frequency of the locked loop. For Fourier frequencies below the loop cutoff frequency, $v'(t)$ is the analog of *frequency* fluctuations. In practice, one should measure the complete servo loop response.

8.9.2. Period Measurement

Assume one has an oscillator whose voltage output may be represented by eq (8.4). If $\left|\frac{\epsilon(t)}{V_0}\right| << 1$ for all t and the total phase

$$\Phi(t) = 2\pi\nu_0 t + \varphi(t), \quad (8.7)$$

is a monotonic function of time (that is, $\left|\frac{\dot{\varphi}(t)}{2\pi\nu_0}\right| \leqslant 1$), then the time t between successive positive-going zero crossings of $V(t)$ is related to the average frequency during the interval, τ; specifically,

$$\frac{1}{\tau} = \nu_0(1+\bar{y}_n)\cdot \quad (8.51)$$

If one lets τ be the time between a positive going zero crossing of $V(t)$ and the Mth successive positive going zero crossing, then

$$\frac{M}{\tau} = \nu_0(1+\bar{y}_n)\cdot \quad (8.52)$$

If the variations $\Delta\tau$ of the period are small compared to the average period τ_0, Cutler and Searle [17] have shown that one may make a reasonable approximation to $\langle\sigma_y^2(N, T, \tau_0)\rangle$ using period measurements.

8.9.3. Period Measurement with Heterodyning

Suppose that $\varphi(t)$ is a monotonic function of time. The output of the filter of Section 8.9.1, eq (8.45) becomes

$$v'(t) \approx \gamma \frac{V_{0r}V_0}{2} \cos\varphi(t), \quad (8.53)$$

if $\left|\frac{\epsilon(t)}{V_0}\right| << 1$. Then one may measure the period τ of two successive positive zero crossings of $v'(t)$. Thus

$$\frac{1}{\tau} = \nu_0\,|\bar{y}_n|, \quad (8.54)$$

and for the Mth positive crossover

$$\frac{M}{\tau} = \nu_0\,|\bar{y}_n|\cdot \quad (8.55)$$

The magnitude bars appear because $\cos\varphi(t)$ is an even function of $\varphi(t)$. It is impossible to determine by this method alone whether φ is increasing or decreasing with time. Since $\bar{y}_n$ may be very small ($\sim 10^{-11}$ or 10^{-12} for very good oscillators), τ may be quite long and thus measurable with a good relative precision. If the phase, $\varphi(t)$, is not monotonic, the true y_n may be near zero but one could still have many zeros of $\cos\varphi(t)$ and thus eqs (8.54) and (8.55) would not be valid.

8.9.4. Frequency Counters

Assume the phase (either Φ or φ) is a monotonic function of time. If one counts the number M of positive going zero crossings in a period of time τ, then the average frequency of the signal is $\frac{M}{\tau}$. If we assume that the signal is $V(t)$ (as defined in eq (8.4)), then

$$\frac{M}{\tau} = \nu_0(1+\bar{y}_n). \quad (8.56)$$

If we assume that the signal is $v'(t)$ (as defined in eq (8.50)), then

$$\frac{M}{\tau} = \nu_0\,|\bar{y}_n|\cdot \quad (8.57)$$

Again, one measures only positive frequencies.

8.9.5. Frequency Discriminators

A frequency discriminator is a device which converts frequency fluctuations into analog voltage fluctuations by means of a dispersive element. For example, by slightly detuning a resonant circuit from the signal $V(t)$ the frequency fluctuations $\frac{1}{2\pi}\dot{\varphi}(t)$ are converted to amplitude fluctuations of the output signal. Provided the input amplitude fluctuations $\frac{\epsilon(t)}{V_0}$ are insignificant, the output amplitude fluctuations can be a good measure of the frequency fluctuations. Obviously, more sophisticated frequency discriminators exist (e.g., the cesium beam). From the analog voltage one may use analog spectrum analyzers to determine $S_y(f)$, the frequency stability. By converting to digital data, other analyses are possible on a computer.

8.9.6. Common Hazards

a. *Errors caused by signal processing equipment.* The intent of most frequency stability measurements is to evaluate the source and not the measuring equipment. Thus, one must know the performance of the measuring system. Of obvious importance are such aspects of the measuring equipment as noise level, dynamic range, resolution (dead time), and frequency range.

It has been pointed out that the noise bandwidth f_h is very essential for the mathematical convergence of certain expressions. Insofar as one wants to measure the signal *source*, one must know that the measuring system is not limiting the frequency response. At the very least, one must recognize that the frequency limit of the measuring system may be a very important, implicit parameter for either $\sigma_y^2(t)$ or $S_y(f)$. Indeed, one must account for any deviations of the measuring system from ideality such as a "non-flat" frequency response of the spectrum analyzer itself.

Almost any electronic circuit which processes a signal will, to some extent, convert amplitude fluctuations at the input terminals into phase fluctuations at the output. Thus, AM noise at the input will cause a time-varying phase (or FM noise) at the output. This can impose important constraints on limiters and automatic gain control (AGC) circuits when good frequency stability is needed. Similarly, this imposes constraints on equipment used for frequency stability measurements.

b. *Analog spectrum analyzers (Frequency Domain).* Typical analog spectrum analyzers are very similar in design to radio receivers of the superheterodyne type, and thus certain design features are quite similar. For example, image rejection (related to predetection bandwidth) is very important. Similarly, the actual shape of the analyzer's frequency window is important since this affects spectral resolution. As with receivers, dynamic range can be critical for the analysis of weak signals in the presence of substantial power in relatively narrow bandwidths (e.g., 60 Hz).

The slewing rate of the analyzer must be consistent with the analyzer's frequency window and the post-detection bandwidth. If one has a frequency window of 1 hertz, one cannot reliably estimate the intensity of a bright line unless the slewing rate is much slower than 1 hertz/second. Additional post-detection filtering will further reduce the maximum usable slewing rate.

c. *Spectral density estimation from time domain data.* It is beyond the scope of this paper to present a comprehensive list of hazards for spectral density estimation; one should consult the literature [12–15]. There are a few points, however, which are worthy of special notice:

(1) Data aliasing (similar to predetection bandwidth problems);
(2) spectral resolution; and
(3) confidence of the estimate.

d. *Variances of frequency fluctuations,* $\sigma_y^2(\tau)$. It is not uncommon to have discrete frequency modulation of a source such as that associated with the power supply frequencies. The existence of discrete frequencies in $S_y(f)$ can cause $\sigma_y^2(\tau)$ to be a very rapidly changing function of τ. An interesting situation results when τ is an exact multiple of the period of the modulation frequency (e.g., one makes $\tau = 1$ second, and there exists 60-Hz frequency modulation on the signal). In this situation, σ_y^2 $(\tau = 1\text{s})$ can be very small relative to values with slightly different values of τ. One also must be concerned with the convergence properties of $\sigma_y^2(\tau)$ since not all noise processes will have finite limits to the estimates of $\sigma_y^2(\tau)$ (see ref. [1], app. I). One must be as critically aware of any "dead time" in the measurement process as of the system bandwidth.

e. *Signal source and loading.* In measuring frequency stability one should specify the exact location in the circuit from which the signal is obtained and the nature of the load used. It is obvious that the transfer characteristics of the device being specified will depend on the load and that the measured frequency stability might be affected. If the load itself is not constant during the measurements, one expects large effects on frequency stability.

f. *Confidence of the estimate.* As with any measurement in science, one wants to know the confidence to assign to numerical results. Thus, when one measures $S_y(f)$ of $\sigma_y^2(\tau)$, it is important to know the accuracies of these estimates.

(1) *The Allan variance* [8]. It is apparent that a single sample variance, e.g., $\sigma_y^2(2, \tau, \tau)$, does not

have good confidence, but by averaging many samples, one can improve greatly the accuracy of the estimate. A detailed discussion of the confidence of the estimate for finite data lengths is given in reference [19]. It has been shown that the infinite average of a set of samples of $\sigma_y^2(N, T, \tau)$ converges for all power law spectral densities as given in eq (8.31) where $\alpha > -3$ [1]. It is worth noting that if we were interested in $\sigma_y^2(N=\infty, T, \tau)$, then convergence only occurs for $\alpha > -1$ [1–3]. Since most real signal generators possess low frequency divergent noises in the range $-1 \geq \alpha > -3$, $\langle \sigma_y^2(N, T, \tau) \rangle$ is more useful than the classical variance, $\sigma_y^2(N=\infty, T, \tau)$.

Although the sample variances, $\sigma_y^2(2, \tau, \tau)$, will not be normally distributed, the variance of the average of M independent (nonoverlapping) samples of $\sigma_y^2(2, \tau, \tau)$ (i.e., the variance estimate of this particular Allan variance) will decrease as $1/M$ provided the conditions on low frequency divergence are met. For sufficiently large M, the distribution of the M-sample-averages of $\sigma_y^2(2, \tau, \tau)$ will tend toward normal (central limit theorem). It is, thus, possible to estimate confidence intervals based on the normal distribution [19].

As always, one may be interested in τ-values approaching the limits of available data. Clearly, when one is interested in τ-values of the order of a year, one is severely limited in the size of M the number of samples of $\sigma_y^2(2, \tau, \tau)$. Unfortunately, there seems to be no substitute for many samples and one extends τ at the expense of confidence in the results. "Truth in packaging" dictates that the sample size M be stated with the results.

(2) *Spectral density.* As before, one is referred to the literature for discussions of spectrum estimation [12–15]. It is worth pointing out, however, that for $S_y(f)$ there are basically two different types of averaging which can be employed: sample averaging of independent estimates of $S_y(f)$ and frequency averaging where the resolution bandwidth is made much greater than the reciprocal data length.

8.10. SUMMARY OF FREQUENCY STABILITY MEASURES

A good measure of frequency stability is the spectral density, $S_y(f)$, of fractional frequency fluctuations, $y(t)$. An alternative is the expected variance of N sample averages of $y(t)$ each taken over a duration τ. With the beginning of successive sample periods spaced every T units of time, the variance is denoted by $\sigma_y^2(N, T, \tau)$. The preferred time domain stability measure, is the expected value of many measurements of $\sigma_y^2(N, T, \tau)$, with $N = 2$ and $T = \tau$, defined as $\sigma_y^2(\tau)$. For all real experiments one has a finite bandwidth. In general, the time domain measure of frequency stability, $\sigma_y^2(\tau)$, is dependent on the noise bandwidth of the system. Thus, there are four important parameters to the time domain measure of frequency stability:

- N, the number of sample averages ($N=2$ for preferred measure);
- T, the repetition time for successive sample averages ($T=\tau$ for preferred measure);
- τ, the duration of each sample average; and
- f_h, the system noise bandwidth.

Translations among the various stability measures for common noise types are possible, but there are significant reasons for choosing $N = 2$ and $T = \tau$ for the preferred measure of frequency stability in the time domain. This measure, the Allan variance for $N = 2$ and $T = \tau$, has been referenced by [19, 22, 29–31] among others. Although $S_y(f)$ appears to be a function of the single variable f, actual experimental estimation procedures for the spectral density involve a great many parameters. Indeed, its experimental estimation can be as involved as the estimation of $\sigma_y^2(\tau)$.

8.11. PRACTICAL APPLICATIONS OF FREQUENCY STABILITY SPECIFICATION AND MEASUREMENT

Up to this point we have considered principally the statistical theory of time and frequency data analysis showing general applications of stability measures. We would now like to show practical laboratory examples to establish standards of terminology and measurement techniques for frequency stability. Emphasis is placed on *details of useful working systems* (apparatus) that could be duplicated by others in the field of frequency stability measurements. Uniformity of data presentation is stressed to facilitate interpretation of stability specifications and to enable one to communicate and compare experimental results on a common base. This part of the chapter is based primarily on the theory given in Sections 8.1–8.10 and the paper by Barnes et al. [1]. We review the terminology for specification of frequency stability and describe the performance of frequency stability measurement systems capable of precise measurements on state-of-the-art sources in both the High Frequency (HF) and Microwave (X-band) regions.

8.12. TERMINOLOGY FOR SPECIFICATION OF FREQUENCY STABILITY

The term *frequency stability* encompasses the concepts of random noise, intended and incidental modulation, and any other fluctuations of the output frequency of a device. (In a very loose sense frequency stability can be considered as the degree to which a signal source (e.g., oscillator) produces the same frequency throughout a specified period of time.) In this chapter we are mainly (but not totally) concerned with random fluctuations corresponding to Fourier frequencies in the 10^0 to 10^6 hertz range. The measurement of frequency stability can be accomplished in both the frequency domain (e.g., spectrum analysis) and the time domain (e.g., gated frequency counter). Previously, in Section 8.6, we described two independent definitions, each related to different but useful measurement methods.

The first definition of *frequency stability* (*frequency domain*) is the one-sided spectral density of the fractional frequency fluctuations, $S_y(f)$, where $y \equiv \delta\nu/\nu_0$ (see ann. 8.A for glossary which defines symbols). The fractional frequency fluctuation spectral density $S_y(f)$ is not to be confused with the radio frequency power spectral density $S_{\sqrt{\mathrm{RFP}}}(\nu)$, or $S_{\delta V}(\nu)$, which is *not* a good primary measure of frequency stability because of fluctuations in amplitude and for other reasons (see sec. 8.6 and ref. [32]). Phase noise spectral density plots (i.e., $S_{\delta\phi}(f)$ versus f) are a common alternative method of data presentation. The spectral density of phase fluctuations is related to $S_y(f)$ by

$$S_{\delta\phi}(f) = [\nu_0^2/f^2]S_y(f). \tag{8.58}$$

The second definition of *frequency stability* (*time domain*) uses the type of sample variance called the Allan variance [8] of y:

$$\langle\sigma_y^2(N, T, \tau, f_h)\rangle \equiv \left\langle \frac{1}{N-1} \sum_{n=1}^{N} \left(\bar{y}_n - \frac{1}{N}\sum_{k=1}^{N} \bar{y}_k\right)^2 \right\rangle. \tag{8.59}$$

Note: Equation (8.59) is identical to eq (8.12) except that the measurement system bandwidth, f_h, has been explicitly inserted in the parenthesis on the left. The particular Allan variance with $N = 2$ and $T = \tau$ is found to be especially useful in practice.

Table 8.1. *Stability measure conversion chart* (frequency domain-time domain).*

$S_y(f) \equiv$ one-sided spectral density of y (dimensions are y^2/f), $0 \le f \le f_h$, $f_h \equiv B$, $2\pi f_h \tau >> 1$; $S_y(f \ge f_h) = 0$

General Definition: $\langle\sigma_y^2(N, T, \tau, f_h)\rangle \equiv \left\langle \frac{1}{N-1} \sum_{n=1}^{N} (\bar{y}_n - \frac{1}{N}\sum_{k=1}^{N} \bar{y}_k)^2 \right\rangle$, $\frac{dx}{dt} \equiv y \equiv \frac{\delta\nu}{\nu_o}$, $r \equiv \frac{T}{\tau}$

Special Case: $\sigma_y^2(\tau) \equiv \langle\sigma_y^2(N=2, T=\tau, \tau, f_h)\rangle = \left\langle \frac{(\bar{y}_{k+1} - \bar{y}_k)^2}{2} \right\rangle$

Useful Relationships:
$(2\pi)^2 = 39.48$
$\ln 2 = 0.693$
$2 \ln 2 = 1.386$
$\ln 10 = 2.303$

Time Domain (Allan variances, ...) / Frequency Domain (Power law spectral densities)	$\sigma_y^2(\tau)$ [N = 2, r = 1]	$\langle\sigma_y^2(N, T=\tau, \tau, f_h)\rangle$ [r = 1]	$\langle\sigma_y^2(N, T, \tau, f_h)\rangle$
WHITE x $S_y(f) = h_2 f^2 \quad \left(S_x(f) = \frac{h_2}{(2\pi)^2}\right)$ $2\pi f_h \tau >> 1$	$h_2 \cdot \frac{3f_h}{(2\pi)^2\tau^2}$	$h_2 \cdot \frac{N+1}{N(2\pi)^2} \cdot \frac{2f_h}{\tau^2}$	$h_2 \cdot \frac{N + \delta_k(r-1)}{N(2\pi)^2} \cdot \frac{2f_h}{\tau^2}$ $\delta_k(r-1) \equiv \begin{cases} 1 \text{ if } r = 1, \\ 0 \text{ otherwise} \end{cases}$
FLICKER x $S_y(f) = h_1 f \quad \left(S_x(f) = \frac{h_1}{(2\pi)^2 f}\right)$ $2\pi f_h \tau >> 1,\ 2\pi f_h T >> 1$	$h_1 \cdot \frac{1}{\tau^2(2\pi)^2}\left[\frac{9}{2} + 3\ln(2\pi f_h \tau) - \ln 2\right]$	$h_1 \cdot \frac{2(N+1)}{N\tau^2(2\pi)^2}\left[\frac{3}{2} + \ln(2\pi f_h \tau) - \frac{\ln N}{N^2-1}\right]$	$h_1 \cdot \frac{2}{(2\pi\tau)^2}\left\{\frac{3}{2} + \ln(2\pi f_h \tau)\right.$ $\left. + \frac{1}{N(N-1)}\sum_{n=1}^{N-1}(N-n) \cdot \ln\left[\frac{n^2r^2}{n^2r^2-1}\right]\right\}$, for r >> 1
WHITE y (Random Walk x) $S_y(f) = h_0 \quad \left(S_x(f) = \frac{h_0}{(2\pi)^2 f^2}\right)$	$h_0 \cdot \frac{1}{2}\tau^{-1}$	$h_0 \cdot \frac{1}{2}\tau^{-1}$	$h_0 \cdot \frac{1}{2}\tau^{-1}$, for $r \ge 1$ $h_0 \cdot \frac{1}{6}r(N+1)\tau^{-1}$, for $Nr \le 1$
FLICKER y $S_y(f) = \frac{h_{-1}}{f} \quad \left(S_x(f) = \frac{h_{-1}}{(2\pi)^2 f^3}\right)$	$h_{-1} \cdot 2\ln 2$	$h_{-1} \cdot \frac{N \ln N}{N-1}$	$h_{-1} \cdot \frac{1}{N(N-1)}\sum_{n=1}^{N-1}(N-n)\left\{-2(nr)^2\ln(nr)\right.$ $\left. + (nr+1)^2\ln(nr+1) + (nr-1)^2\ln\lvert nr-1\rvert\right\}$
RANDOM WALK y $S_y(f) = \frac{h_{-2}}{f^2} \quad \left(S_x(f) = \frac{h_{-2}}{(2\pi)^2 f^4}\right)$	$h_{-2} \cdot \frac{(2\pi)^2\tau}{6}$	$h_{-2} \cdot \frac{(2\pi)^2\tau}{12} \cdot N$	$h_{-2} \cdot \frac{(2\pi)^2\tau}{12}[r(N+1)-1]$, $r \ge 1$

*Adapted from J. A. Barnes et al., "Characterization of Frequency Stability," NBS Technical Note 394 (October 1970); also published in IEEE Trans. on Instrumentation and Measurement IM-20, No. 2, pp. 105-120 (May 1971).

John H. Shoaf, 273.04
National Bureau of Standards
February 1973

This dimensionless measure of stability is denoted by:

$$\sigma_y^2(\tau) \equiv \langle \sigma_y^2(N=2, T=\tau, \tau, f_h) \rangle = \left\langle \frac{(\bar{y}_{k+1} - \bar{y}_k)^2}{2} \right\rangle . \tag{8.60}$$

By way of review, the bar over the y indicates that y has been averaged over a time interval τ; the angular brackets indicate an average of the quantity over time. In the time domain we are concerned with a measure of the square root of each side of eq (8.59) over different time intervals. Plots of $\sigma_y(\tau)$ versus τ ("sigma versus tau") on a log-log scale are commonly used for data presentation. A convenient chart which enables one to translate from frequency domain measures to time domain measures (and often conversely) is given in table 8.1. An example of this translation is shown in Annex 8.B.

An additional frequency domain measure of phase fluctuations (noise, instability, modulation) used in the Time and Frequency Division (T&FD) at NBS is called *Script* $\mathscr{L}(f)$. Script $\mathscr{L}(f)$ is defined as the ratio of the power in one phase modulation sideband (referred to the input carrier frequency, on a spectral density basis) to the total signal power, at Fourier frequency f from the signal's average frequency ν_0, for a single specified signal or device (see ann. 8.C); i.e.,:

$$\mathscr{L}(f) \equiv \frac{\text{Power density (one phase modulation sideband)}}{\text{Power (total signal)}} . \tag{8.61}$$

For small $\delta\phi$,

$$S_{\delta\phi}(f) = 2\mathscr{L}(f). \tag{8.62}$$

A practical system for the measurement of $\mathscr{L}(f)$ or $S_{\delta\phi}(f)$ is described later in Section 8.14.1.

In all known signal sources the output frequency is affected by noises of various types. The random noises [1] include white thermal and shot noises, flicker noise, and integrals of these noises. The noises can be characterized by their frequency dependence as shown in table 8.2. It is the examination of these noise spectra with which we are concerned in the analysis of the frequency stability.

Frequency drift is defined as a systematic (nonrandom, typically-linear) increase or decrease of frequency with time. This is characterized as "aging rate" in crystal oscillators and is expressed in fractional parts per period of time. This section on frequency stability does *not* include a discussion of the so-called "linear drift."

TABLE 8.2. *Frequency dependence of various noise types*

Noise type	Frequency dependence	
	Spectral density—phase fluctuations [1]	Spectral density—frequency fluctuations [2]
White phase noise	f^0	f^2
Flicker phase noise	f^{-1}	f^1
White frequency noise (random walk of phase)	f^{-2}	f^0
Flicker frequency noise	f^{-3}	f^{-1}

[1] i.e., $S_{\delta\phi}(f)$.

[2] i.e., $S_{\delta\nu}(f) = f^2 S_{\delta\phi}(f) = \nu_0^2 S_y(f)$.

8.13. COMPARISON OF MEASUREMENT TECHNIQUES

In this discussion, the primary concern is the measurement of frequency fluctuations, i.e., *instability* or *stability* and not the accuracy of a frequency. It is sometimes convenient to refer to the instability as fractional frequency deviation. The measurement of frequency fluctuations can be accomplished by one or more of several methods. In each method it is essential to use a precise reference which is stable in time. In the case of measuring accurate frequency sources an equally excellent reference source is needed.

A straight forward method is by *direct counting of frequency* (cycles) by means of counters. Here, successive values of frequency are read out directly and can be recorded. (The reference signal controls the counter gate.) Statistical analysis of the results are usually made. When measuring the lower frequencies, high resolution is not possible by this method unless frequency multiplication is used. There are at least two disadvantages of frequency multipliers: a pair of specialized multiplier chains may be needed for each different carrier frequency range and noise from the multiplier itself may be introduced.

Another method involves *mixing the two frequencies and recording the beat or difference frequency.* When the reference and signal frequencies are close in value, this requires determination of the fluctuations in very long beat periods. A quantitative measure of short-term frequency stability is not practical in this case. However, when a large offset frequency is introduced, the method is feasible for assessing stability when a readout device, such as a period counter, is used for observing fluctuations.

A somewhat similar method uses a *phase sensitive detector for determining phase fluctuations* between two signals which are approximately in phase quadrature (and hence must be at the same frequency).

Short-term (or long-term) phase fluctuations may be recorded. In order to facilitate statistical analysis it is advantageous to use an analog-to-digital converter and a printout type counter. This method is related to the NBS system (time domain) which is described in detail in Section 8.14.2.

An interesting and rapid method for comparison of frequencies and also applicable to stability measurement (time domain) is the commercially available *frequency error expander*. Since frequency multiplication is used, the same disadvantages are present as in the first method in which multipliers are employed. The error expander synthesizes one of the signals to a convenient offset frequency which is then mixed and multiplied in stages to obtain higher and higher resolution. Eventually, however, the region is reached where the noise becomes excessive and frequency comparison is no longer possible.

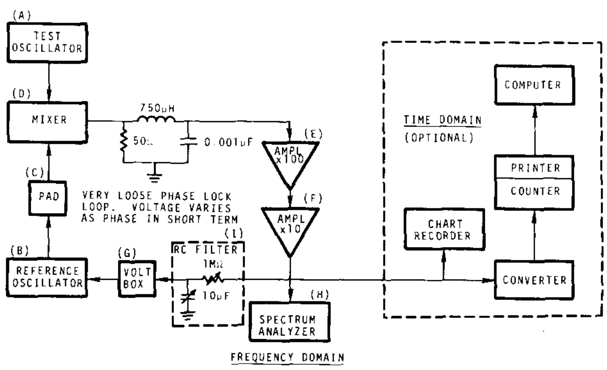
Note that *none* of the above-mentioned conventional methods of measuring frequency stability utilize frequency domain techniques. As indicated previously, we prefer that the measurements involve *both* frequency and time domain techniques for a comprehensive and sufficient indication of frequency stability. This is recommended even though it is possible to compute time domain performance from frequency domain results and often conversely [1]. Table 8.1 shows translation from one domain to the other. At least one manufacturer has made it convenient to determine frequency stability in the time domain automatically through computer programming [33, 34]. Others in the frequency and time community outside NBS have reported on excellent systems for both frequency and time domain measurements of frequency stability [17, 35–37]. In principle, some of these resemble the systems used at NBS; the techniques described by Van Duzer [35] and Meyer [36] are excellent examples. Fortunately, frequency domain and time domain methods for measuring frequency stability require similar apparatus with the following exceptions: frequency domain measurements require a frequency window (spectrum analyzer) following the detector; for time domain measurements a time window (gated counter) must follow the detector. We next describe specific operating procedures at NBS utilizing the aforementioned techniques.

8.14. OPERATIONAL SYSTEMS FOR MEASUREMENT OF FREQUENCY STABILITY AT NBS (HIGH FREQUENCY REGION)

Until recently, most conventional systems for measurement of frequency stability primarily have utilized time domain techniques; however, a complete measure of frequency stability requires use of both frequency and time domain techniques. The introduction of good double-balanced mixers permitted measurement of frequency stability by improved techniques [35–38]. The double-balanced mixer, considered as a phase sensitive detector, provides meaningful frequency stability measurements of high-quality signal sources in both the frequency domain and the time domain. The results are quantitative and may be obtained from a measurement system which is reasonable in cost. The frequency stability measurement systems described below have been used at NBS since 1967 for measurements in the HF region. The functional block diagrams in figures 8.5, 8.11, and 8.13 are referred to in the detailed descriptions of the particular systems. The carrier frequency range 10^3 Hz to 10^9 Hz, and higher, is easily covered with these techniques.

8.14.1. Frequency Domain Measurements

Figure 8.5 illustrates our measurement system typically used for *frequency domain* measurements; it should be noted also that time domain data can be obtained simultaneously, although that is not often done. (For time domain measurements it is often more convenient to use a slightly modified measurement setup described in Section 8.14.2.) In the frequency domain setup of figure 8.5, the oscillator signal under test (A), is fed into one side of a low-noise double-balanced mixer (D) which utilizes Schottky barrier diodes. The other side of the mixer is the reference oscillator signal (B), attenuated about 10 dB by pad (C) shown in figure 8.5. The mixer acts as a phase sensitive detector; when the two signals are identical in frequency and in phase-quadrature, the output is approximately zero volts dc. When this output is returned to the reference



(A) OSCILLATOR UNDER TEST
(B) REFERENCE OSCILLATOR OF HIGH SPECTRAL PURITY (VARACTOR TUNING)
(C) ADJUSTABLE ATTENUATOR (TYPICALLY 10dB)
(D) DOUBLE-BALANCED MIXER (SCHOTTKY BARRIER DIODES)
(E) NBS LOW-NOISE dc AMPLIFIER (SEE FIG. 8.7)
(F) OPERATIONAL AMPLIFIER (SEE FIG. 8.6)
(G) BATTERY BIAS BOX (SEE FIG. 8.9)
(H) SPECTRUM ANALYZER (LOW FREQUENCY, HIGH RESOLUTION)
(I) ADJUSTABLE RC FILTER

FIGURE 8.5. Typical frequency domain measurement of frequency stability (phase sensitive mode).

oscillator via the varactor tuning, phase lock is achieved. The phase lock loop contains proper termination at the output of the mixer followed by operational amplifiers (E, F) with adjustable gain. The time constant of the loop may be adjusted as needed by varying the amplifier gain within the loop and to a lesser extent, by use of the RC filter[4] (I) indicated in the diagram of figure 8.5. Finally, a battery bias box (G) is included at the varactor input to insure operation in a suitably linear portion of the varactor's frequency versus voltage curve.

A *very loose phase lock loop* is indicated inasmuch as the voltage varies as phase (in short term), and in this frequency domain measurement we are observing the small phase variations directly. The phrase—*very loose phase lock loop*—means that the bandwidth of the servo response is small compared to the lowest frequency, f, at which we wish to measure (i.e., the response time is very slow). For convenience in adjusting the gain and the advantage of a self-contained battery supply voltage we use operational amplifiers (stepped gain-commercial) as arranged in the circuit shown in figure 8.6. Special NBS low-noise dc amplifiers used in certain precision measurements are shown in figure 8.7. At NBS we have arranged the adjustable RC or CR filters in a small chassis according to the diagram of figure 8.8; we utilize low-noise components, and rotary switches provide various combinations of R and C. The battery bias box (shown in fig. 8.9) is arranged with a vernier, thus facilitating fine frequency adjustments via the varactor frequency adjustment in one oscillator. A commercial wave analyzer provides the noise plot information relevant to stability (frequency domain). The phase noise sideband levels are read out in rms volts on the analyzer, set to certain chosen values of frequency, f. For typical high quality signal sources, this corresponds to measuring only those phase noise sidebands which are separated from the carrier by the various f intervals chosen. Script $\mathscr{L}(f)$

[4] This filter can cause instabilities if its time constant, RC, is too large.

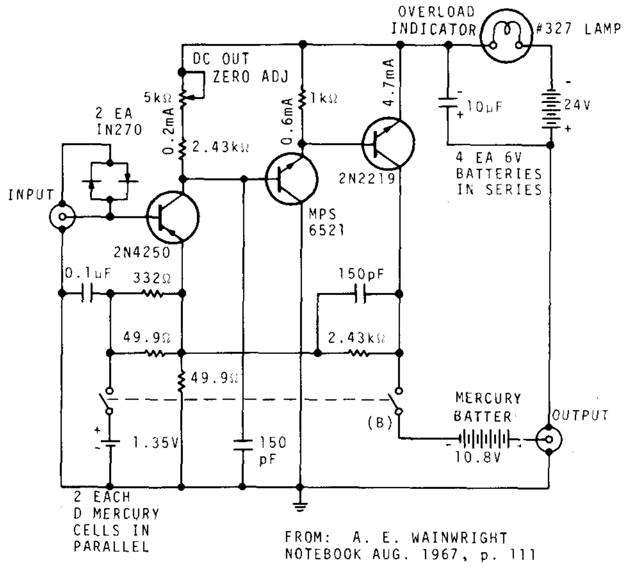
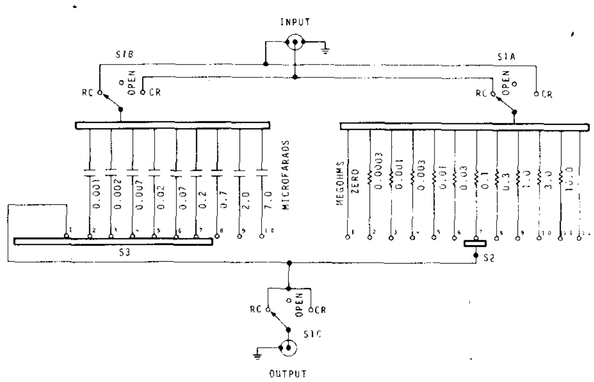


FIGURE 8.7. Low noise amplifier.



S1: FILTER MODE SWITCH (ROTARY, 3 WAFER)
S2: RESISTOR SWITCH (ROTARY, SHORTING TYPE)
S3: CAPACITOR SWITCH (ROTARY, PROGRESSIVE SHORTING)

FIGURE 8.8. Adjustable RC filter.

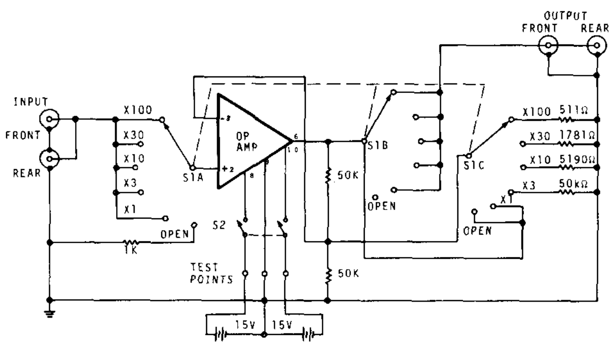
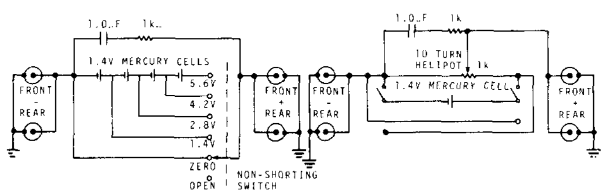


FIGURE 8.6. Stepped-gain operational amplifier.



THE UNITS ARE BUILT ON SEPARATE CHASSIS AND CONNECTED IN SERIES TO FACILITATE FINE ADJUSTMENT BETWEEN STEP VOLTAGES

FIGURE 8.9. Battery bias box.

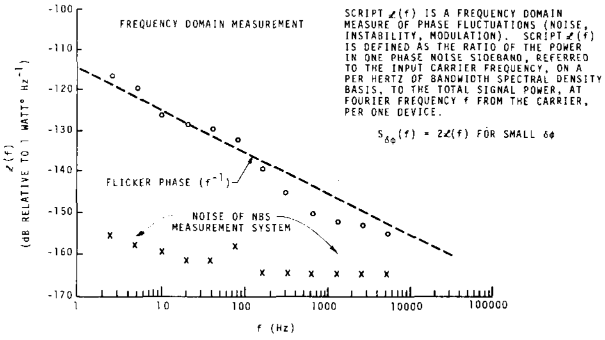
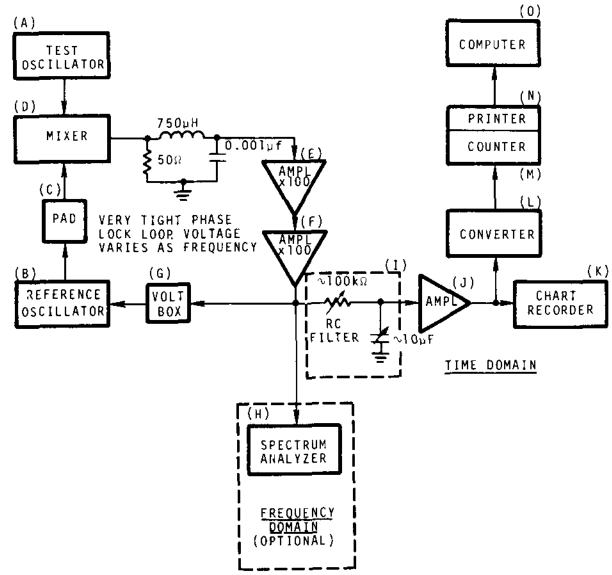


FIGURE 8.10. Script $\mathscr{L}(f)$ versus frequency f.

may be calculated with the assumption that both oscillator sources contribute equally; however, if one source were the major contributor, then the noise of that source would be no worse than 3 dB greater than the value of $\mathscr{L}(f)$ so calculated. A typical plot of script $\mathscr{L}(f)$ versus frequency is shown in figure 8.10; a sample calculation is given in Annex 8.D.

8.14.2. Time Domain Measurements

Figure 8.11 shows a measurement system typically used at NBS for stability measurements in the *time domain.* Note that the principle of operation is similar to that used in the frequency domain measurement wherein the reference oscillator is locked to the test oscillator. However, for the time domain measurement we use a *very tight phase lock loop* and the correction voltage at the oscillator varies as frequency. The phrase—*very tight phase lock loop*—means that the bandwith of the servo response is relatively large (i.e., the response time is much smaller than the smallest time interval, τ, at which we wish to measure). This is a very convenient system for observing frequency fluctuations in longer term. However, with the time constant appropriately adjusted and the means for taking sufficiently short samples, the system is readily used for both short and long term measurements in the time domain. For qualitative observations any suitable oscilloscope or strip chart recorder may be used. For quantitative measurements the system at NBS utilizes a voltage-to-frequency converter, a frequency counter, and a printer capable of recording rapid samples of data with very short dead time. The data are analyzed typically by computer via a program designed to compute the appropriate Allan variance [1, 8]. In our computer program log $\langle\sigma_y^2(N, T, \tau, f_h)\rangle^{1/2}$ versus log τ (σ versus τ) along with the associated confidence in sigma are automatically plotted on microfilm. For small batches of



ITEMS (A) THROUGH (I) SIMILAR TO FIGURE 8.5.
(J) OPERATIONAL AMPLIFIER
(K) STRIP CHART RECORDER FOR QUALITATIVE OBSERVATION
(L) VOLTAGE-TO-FREQUENCY CONVERTER
(M) FREQUENCY COUNTER WITH LOW DEAD TIME
(N) DIGITAL RECORDER WITH FAST RECORDING SPEED (INHIBIT TIME COMPATIBLE WITH COUNTER DEAD TIME)
(O) COMPUTER (OPTIONAL METHOD OF DATA ANALYSIS)

FIGURE 8.11. Typical time domain measurement of frequency stability (frequency sensitive mode).

data a desk calculator could be used, and the computer analysis would not be necessary. An example of a specific Allan variance computation is shown in Annex 8.E. Figure 8.12 gives a typical plot of σ versus τ. The dashed lines indicate the *slopes* which are characteristic of the types of noise indicated.

The convenience of obtaining time domain data has been greatly enhanced by utilizing recently developed counters [33, 34] which are programmable to automatically compute σ versus τ. A block diagram (fig. 8.13) shows a measurement system for determining frequency stability by using a comput-

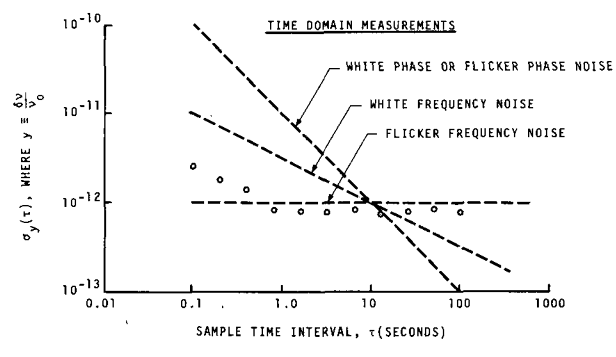


FIGURE 8.12. Sigma $\sigma_y(\tau)$ versus tau, τ.

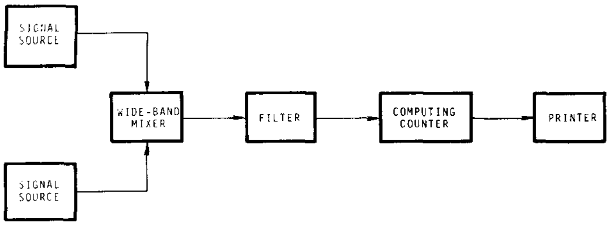


FIGURE 8.13. *Frequency stability measurements using a computing counter.*

ing counter programmed to estimate $\langle\sigma_y^2\ (N=2, T, \tau, f_h)\rangle^{1/2}$. A typical program for such use is given in Annex 8.F. Certain limitations of deadtime are inherent in the use of this time domain method. However, in general (except for very short τ), frequency stability in the time domain may be measured quickly and accurately using a computing counter.

8.14.3. Differential Phase Noise Measurements

An additional useful system, illustrated in figure 8.14, measures *differential phase noise* of various discrete components which are frequently found in stability measurement systems. In this system only one frequency source is used. Its output is split so that part of the signal passes through the component to be tested. The signal is adjusted via a phase shifter (fig. 8.15), so that it is in phase quadrature with the other part of the original signal and is downconverted in the Schottky barrier diode mixer as described for previous systems. The switchable 50Ω load in figure 8.15 is not essential but is included for convenience. A low-pass filter is included before the signal is amplified in special low-noise, low-level dc amplifiers and observed on the spectrum analyzer. Script $\mathscr{L}(f)$ values are calculated at various frequency values, f, and plotted. A sample calculation is shown in Annex 8.D.

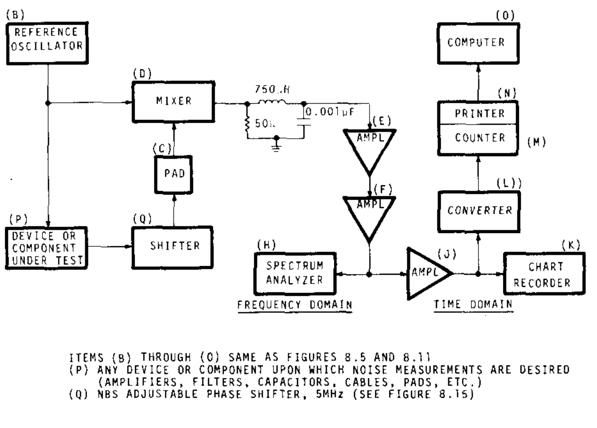


FIGURE 8.14. Differential phase noise measurement.

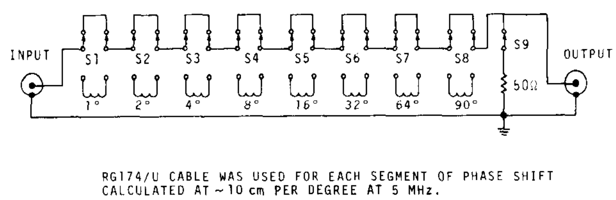


FIGURE 8.15. Adjustable phase shifter (5-MHz delay line).

The measurement system noise level (e.g., see fig. 8.10) is easily evaluated. Using the differential phase noise measurement system shown in figure 8.14, one can use a short length of coaxial cable (which is itself not a source of significant noise) as the "device under test." The small amount of noise observed on the spectrum analyzer represents the system noise, mainly due to the mixer (O) or the first amplifier (E) in figure 8.14. The calculation of the system noise is the procedure illustrated in Annex 8.D.

8.15. OPERATIONAL SYSTEMS FOR MEASUREMENT OF FREQUENCY STABILITY AT NBS (MICROWAVE REGION)

The systems we have described up to this point have been used extensively for determining frequency stability of 1 to 100-MHz frequency sources. However, we anticipate that these measurement systems would provide useful data for frequency sources up to 2 GHz, using the identical system illustrated in figures 8.5 and 8.11, with some modifications for the higher frequencies. Section 8.15 describes measurement procedures for microwave frequency sources at X-band (5–12.4 GHz). For the frequency band 2 to 5 GHz we expect that either the HF or microwave (wave guide components) systems may be used; the wave guide system would be more expensive, and the HF method would require more sophisticated circuitry for good phase lock.

Thorough investigation of stability measurement techniques in the X-band region (5–12.4 GHz), revealed that it generally was desirable to use a method different from that previously described for HF measurements. The recommended measurement system for the microwave band follows; other techniques of stability measurements in X-band will be discussed to a lesser extent. The recommended system[5] is a single-oscillator system as shown in the photograph of figure 8.16 and in the block diagram of figure 8.17. These measurement systems are based on earlier work by Ashley et al. [38] and Ondria [39].

[5] The recommended system is discussed here as a frequency domain measurement. However, time domain measurements can also be made.

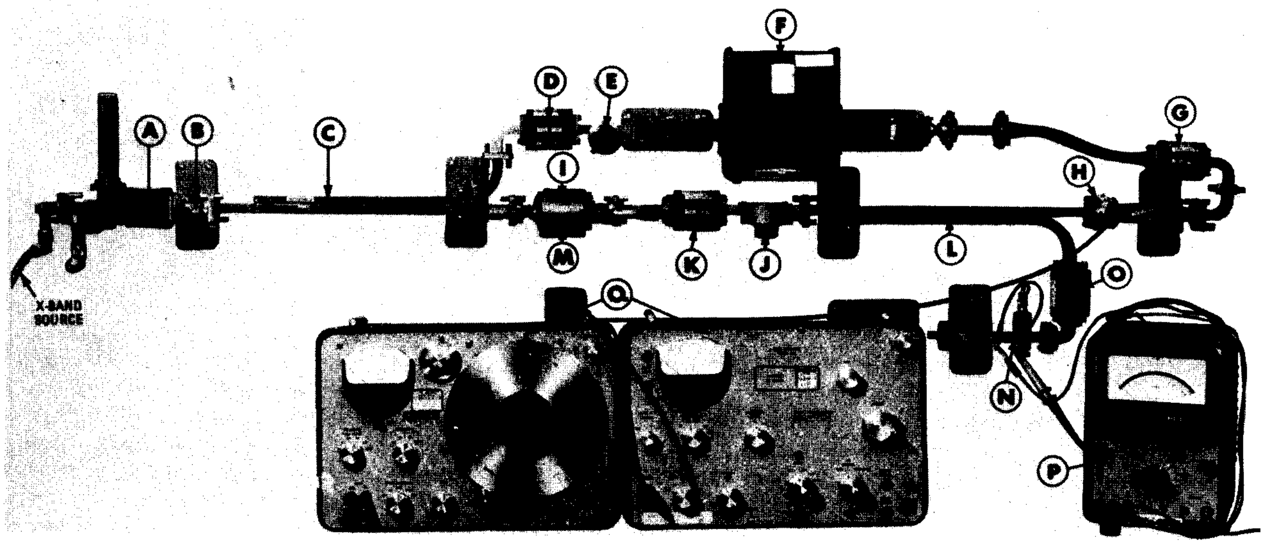


FIGURE 8.16. Single oscillator frequency stability measurement system. (Pictorial).

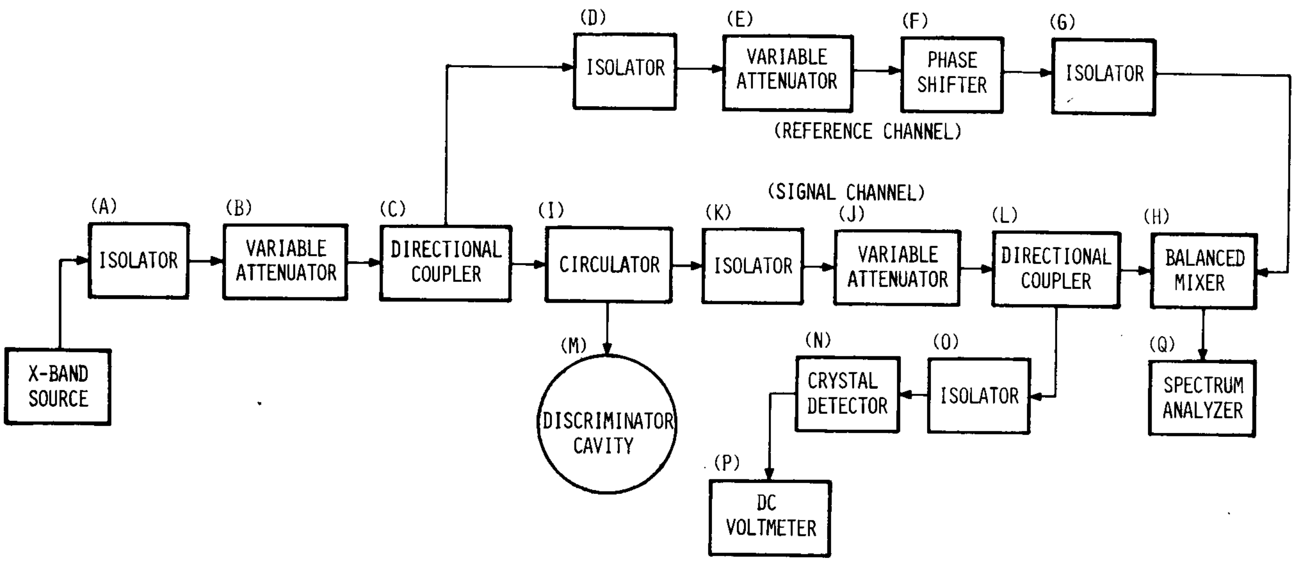


FIGURE 8.17. Single oscillator frequency stability measurement system. (Block diagram).

8.15.1. Basic Considerations of the Microwave (Frequency Stability) Measurement System

The single-oscillator frequency stability measurement system is basically a frequency modulation (FM) demodulator. That is, it can retrieve from the modulated carrier the signal with which the carrier was originally frequency modulated.

An important consideration when making these measurements is maintenance of the quadrature condition—a 90° average phase difference between the reference channel and the signal channel as seen at the mixer. Unfortunately, there is a fairly high probability that, during the course of a measurement, the average phase difference will fluctuate a few degrees about the desired 90° setting. Therefore, it is recommended that an occasional check of the quadrature condition be made. (In a two-oscillator system of measurement, discussed later, the quadrature condition—in long term—is established and maintained by phase-locking one source to the other. A similar procedure could be used here, but we consider it to be unnecessary in practice.)

In practice, there is a low frequency limit to the usefulness of this method for the measurement of FM noise. We have seen limiting values of f ranging from as high as 500 Hz to as low as 2.5 Hz. The single-oscillator system and the two-oscillator system each have an upper frequency limit; i.e., a value of f above which frequency stability measurements cannot meaningfully be made. For the single-oscillator system, as described here, this upper limit is $f \approx W_c$, where W_c is the 3-dB resonance linewidth of the loaded discriminator cavity. In the NBS single-oscillator system, measurements were made at values of f as high as 100 kHz.

8.15.2. Description of the Microwave (Frequency Stability) Measurement System

Operation of the single-oscillator measuring system may readily be followed by referring to the pictorial view in figure 8.16 and the block diagram, given in figure 8.17. The X-band source under test is connected at the left-hand side of the system. The signal passes through an isolator (A) and variable attenuator (B) before it is split via a 3-dB directional coupler (C). (It should be noted that isolators (A), (D), (G), (K), and (O) are used at several points throughout the system as a means of preventing any serious reflections which might otherwise exist.) The signal from output 1 of the coupler enters the reference channel (upper arm), passing through a phase shifter (F) via a variable attenuator (E) and eventually through a 90° twist into the balanced mixer (H). Output 2 from the coupler enters the signal channel (lower arm), passing through a three-port circulator (I) connected to a discriminator cavity (M) at one port. A variable attenuator (J) reduces the circulator output in the signal channel before it reaches the balanced mixer (H). The output of the mixer goes to either of two spectrum analyzers (Q). A 10-dB directional coupler (L) is utilized in the signal channel to facilitate detection of resonance tuning of the cavity. This is observed via a detector (N) with a dc voltmeter readout (P).

The only component in the system, not readily available as a commercial stock item, is the discriminator cavity (M). It is a TE_{011} right circular one-port cavity. A movable end wall provides the coarse tuning. The fine tuner is a small diameter rod which can be moved coaxially in the cavity. The diameter of the rod should be such that the cavity frequency changes no more than 1.5 MHz for 0.05 inch (~ 1 millimeter) change in depth of insertion. At any desired frequency the coupling of the cavity should be such that the absorption is very nearly complete. For further discussion of cavity coupling see reference [38]. The cavity Q must be both high enough for good sensitivity and sufficiently low for the required bandwidth. The cavity used in the NBS frequency stability measurement system has an unloaded Q of approximately 20,000.

8.15.3. Calibration Procedure

Initial calibration of the measurement system is necessary for assignment of an absolute scale to the stability measurements. To facilitate calibration, a sinusoidally-modulated X-band source is used to drive the system. The frequency-modulated signal is observed on an RF power spectral density analyzer (not shown in figs. 8.16 and 8.17) and the modulation level is adjusted to a value sufficient to completely suppress the X-band carrier. For sinusoidal modulation, the first carrier null corresponds to a modulation index of 2.4. Modulation at 5 kHz was convenient because of the particular dispersion and bandwidth settings which were available on the particular spectrum analyzer used to display the carrier suppression. The detailed procedure for obtaining the calibration factor is given in Annex 8.H.

8.15.4. Measurement Procedure

The procedure for obtaining data for the spectral density plot is quite similar to the calibration procedure except that the X-band carrier is not subjected to intentional modulation. Detailed steps for X-band frequency stability measurements are given in Annex 8.I. Included are techniques for calculating values of $S_{\delta\phi}(f)$. Results of such calculations for an X-band Gunn diode oscillator signal source is given in figure 8.18.

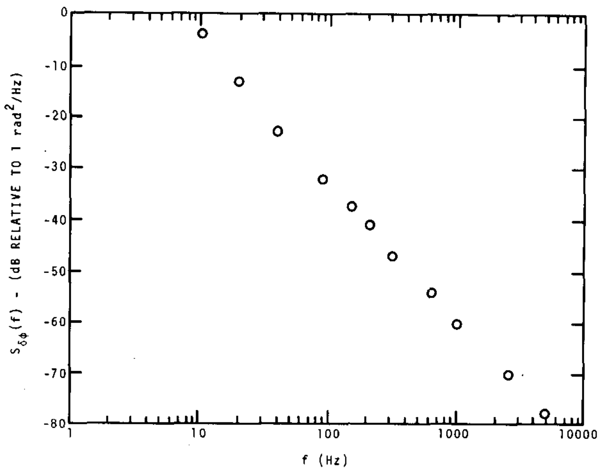


FIGURE 8.18. Frequency domain plot of phase noise of X-band Gunn diode oscillator signal source (single oscillator method).

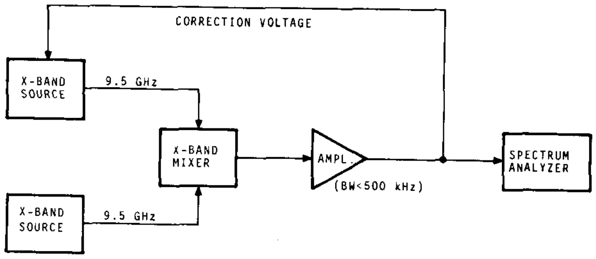


FIGURE 8.19. Frequency stability measurement system (phase-lock servo loop).

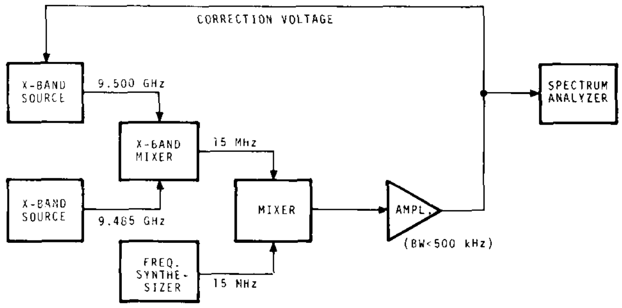


FIGURE 8.20. Frequency stability measurement system (offset-frequency phase-lock servo loop).

8.15.5. Additional Techniques for Frequency Stability Measurements at X-Band

It has been found convenient and desirable, under certain circumstances, to use other techniques for measuring frequency stability at X-band. Where two X-band sources are available, phase- or frequency-locking techniques similar to those used at HF can be used. (See figs. 8.19–8.21). Good wide-band double-balanced mixers with coaxial connectors are available [40] which permit many of the measurements to be performed without use of waveguide components.

The measurement setup, shown in the block diagram of figure 8.13, also can be used at microwave frequencies; this system utilizes a computing counter for time domain measurements. Extensive

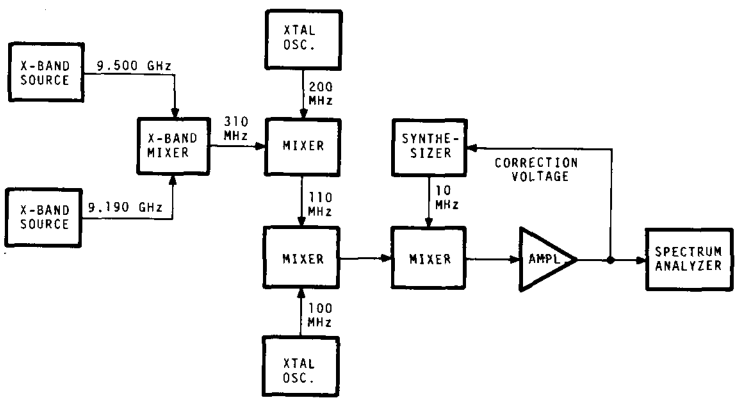


FIGURE 8.21. Frequency stability measurement system (large frequency-offset phase-lock servo loop).

measurements of frequency stability have been made on stabilized X-band sources [40]. Time domain data obtained via the computing counter have been compared with frequency domain data obtained via several methods. The example shown in Annex 8.D, the frequency domain data of figure 8.18, are translated into time domain data; these data are plotted in figure 8.22. In the same figure we have plotted time domain data taken directly via a computing counter using the system shown in figure 8.13.

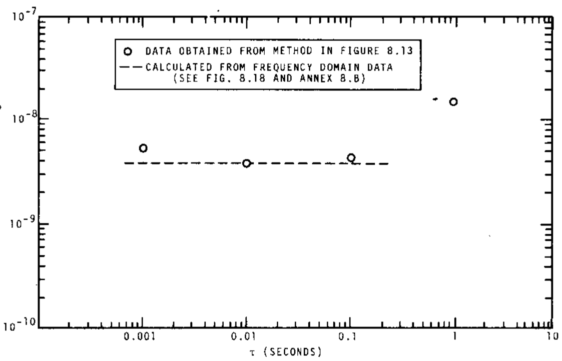


FIGURE 8.22. Time domain plot of X-band Gunn diode oscillator signal source.

8.16. CONCLUSIONS/SUMMARY

Concise definitions for specifying frequency stability have been given for measurements in the frequency domain and time domain. The first part of this chapter gives an in-depth characterization of frequency stability to enable understanding of the basic concepts. This is followed by a description of operational systems for measurement of frequency stability in detail sufficient for duplication of the required instrumentation. Uniform methods of reporting data and techniques of measurement both are recommended as advantageous and desirable for better interpretation of frequency stability specifications. The methods are applicable for both HF and microwave frequency sources.

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ANNEX 8.A

GLOSSARY OF SYMBOLS

A_{ptp}	Peak-to-peak voltage of a beat frequency at output of mixer.
B	High frequency cutoff f_h (bandwidth).
B_a	Analysis bandwidth (frequency window) of the spectrum analyzer.
$B_1(N, r, \mu)$; $B_2(r, \mu)$	Bias functions for variances based on finite samples of a process with a power-law spectral density (see ref. [3]).
C_α	A real constant defined in reference [1].
c_0, c_1	Real constants.
$c(t)$	A real, deterministic function of time.
$D_x^2(\tau)$	Expected value of the squared second difference of $x(t)$ **with lag time τ. (see ref. [1]).**
$f \equiv \omega/2\pi$	Fourier frequency of fluctuations.
f_h	High-frequency cutoff of an idealized infinitely sharp cutoff, low-pass filter (see symbol B).
f_l	Low-frequency cutoff of an idealized infinitely sharp cutoff, high-pass filter.
$g(t)$	A real function of time.
h_α	Positive real coefficient of f^α in a power series expansion of the spectral density of the function $y(t)$.
i, j, k, m, n	Integers, often a dummy index of summation.
K	Calibration factor used in the single oscillator stability measurement system for microwave frequencies, $K = (\Delta\nu)_{\text{rms}}/V_{\text{rms}}$.
$\mathscr{L}(f)$	Frequency domain measure of phase fluctuation sidebands; **Script $\mathscr{L}(f)$ is defined as the ratio of** $$\frac{\text{Power density (one phase modulation sideband)}}{\text{Power (total signal)}}.$$ For small $\delta\phi$, $S_{\delta\phi}(f) = 2\mathscr{L}(f)$.
M	Total number of data values available (usually $M \gg N$).

Also, positive integer giving the number of cycles averaged.

N — Positive integer giving the number of data values used in obtaining a sample variance.

$n(t)$ — A nondeterministic function of time.

P_{total} — Total power of signal.

$R_y(\tau)$ — Autocovariance function of $y(t)$ (see ref. [1]).

r — Parameter related to dead time $(T-\tau)$; $r \equiv T/\tau$.

$S_{\delta\nu}(f)$ — Spectral density of frequency fluctuations.

$S_{\delta V}(f)$ — Spectral density of voltage fluctuations.

$S_{\delta\phi}(f)$ — Spectral density of phase fluctuations:

$$S_{\delta\phi}(f)=\frac{S_{\delta\nu}(f)}{f^2}=S_y(f)\,\frac{\nu_0^2}{f^2}\,.$$

$S_{\sqrt{RFP}}(\nu)$ — Spectral density of the (square root of the) radio frequency power.

$S_g(f)$ — One-sided (power) spectral density on a per hertz basis of the pure real function $g(t)$. The dimensions of $S_g(f)$ are the dimensions of $g^2(t)/f$.

$S_y(f)$ — A definition for the measure of frequency stability. One-sided (power) spectral density of $y(t)$ on a per hertz basis. The dimensions of $S_y(f)$ are Hz^{-1} (spectral density of fractional frequency fluctuations).

T — Time interval between the beginnings of two successive measurements.

t — Time variable.

t_0 — An arbitrary fixed instant of time.

t_k — The time coordinate of the beginning of the kth measurement of average frequency. By definition $t_{k+1}=t_k+T$, $k=0, 1, 2 \ldots .$

u — Dummy variable of integration; $u \equiv \pi f \tau$.

$V(t)$ — Instantaneous output voltage of signal generator; see eq (8.4).

V_0 — Nominal peak amplitude of signal generator output; see eq (8.4).

V_{0r} — Peak amplitude of reference signal; see eq (8.42).

$V_{r(t)}$ — Instantaneous voltage of reference signal; see eq (8.42).

V_{rms} — Root-mean-square voltage of the output of an FM demodulator due to intentional modulation.

v — Root-mean-square (noise) voltage at output of mixer as measured by a spectrum analyzer.

$v(t)$ — Voltage output of ideal product detector.

$v'(t)$ — Low-pass filtered output of product detector.

W_c — The -3 dB resonance linewidth of the loaded discriminator cavity.

$x(t)$ — Real function of time related to the phase of the signal $V(t)$ by the eq $x(t) \equiv [\varphi(t)]/(2\pi\nu_0)$.

$\hat{x}(t)$ — A predicted value for $x(t)$.

x — Time interval fluctuations; $\frac{dx}{dt} \equiv y$, hence $x=\delta\tau$.

y — Fractional frequency fluctuations, $y \equiv \frac{\delta\nu}{\nu_0}\cdot$

$\bar{y}$ — Average of y over a specified time interval, τ.

$y(t)$ — Fractional frequency offset of $V(t)$ from the nominal frequency; see eq (8.9).

$\bar{y}_k$ — Average fractional frequency offset during the kth measurement interval; see eq (8.11).

$\langle \bar{y} \rangle_N$ — The sample average of N successive values of $\bar{y}_k$; see reference [1].

$z_n(t)$ — Nondeterministic (noise) function with (power) spectral density; see eq (8.28).

$\langle\ \rangle$ — Infinite time average operator.

α — Exponent of f for a power-law spectral density.

γ — Positive real constant.

Δ — Difference operator.

δ — Fluctuation operator.

$\delta\nu$ — Frequency fluctuations.

$\delta\phi$ — Phase fluctuations and is equivalent to $\phi(t)$.

$\delta_k(r-1)$	The Kronecker δ function defined by $$\delta_k(r-1) \equiv \begin{cases} 1, & \text{if } r=1 \\ 0, & \text{otherwise.} \end{cases}$$
$\delta V, v$	Voltage fluctuations.
$\epsilon(t)$	Amplitude fluctuations of signal; see eq (8.4).
μ	Exponent of τ; see eq (8.32).
v	Signal frequency (carrier frequency) variable.
$\nu(t)$	Instantaneous frequency of V(t). Defined by $$\nu(t) \equiv \frac{1}{2\pi}\frac{d}{dt}\Phi(t).$$
ν_0	Nominal (constant) frequency of $V(t)$.
$\kappa(f)$	The Fourier transform of $n(t)$.
σ	Square root of a variance.
$\sigma_y^2(N, T, \tau, f_h)$	Sample variance of N averages of $y(t)$, each of duration τ, and repeated every T units of time measured in a post-detection noise bandwidth of f_h; see eq (8.59).
$\langle\sigma_y^2(N, T, \tau, f_h)\rangle$	Average value of the sample variance $\sigma_y^2(N, T, \tau, f_h)$; (Allan variance).
$\sigma_y^2(\tau)$	Specific Allan variance where $N=2$, $T=\tau$; see eq (8.60); Allan variances.
τ	Sampling time interval; see eq (8.11).
τ_a	Post-detection averaging time of the spectrum analyzer.
$\Phi(t)$	Instantaneous phase of $V(t)$. Defined by $\Phi(t) \equiv 2\pi\nu_0 t + \varphi(t)$.
$\varphi(t)$	Instantaneous phase fluctuations about the ideal phase $2\pi\nu_0 t$; see eq (8.4) and is equivalent to $\delta\phi$.
$\psi_x^2(T, \tau)$	Mean-square time error for Doppler radar; see reference [1].
Ω	Signal angular frequency (carrier angular frequency), $\Omega \equiv 2\pi\nu$.
ω	Fourier angular frequency of fluctuations, $\omega \equiv 2\pi f$.

ANNEX 8.B

TRANSLATION OF DATA FROM FREQUENCY DOMAIN TO TIME DOMAIN USING THE CONVERSION CHART (table 8.1)

Referring to the frequency domain plot in figure 8.18 it is determined that $S_{\delta\phi}(f)$ indicates f^{-3} behavior over the total range plotted. Therefore $S_{\delta\nu}(f)$ is proportional to f^{-1} (i.e., flicker frequency noise). At $f=1000$ Hz, $S_{\delta\nu}(f)$ is equal to -0.3 dB relative to 1 Hz (see table 8.2). The carrier frequency, ν_0, is 9.5 GHz.

$$S_y(f) = \frac{S_{\delta\nu}(f)}{\nu_0^2} = \frac{(10^{-0.03}\text{Hz})}{(9.5\times 10^9\text{Hz})^2} = 1.04\times 10^{-20}\text{Hz}^{-1} \tag{8.B.1}$$

$$S_y(f) = \frac{h_{-1}}{f} \tag{8.B.2}$$

(see conversion chart-table 8.1).

$$h_{-1} = S_y(f)\times f = (1.04\times 10^{-20}\text{Hz}^{-1}) \times (10^3\text{Hz}) = 10.4\times 10^{-18} \tag{8.B.3}$$

$$\sigma_y^2(\tau) = h_{-1}\cdot 2\ln 2 = 10.4\times 10^{-18} \times 1.39 = 14.5\times 10^{-18} \tag{8.B.4}$$

$$\sigma_y(\tau) = 3.8\times 10^{-9}. \tag{8.B.5}$$

For flicker frequency noise there is no τ dependence. A dashed line at this calculated value is plotted on the same graph as data taken directly in the time domain (fig. 8.22).

ANNEX 8.C

SPECTRAL DENSITIES: FREQUENCY DOMAIN MEASURES OF STABILITY

Stability in the frequency domain is commonly specified in terms of spectral densities. We have used the concept of spectral density extensively in this chapter. The spectral density concept is simple and very useful, but care must be exercised in its use. There are at least four different, but related, types of spectral densities which are used in this chapter. In this Annex, we state and explain some of the simple, often-needed relations among these often-used types of spectral densities.

8.C.1. Some Types of Spectral Densities

Four types of spectral densities which are most relevant to frequency and phase fluctuations are

- $S_y(f)$ Spectral density of fractional frequency fluctuations, y (noise, instability, modulation). The dimensionality is per hertz. The range of f is from zero to infinity.
- $S_{\delta\nu}(f)$ Spectral density of frequency fluctuations $\delta\nu$ (noise, instability, modulation). The dimensionality is hertz squared per hertz. The range of f is from zero to infinity.
- $S_{\delta\phi}(f)$ Spectral density of phase fluctuations $\delta\phi$ (noise, instability, modulation). The dimensionality is radians squared per hertz. The range of f is from zero to infinity.
- $\mathscr{L}(f)$ Script $\mathscr{L}(f)$ is a frequency domain measure of phase fluctuation sidebands (noise, instability, modulation). Script $\mathscr{L}(f)$ is defined as the ratio of the power in one phase modulation sideband, referred to the input carrier frequency, on a spectral density basis, to the total signal power, at Fourier frequency f from the signal's average frequency ν_0, for a single specified signal or device. The dimensionality is per hertz. The range of f is from minus ν_0 to plus infinity.

Each of these spectral densities is one-sided and is on a per hertz of bandwidth density basis. This means that the total mean-square fluctuation (the total variance) of frequency, for example, is given mathematically by

$$\int_0^{\otimes} S_{\delta\nu}(f)\, df,$$

and, as another example, since Script $\mathscr{L}(f)$ is a normalized density, that

$$\int_{-\nu_0}^{+\otimes} \mathscr{L}(f)\, df$$

is equal to unity.

Two-sided spectral densities are defined such that the range of integration is from minus infinity to plus infinity. For specification of noise as treated in this chapter, our one-sided spectral density is twice as large as the corresponding two-sided spectral density. That is,

$$\int_{-\infty}^{+\infty} [S^{\text{Two-Sided}}]\, df = 2\int_0^{+\infty} [S^{\text{Two-Sided}}]\, df = \int_0^{+\infty} [S^{\text{One-Sided}}]\, df. \qquad (8.C.1)$$

Two-sided spectral densities are useful mainly in pure mathematical analysis involving Fourier transformations. We recommend and use one-sided spectral densities for experimental work.

We use the definition

$$y \equiv \frac{\delta\nu}{\nu_0}, \qquad (8.C.2)$$

and it follows that

$$S_y(f) \equiv S_{\frac{\delta\nu}{\nu_0}}(f) = \left(\frac{1}{\nu_0}\right)^2 S_{\delta\nu}(f). \qquad (8.C.3)$$

To relate frequency, angular frequency, and phase we use

$$2\pi[\nu(t)] = \Omega(t) = \frac{d\Phi(t)}{dt}. \qquad (8.C.4)$$

This may be regarded as a definition of instantaneous frequency $\nu(t)$. From eq (8.C. 4), a direct result of transform theory is

$$S_{\delta\phi}(f) = \left(\frac{1}{\omega}\right)^2 S_{\delta\Omega}(f) = \left(\frac{1}{f}\right)^2 S_{\delta\nu}(f). \qquad (8.C.5)$$

Script $\mathscr{L}(f)$ can be related in a simple way to $S_{\delta\phi}(f)$, but only for the condition that the phase fluctuations occurring at rates f and faster are small compared to one radian. Otherwise Bessel function algebra must be used to relate Script $\mathscr{L}(f)$ to $S_{\delta\phi}(f)$. Fortunately, the "small angle condition" is often met in random noise problems. Specifically we find

$$\mathscr{L}(f) \approx \left(\frac{1}{2\ \text{rad}^2}\right) S_{\delta\phi}(f), \qquad (8.C.6)$$

provided that

$$\int_f^{+\infty} S_{\delta\phi}(f')\,df' \ll 1\ \text{rad}^2. \qquad (8.C.7)$$

For the types of signals under discussion and for $|f| < \nu_0$, we use as a good approximation

$$\mathscr{L}(-f) \approx \mathscr{L}(f). \qquad (8.C.8)$$

Script $\mathscr{L}(f)$ is the normalized version of the phase modulation (PM) portion of $S_{\sqrt{\text{RFP}}}(\nu)$, with its frequency parameter f referenced to the signal's average frequency ν_0 as the origin such that f equals $\nu - \nu_0$. In the absence of amplitude modulation (AM), all of the radio frequency power (RFP) in the sidebands is associated with phase modulation. In high quality signal sources, it is often found that the AM is negligible compared to the PM.

8.C.2. Some Mathematics of Phase Sideband Power as Related to Phase Fluctuations

A simple derivation of eq (8.C.6) is possible. We combine the derivation with an example which illustrates the operation of a double-balanced mixer as a phase detector. Consider two sinusoidal 5-MHz signals (having negligible amplitude modulation) feeding the two input ports of a double-balanced mixer. When the two signals are slightly out of zero beat, a slow sinusoidal beat with a period of several seconds at the output of the mixer is measured to have a peak-to-peak swing of A_{ptp}.

Without changing their amplitudes, the two signals are retuned to be at zero beat and in phase quadrature (that is, $\pi/2$ out of phase with each other), and the output of the mixer is a small fluctuating voltage centered on zero volts. Provided this fluctuating voltage is small compared to $A_{ptp}/2$, the phase quadrature condition is being closely maintained, and the "small angle condition" is being met. Further details on this measurement procedure are given in Section 8.14.1 and figure 8.5.

Phase fluctuations $\delta\phi$ between the two signals of phases ϕ_2 and ϕ_1, respectively, where

$$\delta\phi \equiv \delta(\phi_2 - \phi_1), \tag{8.C.9}$$

will give rise to voltage fluctuations δV at the output of the mixer

$$\delta V \approx \frac{A_{ptp}}{2}\,\delta\phi, \tag{8.C.10}$$

where we have used radian measure for phase angles, and we have used

$$\sin\delta\phi \approx \delta\phi, \tag{8.C.11}$$

for small $\delta\phi$ ($\delta\phi \ll 1$ rad). We solve eq (8.C.10) for $\delta\phi$, square both sides, and take a time average

$$\langle(\delta\phi)^2\rangle \approx 4\,\frac{\langle(\delta V)^2\rangle}{(A_{ptp})^2}\cdot \tag{8.C.12}$$

If we interpret the mean-square fluctuations of $\delta\phi$ and of δV, respectively, in eq (8.C.12) in a spectral density fashion, we may write

$$S_{\delta\phi}(f) \approx \frac{S_{\delta V}(f)}{2(A_{\rm rms})^2}, \tag{8.C.13}$$

where we have used

$$(A_{ptp})^2 = 8(A_{\rm rms})^2, \tag{8.C.14}$$

which is valid for the sinusoidal beat signal.

For the types of signals under consideration, by definition the two phase noise sidebands (lower sideband and upper sideband, at $-f$ and $+f$ from ν_0, respectively) of a signal are coherent with each other. As already expressed in eq (8.C.8), they are of equal intensity also. The operation of the mixer, when it is driven at quadrature, is such that the amplitudes of the two phase sidebands add linearly in the output of the mixer, resulting in four times as much power in the output as would be present if only one of the phase sidebands were allowed to contribute to the output of the mixer. Hence for $|f| < \nu_0$ we obtain

$$\frac{S_{\delta V}(|f|)}{(A_{\rm rms})^2} \approx 4\,\frac{[S_{\sqrt{\rm RFP}}(\nu_0+f)]}{P_{\rm total}}\ \text{PM}, \tag{8.C.15}$$

and, using the definition of Script $\mathscr{L}(f)$,

$$\mathscr{L}(f) \equiv \frac{[S_{\sqrt{\rm RFP}}(\nu_0+f)]}{P_{\rm total}}\ \text{PM} \approx \frac{1}{2}S_{\delta\phi}(|f|), \tag{8.C.16}$$

provided the phase quadrature condition is approximately valid. Recall that we assumed the signals have negligible amplitude modulation. The phase quadrature condition will be met for a time interval at least τ long, provided

$$\int_{(2\pi\tau)^{-1}}^{\infty} S_{\delta\phi}(f')\,df' << 1\ \text{rad}^2, \tag{8.C.17}$$

and hence eq (8.C.16) is useful for values of f at least as low as $(2\pi\tau)^{-1}$. Equations (8.C.16) and (8.C.17) correspond to eq (8.C.6) and (8.C.7), respectively.

ANNEX 8.D

A SAMPLE CALCULATION OF SCRIPT $\mathscr{L}$

For convenience of computation and plotting it often is advantageous to set the beat frequency voltage (before locking) to $\frac{1}{\sqrt{10}}$ volts (0.316 V) peak-to-peak at the mixer output. Then (after lock) with the output of the phase detector expressed in rms nanovolts per root hertz, direct plotting is facilitated for Script $\mathscr{L}(f)$ in decibels versus frequency in hertz. In this case 1000, 100, and 10 nanovolts per root hertz correspond to -110, -130, and -150 dB respectively. A sample calculation demonstrating this convenience is shown below. Equation (8.D.3) given for Script $\mathscr{L}(f)$ is valid for the case where two equally noisy signals are driving the mixer.

Given:

$$A_{ptp} = 0.316\ V\left(\text{i.e., } \frac{1}{\sqrt{10}} V\right) \quad (8.D.1)$$

$$v = 100 \text{ nV Hz}^{-1/2} \text{ @ } f = 20 \text{ Hz}, \quad (8.D.2)$$

from a pair of equally noisy signals.

$$\mathscr{L}(20 \text{ Hz}) = \left(\frac{v}{A_{ptp}}\right)^2 \quad (8.D.3)$$

$$= \left(\frac{100 \text{ nV Hz}^{-1/2}}{0.316 \text{ V}}\right)^2 = \left(\frac{10^{-7}}{\sqrt{10^{-1}}}\right) \text{Hz}^{-1} = \frac{10^{-14}}{10^{-1}} \text{Hz}^{-1}$$

$$= 10^{-13} \text{ Hz}^{-1} = -130 \text{ dB},$$

or using logarithms:

$$\mathscr{L}(20 \text{ Hz}) = 20 \log_{10}\left(\frac{v}{A_{ptp}}\right) \quad (8.D.4)$$

$$= 20 \log_{10} \frac{(10^{-7} \text{ V} \cdot \text{Hz}^{-1/2})}{(10^{-1/2} \text{ V})} = 20(-7+0.5)$$

$$= -130 \text{ dB}.$$

If the phase noise follows flicker law, at $f = 1$ Hz it is 20 times worse (or 13 dB greater); that is

$$\mathscr{L}(1\text{Hz}) = -130 \text{ dB} + 13 \text{ dB} = -117 \text{ dB}. \quad (8.D.5)$$

ANNEX 8.E

A SAMPLE CALCULATION OF ALLAN VARIANCE, $\sigma_y^2(\tau)$

$$\sigma_y^2(\tau) \equiv \langle \sigma_y^2(N=2, T=\tau, \tau)\rangle = \left\langle \frac{(\bar{y}_{k+1} - \bar{y}_k)^2}{2} \right\rangle$$

$$\approx \frac{1}{2(M-1)} \sum_{k=1}^{M-1} (\bar{y}_{k+1} - \bar{y}_k)^2. \quad (8.E.1)$$

in the example below:

Number of data values available, $M = 9$
Number of differences averaged, $M - 1 = 8$
Sampling time interval $\tau = 1$s

TABLE 8.E.1. *Sample data tabulation*

Data values $(\bar{y})$	First differences $(\bar{y}_{k+1} - \bar{y}_k)$	First differences squared $(\bar{y}_{k+1} - \bar{y}_k)^2$
892	—	—
809	−83	6889
823	14	196
798	−25	625
671	−127	16129
644	−27	729
883	239	57121
903	20	400
677	−226	51076
		$\sum_{k=1}^{M-1} (\bar{y}_{k+1} - \bar{y}_k)^2 = 133165$

Based on these data:

$$\sigma_y^2(\tau) = \frac{133165}{2(8)} = 8322.81, \quad (8.E.2)$$

$$[\sigma_y^2(\tau)]^{1/2} = \sqrt{8322.81} = 91.23, \ N = 2, \ T = \tau = 1 \text{ s}. \quad (8.E.3)$$

In this example, the data values may be understood to be expressed in parts in 10^{12}; the data may have been taken as the counted number of periods, in the time interval τ, of the beat frequency between the oscillator under test and a reference oscillator, divided by the nominal carrier frequency ν_0, and multiplied by the factor 10^{12}.

Using the same data as in the above example it is possible to calculate the Allan variance for $\tau = 2$ s by averaging pairs of adjacent data values and using these averaged values as new data values to proceed with the calculation as before. Allan variance values may be obtained for $\tau = 3$ s by averaging three adjacent data values in a similar manner, etc., for larger values of τ.

Ideally the calculation is done via a computer and a large number, M, of data values should be used. (Typically $M = 256$ data values are used in the NBS computer program.) The statistical confidence of the calculated Allan variance improves nomially as the square root of the number, M, of data values used [19]. For $M = 256$, the confidence of the Allan variance is expected to be approximately $\pm \frac{1}{\sqrt{256}} \times 100$ percent $\approx \pm 7$ percent of its value. The use of $M >> 1$ is logically similar to the use of $B_a \cdot \tau_a >> 1$ in spectrum analysis measurements, where B_a is the analysis bandwidth (frequency window) of the spectrum analyzer, and τ_a is the post-detection averaging time of the spectrum analyzer.

ANNEX 8.F

COMPUTING COUNTER PROGRAM USING AN EFFICIENT OVERLAPPING ESTIMATOR FOR $\langle \sigma_y^2(N=2, T, \tau, f_h) \rangle^{1/2}$

(1) MANUAL
(2) Enter carrier or basic frequency
(3) $c \rightleftarrows x$ [skip to (33) if program is already in]
(4) LEARN
(5) CLEAR x
(6) $b \rightleftarrows x$
(7) MODULE or PLUG-IN
(8) $a \rightleftarrows x$
(9) X FER PROGRAM
(10) MODULE or PLUG-IN
(11) $a \rightleftarrows x$
(12) $\overrightarrow{axy}$
(13) − (subtract)
(14) $\overrightarrow{xy}$
(15) × (multiply)
(16) $\overrightarrow{bxy}$
(17) + (add)
(18) $b \rightleftarrows x$
(19) REPEAT
(20) X FER PROGRAM
(21) $\overrightarrow{Nxy}$
(22) $\overrightarrow{Nxy}$
(23) + (add)
(24) $a \rightleftarrows x$
(25) $\overrightarrow{bxy}$
(26) $\overrightarrow{axy}$
(27) ÷ (divide)
(28) 1
(29) $\sqrt{x}$
(30) $\overrightarrow{cxy}$
(31) ÷ (divide)
(32) PAUSE
(33) RUN
(34) START

Program will automatically repeat unless right hand PAUSE switch is in HALT position. PAUSE DISPLAY switch must be on.

$\tau \equiv$ Sample time (computing counter "**measurement** time").
$T-\tau \simeq 0.003$ seconds (compute + cycle time)
$N=2$
$f_h \equiv$ Measurement system bandwidth

Number set on repeat loop corresponds to the number of estimates of the variance. For good confidence levels 100 or more estimates usually are required.

ANNEX 8.G

SELECTED FREQUENCY STABILITY REFERENCES: BIBLIOGRAPHY[6]

ANNEX 8.G.1

Selected References (Proceedings, Special Issues, etc.) Applicable to Measurement and Specification of Frequency Stability

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[R5] *IEEE Trans. on Microwave Theory and Techniques, Special Issue on Noise,* **MTT-16,** No. 9 (September 1968).
[R6] *Proc. IEEE, Special Issue on Time and Frequency,* **60,** No. 5, (May 1972).

[6] Annotated when additional information is pertinent.
[7] R denotes reference listing.

[R7] U.S. Army Electronics Command, *Proc. Ann. Symp. on Frequency Control* (U.S. Army Electronics Command, Ft. Monmouth, NJ 07703). These symposia are held usually at Atlantic City during the spring of each year. The Proceedings contain a wide variety of frequency/time papers, including general interest, progress reports, and well-documented state-of-the-art accounts. Information about the Proceedings for the last years is given on page 178, together with the source of availability.

Symposium Number	Date	Proceedings Accession No.	Pages	Availability	Cost
10*	May 15–17, 1956	AD 298322	585	National Technical Information Service, Sills Building, 5285 Port Royal Road, Springfield, VA 22151.	$3.00
11	May 7–9, 1957	AD 298323	634	NTIS	3.00
12	May 6–8, 1958	AD 298324	666	NTIS	3.00
13	May 12–14, 1959	AD 298325	723	NTIS	3.00
14	May 31–June 2, 1960	AD 246500	443	NTIS	3.00
15	May 31–June 2, 1961	AD 265455	335	NTIS	3.00
16	April 25–27, 1962	AD 285086	455	NTIS	3.00
17	May 27–29, 1963	AD 423381	618	NTIS	3.00
18	May 4–6, 1964	AD 450341	597	NTIS	3.00
19	April 20–22, 1965	AD 471229	673	NTIS	3.00
20	April 19–21, 1966	AD 800523	679	NTIS	3.00
21	April 24–26, 1967	AD 659792	579	NTIS	3.00
22	April 22–24, 1968	AD 844911	620	NTIS	3.00
23	May 6–8, 1969	AD 746209	313	NTIS	3.00
24	April 27–30, 1970		361	Electronics Industries Assn., 2001 Eye Street, NW, Washington, DC 20006.	6.50
25	April 26–28, 1971	AD 746211	351	NTIS	3.00
26	June 6–8, 1972		322	Electronics Industries Assn.	6.50
27	June 12–14, 1973		446	Electronics Industries Assn.	6.50

*First published proceedings.

[R8] Blair, B. E., and Morgan, A. H. (Eds.), *Precision Measurement and Calibration-Selected NBS Papers on Frequency and Time, Nat. Bur. Stand. (U.S.) Spec. Publ. 300*, **V**, 555 pages (USGPO, June 1972). This volume contains important reprints of NBS papers written during 1960 to 1970 (February) on various time and frequency subjects including statistics of frequency and time measurements.

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ANNEX 8.H

DETAILED PROCEDURE FOR CALIBRATING MICROWAVE (FREQUENCY-STABILITY) MEASUREMENT SYSTEM

The reader is referred to figure 8.17 in following the step-wise calibration procedure below. Note that a *sinusoidally-modulated* X-band frequency source is used to drive the system to facilitate the calibration. (For a general overview of the method see sec. 8.15.3.)

(a) With the discriminator cavity (M) far off resonance, set the level of the X-band signal (as determined with a dc voltmeter (P) at the detector (N)) to a convenient value and *record* the value. The first variable attenuator (B) should be used for this adjustment. Any convenient level may be chosen provided an equal amount of power also will be available from the signal source which is to be evaluated.

(b) Attenuate the intentional modulation so that no sidebands are visible on the power spectral density analyzer (see sec. 8.15.3).

(c) Adjust the cavity to resonance. The dc voltmeter at the detector is used to determine resonance. Place the dc voltmeter at the output of the mixer (H) and adjust the phase shifter (F) in the reference channel until the dc output at the mixer is zero (phase quadrature). Remove the dc voltmeter and connect the output of the mixer to a spectrum analyzer (Q) tuned to the modulation frequency, 5 kHz. Replace the intentional modulation so that the carrier is, again, fully suppressed. Record the rms voltage reading (V_{rms}) of the spectrum analyzer.

(d) It is now possible to calculate the calibration factor K.

$$K=\frac{(\Delta\nu)_{rms}}{V_{rms}}, \tag{8H.1}$$

where $(\Delta\nu)_{rms}$ is the rms frequency deviation of the carrier due to intentional frequency modulation. This deviation is calculated using the equation

$$(\Delta\nu)_{rms}=0.707(\Delta\nu)_{peak}, \tag{8H.2}$$

where $(\Delta\nu)_{peak}$ is the product of the modulation index with the frequency of sinusoidal modulation, i.e., 2.405×5 kHz$=12.025$ kHz. Therefore the calibration factor in our case is

$$K=\frac{0.707(12.025\text{ kHz})}{V_{rms}}=\frac{8.51\text{ kHz}}{V_{rms}}. \tag{8H.3}$$

ANNEX 8.I

MEASUREMENT PROCEDURE FOR DETERMINING FREQUENCY STABILITY IN THE MICROWAVE REGION

The ensuing detailed steps are given to aid one in determining frequency stability of microwave sources. Referral to figure 8.17 will help in following the sequence of the test procedure. The procedure includes instruction and example on the calculation and presentation of $S_{\delta\phi}(f)$.

(a) Be sure the intentional modulation has been completely removed from the X-band source. With the cavity far off resonance, set the level at the detector to the same value obtained during calibration. Use the variable attenuator (B) for this adjustment. The other variable attenuators (E) and (J) are set to zero.

(b) Adjust the cavity frequency to that frequency of the X-band source. Adjust the phase shifter (F) so that the dc output at the mixer (H) is zero. Attach the spectrum analyzer to the output of the mixer and record rms voltage readings (V'_{rms}) for various frequency settings of the spectrum analyzer. A low noise amplifier may be necessary to obtain useful readings at large Fourier frequencies. A second reading (V''_{rms}) should be taken at each value of f with the signal strongly attenuated in the signal channel. This is to record the residual additive background noise not attributable to actual phase noise on the carrier. This attenuation is accomplished by inserting all (> 20 dB) of the attenuation in the variable attenuator (J).

(c) In order to calculate values of $S_{\delta\phi}(f)$ for plotting at various Fourier frequencies, it is convenient to make a tabulation of results. An example of some typical results is given in table 8.I.1. The following relations are used:

$$V_{rms}=\sqrt{(V'_{rms})^2-(V''_{rms})^2}. \tag{8.I.1}$$

$$\delta\nu_{rms}=V_{rms}\times K. \tag{8.I.2}$$

$$S_{\delta\nu}(f)=\frac{(\delta\nu_{rms})^2}{B}. \tag{8.I.3}$$

where B is the bandwidth at which the readings were made on the spectrum analyzer, and

$$S_{\delta\phi}(f)=\frac{S_{\delta\nu}(f)}{f^2}. \tag{8.I.4}$$

Values of $S_{\delta\phi}(f)$ which were calculated this way (table 8.I.1) are plotted in figure 8.18.

Table 8.I.1. *Tabulation of $S_{\delta\phi}(f)$ at various Fourier frequencies*

f(Hz)	f^2 (Hz^2)	B (Hz)	v_{rms} (μV)	$\delta\nu_{rms}$ (Hz)	$(\delta\nu_{rms})^2$ (Hz^2)	$S_{\delta\nu}(f)$ (Hz)	$S_{\delta\phi}(f)$ (dB)[a]
5×10^3	2.5×10^7 (74 dB)[b]	100	660	6.38	40.7	0.41 (−3.9 dB)[c]	−77.9
2.5×10^3	6.2×10^6 (68 dB)	100	780	7.54	56.9	0.57 (−2.4 dB)	−70.4
1×10^3	1.0×10^6 (60 dB)	100	1000	9.67	93.5	0.94 (−0.3 dB)	−60.3
640	4.1×10^5 (56 dB)	10	1200	11.6	135.0	1.4 (1.5 dB)	−54.5
320	1.0×10^5 (50 dB)	10	460	4.45	19.8	2.0 (3.0 dB)	−47.0
210	4.4×10^4 (46 dB)	10	580	5.61	31.5	3.2 (5.1 dB)	−40.9
150	2.2×10^4 (44 dB)	10	680	6.58	43.3	4.3 (6.3 dB)	−37.7
90	8.1×10^3 (39 dB)	1	230	2.22	4.93	4.9 (6.9 dB)	−32.1
40	1.6×10^3 (32 dB)	1	300	2.90	8.41	8.4 (9.3 dB)	−22.7
20	400 (26 dB)	1	450	4.35	18.9	19.0 (12.8 dB)	−13.2
10	100 (20 dB)	1	700	6.77	45.8	46.0 (16.6 dB)	−3.4

[a] $S_{\delta\phi}(f)$ is tabulated in decibels relative to 1 radian^2 Hz^{-1}
[b] dB relative to 1 Hz^2
[c] dB relative to 1 Hz^{2}Hz^{-1}

$$\text{Calibration factor } K=\frac{8.51\times10^3\ \text{Hz}}{0.88\ \text{V}}=9.67\times10^3\ \text{Hz/V}.$$

ANNEX 8.J

TABLES OF BIAS FUNCTIONS, B_1 AND B_2, FOR VARIANCES BASED ON FINITE SAMPLES OF PROCESSES WITH POWER LAW SPECTRAL DENSITIES[9]

8.J.1. Description of the Bias Functions, B_1 and B_2

Following Allan [8] consider a random process $y(t)$ with continuous sample functions. We assume that $y(t)$ has a spectral density, $S_y(f)$, which obeys the law

$$S_y(f) = h|f|^{\alpha}, f_l < |f| < f_h \quad , \qquad (8.J.1)$$

where h is a constant, the limit frequencies f_l and f_u satisfy the relations

$$0 \leqslant f_l \ll f_h < \infty,$$

and any intervals of time, Δt, of any significance satisfy the relations

$$\frac{1}{f_h} \ll \Delta t \ll \frac{1}{f_l}.$$

In short, $y(t)$ has a power law spectral density over the entire range of significance.

Consider a measurement process which determines an average value of $y(t)$ over the interval t to $t+\tau$. That is,

$$\bar{y}(t) = \frac{1}{\tau}\int_t^{t+\tau} y(t')\, dt'. \qquad (8.J.2)$$

One, now, may determine an estimated variance from a group of N such measurements spaced every T units of time; that is,

$$\sigma_y^2(N, T, \tau) = \frac{1}{N-1}\sum_{n=1}^{N}\left\{\bar{y}(t+nT) - \frac{1}{N}\sum_{k=1}^{N}\bar{y}(t+kT)\right\}^2. \qquad (8.J.3)$$

Allan [8] has shown that under these conditions,

$$\langle \sigma_y^2(N, T, \tau)\rangle \propto \tau^{\mu}, N \text{ and } T/\tau \text{ constant},$$

where μ is related[10] to α according to the mapping shown in figure 8.1 (see ref. [8] and [26]). The relation between μ and α may be given as

$$\mu = \begin{cases} -2 \text{ if } \alpha \geqslant 1 \\ -\alpha - 1 \text{ if } -3 < \alpha \leqslant 1 \\ \text{not defined otherwise.} \end{cases}$$

This mapping involves a simple extension of Allan's work [8] to the range $0 < \mu < 2$. This extension was also mentioned in [20].

Allan [8] considered in some detail the case where $T = \tau$. This is the case of exactly adjacent sample averages—no "dead time" between measurements. Allan defined a function, $\chi(N, \mu)$, as follows

$$\chi(N, \mu) \equiv \frac{\langle \sigma_y^2(N, \tau, \tau)\rangle}{\langle \sigma_y^2(2, \tau, \tau)\rangle}, \qquad (8.J.4)$$

where it is again assumed that

$$\langle \sigma_y^2 N, \tau, \tau)\rangle \propto \tau^{\mu}, N \text{ constant}.$$

Allan shows [8] that experimental evaluations of $\chi(N, \mu)$ may be used to infer μ and hence the spectral type by use of the mapping of figure 8.1.

Since many experiments actually have dead time present, it is of value to make two different extensions of this function, $\chi(N, \mu)$. First, define $B_1(N, r, \mu)$ by the relations

$$B_1(N, r, \mu) \equiv \frac{\langle \sigma_y^2(N, T, \tau)\rangle}{\langle \sigma_y^2(2, T, \tau)\rangle}, \qquad (8.J.5)$$

where $r \equiv T/\tau$ and

$$\langle \sigma_y^2(N, T, \tau)\rangle \propto \tau^{\mu}, N \text{ and } r \text{ constant}.$$

The second function, $B_2(r, \mu)$, is defined according to the relation

$$B_2(r, \mu) \equiv \frac{\langle \sigma_y^2(2, T, \tau)\rangle}{\langle \sigma_y^2(2, \tau, \tau)\rangle}, \qquad (8.J.6)$$

where $r \equiv T/\tau$. In words, B_1 is the ratio of the expected variance for N samples to the expected variance for 2 samples (everything else fixed); while B_2 is the ratio of the expected variance with

[9] From Barnes, J. A., Nat. Bur. Stand. (U.S.) Tech. Note 375 (January 1968) [3]; references are identical to those given in the main body of Chapter 8.

[10] It should be noted that in reference [8] the exponent, α, corresponds to the spectrum of *phase* fluctuations while variances are taken over average *frequency* fluctuations. In the present paper, α is equal to the exponent, α, in [8] plus two. Thus, in this paper, all considerations are confined to one variable, $y(t)$ (analogous to *frequency* in [8]) and the spectral density of y, $S_y(f)$. This paper does not consider the spectrum of the integral of $y(t)$.

dead time to that of no dead time (with $N=2$ and τ held constant). The B's, then, reflect bias relative to $N=2$ rather than $N=\infty$. It is apparent that $B_1(N, r=1, \mu) \equiv \chi(N, \mu)$.

For the conditions given above and with reference to Allan [8], one may write expressions for both B_1 and B_2, as follows:

$$B_1(N,r,\mu) = \frac{1+\sum_{n=1}^{N-1} \frac{N-n}{N(N-1)} \left[2|nr|^{\mu+2}-|nr+1|^{\mu+2}-|nr-1|^{\mu+2}\right]}{1+\frac{1}{2}\left[2|r|^{\mu+2}-|r+1|^{\mu+2}-|r-1|^{\mu+2}\right]}; \tag{8.J.7}$$

in particular for $r=1$,

$$B_1(N,1,\mu) = \frac{N(1-N^{\mu})}{2(N-1)\ (1-2^{\mu})}; \tag{8.J.8}$$

and

$$B_2(r,\mu) = \frac{1+\frac{1}{2}\left[2|r|^{\mu+2}-|r+1|^{\mu+2}-|r-1|^{\mu+2}\right]}{2(1-2^{\mu})}, \tag{8.J.9}$$

except that by definition, $B_2(1,\mu) \equiv 1$. The magnitude bars are essential on the $r-1$ term when $r<1$, and, indeed, proper. Since Allan was involved with $r \geqslant 1$ the magnitude bars were dropped in reference [8].

For $\mu=0$, eqs (8.J.7), (8.J.8), and (8.J.9) are indeterminate of form 0/0 and must be evaluated by L'Hospital's rule. Special attention must also be given when expressions of the form 0^0 arise.

One may obtain the following results:

$$B_1(2,r,\mu) \equiv 1$$
$$B_1(N,r,2) = \frac{N(N+1)}{6}$$
$$B_1(N,1,1) = \frac{N}{2}$$
$$B_1(N,r,-1) = 1 \text{ if } r \geqslant 1$$
$$B_1(N,r,-2) = 1 \text{ if } r \neq 1 \text{ or } 0$$
$$B_2(0,\mu) \equiv 0$$
$$B_2(1,\mu) \equiv 1. \quad B_2(r,2) = r^2$$
$$B_2(r,1) = \frac{1}{2}(3r-1) \text{ if } r \geqslant 1$$
$$B_2(r,-1) = \begin{cases} r \text{ if } 0 \leqslant r \leqslant 1 \\ 1 \text{ if } r \geqslant 1 \end{cases}$$
$$B_2(r,-2) = \begin{cases} 0 \text{ if } r=0 \\ 1 \text{ if } r=1 \\ 2/3 \text{ otherwise} \end{cases}$$

Values of the functions $B_1(N, r, \mu)$ and $B_2(r, \mu)$ are tabulated on the following pages for values of N, r, μ as shown below:

$\mu = -2.0$ to 2.0 in steps of 0.2;
$N = 4, 8, 16, 32, 64, 128, 256, 512, 1024, \infty$;
$r = 0.001, 0.003, 0.01, 0.03, 0.1, 0.2, 0.4, 0.8, 1, 1.01, 1.1, 2, 4, 8, 16, 32, 64, 128, 256, 512, 1024, 2048, \infty$.

Figure 8.3 is a graphical representation of $B_2(r, \mu)$ for $0 \leqslant r \leqslant 2$ and $-2 \leqslant \mu \leqslant 2$.

8.J.1.1. Examples of the Use of the Bias Function

The spectral type, that is, the value of μ, may be inferred by varying τ, the sample time [8, 26]. Another convenient way, however, of determining the value of μ is by using $B_1(N, r, \mu)$ as follows: calculate an estimate of $\langle\sigma_y^2(N, T, \tau)\rangle$ and of $\langle\sigma_y^2(2, T, \tau)\rangle$ and hence $B_1(N, r, \mu)$; then by use of the tables the value of μ may be inferred.

Suppose one has an experimental value of $\langle\sigma_y^2(N_1, T_1, \tau_1)\rangle$ and its spectral type is known—that is, μ is known. Suppose also that one wishes to know the variance at some other set of measurement parameters, N_2, T_2, τ_2. An unbiased estimate of $\langle\sigma_y^2(N_2, T_2, \tau_2)\rangle$ may be calculated by the equation:

$$\langle\sigma_y^2(N_2, T_2, \tau_2)\rangle = \left(\frac{\tau_2}{\tau_1}\right)^{\mu} \left[\frac{B_1(N_2, r_2, \mu) B_2(r_2, \mu)}{B_1(N_1, r_1, \mu) B_2(r_1, \mu)}\right] \langle\sigma_y^2(N_1, T_1, \tau_1)\rangle, \tag{8.J.10}$$

where $r_1 = T_1/\tau_1$ and $r_2 = T_2/\tau_2$.

Obviously one might be interested in $N_2=\infty$. In this case if $\mu \geqslant 0$, the expected value of $\sigma_y^2(\infty, T_2, \tau_2)$ is also infinite. This is true because,

$$\lim_{N_2 \to \infty} B_1(N_2, r_2, \mu) = \infty,$$

for $\mu \geqslant 0$.

Also, it should be noted that, for $\mu = 2$, $\langle\sigma_y^2(N, T, \tau)\rangle$ is a function of f_l for any $N \geqslant 2, T, \tau$, even though $B_1(N, r, 2)$ and $B_2(r, 2)$ as determined from eqs (8.J.7), (8.J.8), and (8.J.9) are finite and well behaved [20]. In this region, $\mu \approx 2$, the low frequency behavior is critically important.

8.J.2. ACKNOWLEDGEMENTS

Acknowledgement is given to Mrs. Bernice Bender who capably assisted in writing the computer programs and evaluated the functions.

TABLE 8.J.1. *Functions of* B_1 *and* B_2

The tables are typeset from a computer printout. Each entry for the value of the functions, B_1 and B_2 consists of a decimal number followed by an integer which is the exponent of 10. Ten raised to this power should multiply the decimal number. Thus the table entry "2.752+003" could be written 2.752×10^3 or, simply 2752. Similarly, "9.869−001" $=9.869\times 10^{-1}=0.9869$.

Note for the specific case where $\mu=-2$: As can be seen from figure 8.1, the value of α is ambiguous for this case ($\alpha \geq +1$). The values tabulated for $\mu=-2$ are for $\alpha=+2$ (white noise phase modulation). Slight variations occur (of a few percent) in $B_1(N, r, \mu=-2)$ and $B_2(r, \mu=-2)$ if $\alpha=+1$, and if $2\pi\tau f_h$ is not very large compared to 1. If very precise variance information is desired in such a case as $\alpha=+1$ (flicker noise phase modulation), it is recommended that one use the fundamental equations given in table 8.1.

$B_1(N, r, \mu)$ for $r=0.001$

	N									
μ	4	8	16	32	64	128	256	512	1024	∞
2.00	3.333+000	1.200+001	4.533+001	1.760+002	6.933+002	2.752+003	1.097+004	4.378+004	1.749+005	∞
1.80	3.333+000	1.200+001	4.533+001	1.760+002	6.931+002	2.749+003	1.092+004	4.326+004	1.690+005	∞
1.60	3.333+000	1.200+001	4.533+001	1.759+002	6.926+002	2.743+003	1.087+004	4.265+004	1.628+005	∞
1.40	3.333+000	1.200+001	4.532+001	1.758+002	6.916+002	2.734+003	1.078+004	4.187+004	1.560+005	∞
1.20	3.333+000	1.200+001	4.529+001	1.756+002	6.894+002	2.716+003	1.064+004	4.081+004	1.484+005	∞
1.00	3.332+000	1.199+001	4.520+001	1.749+002	6.847+002	2.682+003	1.041+004	3.931+004	1.392+005	∞
0.80	3.328+000	1.195+001	4.497+001	1.733+002	6.743+002	2.617+003	1.001+004	3.708+004	1.276+005	∞
0.60	3.317+000	1.186+001	4.435+001	1.695+002	6.519+002	2.491+003	9.344+003	3.373+004	1.124+005	∞
0.40	3.284+000	1.161+001	4.280+001	1.608+002	6.058+002	2.259+003	8.233+003	2.877+004	9.247+004	∞
0.20	3.200+000	1.101+001	3.943+001	1.435+002	5.220+002	1.875+003	6.563+003	2.200+004	6.786+004	∞
−0.00	3.027+000	9.877+000	3.354+001	1.157+002	3.985+002	1.355+003	4.491+003	1.429+004	4.201+004	∞
−0.20	2.761+001	8.268+000	2.585+001	8.231+001	2.624+002	8.281+002	2.558+003	7.622+003	2.120+004	3.066+005
−0.40	2.450+000	6.549+000	1.837+001	5.269+001	1.520+002	4.365+002	1.235+003	3.399+003	8.860+003	6.507+004
−0.60	2.150+000	5.053+000	1.250+001	3.172+001	8.138+001	2.089+002	5.325+002	1.335+003	3.224+003	1.592+004
−0.80	1.888+000	3.881+000	8.397+000	1.868+001	4.211+001	9.545+001	2.162+002	4.865+002	1.076+003	3.983+003
−1.00	1.667+000	3.000+000	5.667+000	1.100+001	2.167+001	4.300+001	8.567+001	1.710+002	3.417+002	1.000+003
−1.20	1.482+000	2.344+000	3.868+000	6.547+000	1.124+001	1.945+001	3.386+001	5.936+001	1.057+002	2.512+002
−1.40	1.327+000	1.854+000	2.680+000	3.959+000	5.921+000	8.924+000	1.353+001	2.068+001	3.236+001	6.310+001
−1.60	1.198+000	1.487+000	1.890+000	2.442+000	3.185+000	4.179+000	5.511+000	7.317+000	9.946+000	1.585+001
−1.80	1.091+000	1.210+000	1.360+000	1.541+000	1.757+000	2.010+000	2.306+000	2.656+000	3.106+000	3.981+000
−2.00	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000

$B_1(N, r, \mu)$ for $r=0.003$

	N									
μ	4	8	16	32	64	128	256	512	1024	∞
2.00	3.333+000	1.200+001	4.533+001	1.760+002	6.933+002	2.752+003	1.097+004	4.378+004	1.749+005	∞
1.80	3.333+000	1.200+001	4.532+001	1.759+002	6.917+002	2.732+003	1.072+004	1.112+004	1.514+005	∞
1.60	3.333+000	1.200+001	4.530+001	1.756+002	6.892+002	2.706+003	1.046+004	3.854+004	1.311+005	∞
1.40	3.333+000	1.199+001	4.526+001	1.752+002	6.853+002	2.671+003	1.015+004	3.596+004	1.132+005	∞
1.20	3.332+000	1.198+001	4.516+001	1.744+002	6.788+002	2.620+003	9.774+003	3.330+004	9.717+004	∞
1.00	3.329+000	1.195+001	4.494+001	1.728+002	6.674+002	2.543+003	9.290+003	3.043+004	8.254+004	∞
0.80	3.321+000	1.189+001	4.446+001	1.696+002	6.473+002	2.425+003	8.645+003	2.721+004	6.875+004	∞
0.60	3.302+000	1.173+001	3.342+001	1.634+002	6.124+002	2.242+003	7.765+003	2.346+004	5.536+004	∞
0.40	3.256+000	1.138+001	4.132+001	1.519+002	5.544+002	1.967+003	6.584+003	1.907+004	4.210+004	∞
0.20	3.157+000	1.069+001	3.753+001	1.330+002	4.667+002	1.589+003	5.103+003	1.414+004	2.927+004	∞
−0.00	2.981+000	9.558+000	3.177+001	1.066+002	3.541+002	1.143+003	3.490+003	9.224+003	1.794+004	∞
−0.20	2.729+000	8.054+000	2.474+001	7.694+001	2.377+002	7.169+002	2.061+003	5.175+003	9.470+003	4.488+004
−0.40	2.435+000	6.454+000	1.790+001	5.052+001	1.425+002	3.955+002	1.058+003	2.511+003	4.330+003	1.142+004
−0.60	2.145+000	5.024+000	1.236+001	3.112+001	7.881+001	1.982+002	4.870+002	1.085+003	1.765+003	3.434+003
−0.80	1.887+000	3.876+000	8.372+000	1.857+001	4.167+001	9.363+001	2.085+002	4.326+002	6.638+002	1.066+003
−1.00	1.667+000	3.000+000	5.667+000	1.100+001	2.167+001	4.300+001	8.567+001	1.637+002	2.368+002	3.333+002
−1.20	1.482+000	2.344+000	3.870+000	6.556+000	1.128+001	1.960+001	3.451+001	6.001+001	8.168+001	1.043+002
−1.40	1.328+000	1.854+000	2.681+000	3.964+000	5.941+000	9.001+000	1.386+001	2.159+001	2.756+001	3.264+001
−1.60	1.199+000	1.487+000	1.890+000	2.443+000	3.191+000	4.205+000	5.623+000	7.711+000	9.174+000	1.021+001
−1.80	1.091+000	1.210+000	1.360+000	1.542+000	1.759+000	2.016+000	2.331+000	2.759+000	3.031+000	3.196+000
−2.00	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000

$B_1(N, r, \mu)$ for $r=0.010$

μ	N: 4	8	16	32	64	128	256	512	1024	∞
2.00	3.333+000	1.200+001	4.533+001	1.760+002	6.933+002	2.752+003	1.097+004	4.378+004	1.749+005	∞
1.80	3.333+000	1.199+001	4.526+001	1.750+002	6.819+002	2.623+003	9.737+003	3.493+004	1.231+005	∞
1.60	3.332+000	1.198+001	4.514+001	1.738+002	6.688+002	2.493+003	8.637+003	2.793+004	8.697+004	∞
1.40	3.330+000	1.196+001	4.494+001	1.720+002	6.530+002	2.359+003	7.640+003	2.234+004	6.169+004	∞
1.20	3.327+000	1.192+001	4.461+001	1.693+002	6.330+002	2.215+003	6.717+003	1.782+004	4.383+004	∞
1.00	3.319+000	1.185+001	4.403+001	1.653+002	6.066+002	2.055+003	5.842+003	1.412+004	3.109+004	∞
0.80	3.302+000	1.170+001	4.303+001	1.590+002	5.708+002	1.869+003	4.991+003	1.104+004	2.189+004	∞
0.60	3.268+000	1.143+001	4.133+001	1.495+002	5.221+002	1.649+003	4.142+003	8.419+003	1.514+004	∞
0.40	3.203+000	1.095+001	3.857+001	1.354+002	4.572+002	1.390+003	3.287+003	6.172+003	1.014+004	∞
0.20	3.089+000	1.018+001	3.448+001	1.162+002	3.763+002	1.097+003	2.446+003	4.263+003	6.453+003	∞
−0.00	2.912+000	9.076+000	2.909+001	9.282+001	2.857+002	7.951+002	1.672+003	2.719+003	3.826+003	∞
−0.20	2.677+000	7.714+000	2.296+001	6.836+001	1.976+002	5.220+002	1.036+003	1.580+003	2.084+003	5.581+003
−0.40	2.407+000	6.279+000	1.703+001	4.655+001	1.247+002	3.106+002	5.814+002	8.357+002	1.043+003	1.715+003
−0.60	2.134+000	4.959+000	1.205+001	2.977+001	7.296+001	1.697+002	2.995+002	4.076+002	4.851+002	6.423+002
−0.80	1.884+000	3.860+000	8.303+000	1.828+001	4.041+001	8.691+001	1.443+002	1.868+002	2.137+002	2.519+002
−1.00	1.667+000	3.000+000	5.667+000	1.100+001	2.167+001	4.255+001	6.628+001	8.190+001	9.064+001	1.000+002
−1.20	1.482+000	2.346+000	3.880+000	6.598+000	1.145+001	2.026+001	2.947+001	3.486+001	3.754+001	3.980+001
−1.40	1.328+000	1.856+000	2.688+000	3.991+000	6.054+000	9.500+000	1.282+001	1.456+001	1.532+001	1.585+001
−1.60	1.199+000	1.487+000	1.893+000	2.455+000	3.239+000	4.440+000	5.505+000	6.002+000	6.198+000	6.309+000
−1.80	1.091+000	1.210+000	1.360+000	1.545+000	1.772+000	2.089+000	2.349+000	2.456+000	2.494+000	2.512+000
−2.00	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000

$B_1(N, r, \mu)$ for $r=0.030$

μ	N: 4	8	16	32	64	128	256	512	1024	∞
2.00	3.333+000	1.200+001	4.533+001	1.760+002	6.933+002	2.752+003	1.097+004	4.378+004	1.749+005	∞
1.80	3.331+000	1.196+001	4.487+001	1.706+002	6.374+002	2.304+003	8.142+003	2.849+004	9.936+004	∞
1.60	3.327+000	1.191+001	4.431+001	1.650+002	5.851+002	1.930+003	6.068+003	1.866+004	5.684+004	∞
1.40	3.320+000	1.184+001	4.360+001	1.588+002	5.352+002	1.615+003	4.535+003	1.228+004	3.276+004	∞
1.20	3.309+000	1.172+001	4.267+001	1.518+002	4.867+002	1.348+003	3.392+003	8.128+003	1.903+004	∞
1.00	3.290+000	1.155+001	4.141+001	1.437+002	4.382+002	1.117+003	2.533+003	5.395+003	1.114+004	∞
0.80	3.257+000	1.128+001	3.966+001	1.339+002	3.887+002	9.152+002	1.879+003	3.580+003	6.554+003	∞
0.60	3.203+000	1.087+001	3.728+001	1.220+002	3.372+002	7.357+002	1.376+003	2.364+003	3.868+003	∞
0.40	3.118+000	1.027+001	3.412+001	1.077+002	2.835+002	5.750+002	9.868+002	1.541+003	2.277+003	∞
0.20	2.992+000	9.460+000	3.013+001	9.127+001	2.286+002	4.324+002	6.856+002	9.834+002	1.328+003	∞
−0.00	2.819+000	8.430+000	2.546+001	7.348+001	1.748+002	3.096+002	4.568+002	6.082+002	7.611+002	∞
−0.20	2.604+000	7.242+000	2.047+001	5.581+001	1.259+002	2.094+002	2.897+002	3.619+002	4.256+002	8.571+002
−0.40	2.362+000	6.005+000	1.566+001	3.993+001	8.524+001	1.335+002	1.744+002	2.066+002	2.314+002	3.099+002
−0.60	2.114+000	4.838+000	1.147+001	2.706+001	5.441+001	8.049+001	9.997+001	1.134+002	1.225+002	1.403+002
−0.80	1.878+000	3.825+000	8.141+000	1.753+001	3.304+001	4.626+001	5.499+001	6.029+001	6.341+001	6.770+001
−1.00	1.667+000	3.000+000	5.667+000	1.100+001	1.928+001	2.560+001	2.929+001	3.127+001	3.229+001	3.333+001
−1.20	1.484+000	2.355+000	3.918+000	6.768+000	1.093+001	1.378+001	1.525+001	1.595+001	1.626+001	1.651+001
−1.40	1.329+000	1.863+000	2.719+000	4.133+000	6.080+000	7.268+000	7.815+000	8.044+000	8.135+000	8.191+000
−1.60	1.199+000	1.491+000	1.910+000	2.532+000	3.341+000	3.782+000	3.963+000	4.030+000	4.053+000	4.064+000
−1.80	1.091+000	1.211+000	1.366+000	1.573+000	1.827+000	1.950+000	1.995+000	2.010+000	2.015+000	2.016+000
−2.00	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000

$B_1(N, r, \mu)$ for $r=0.100$

μ	N: 4	8	16	32	64	128	256	512	1024	∞
2.00	3.333+000	1.200+001	4.533+001	1.760+002	6.933+002	2.752+003	1.097+004	4.378+004	1.749+005	∞
1.80	3.315+000	1.174+001	4.252+001	1.517+002	5.330+002	1.858+003	6.467+003	2.250+004	7.830+004	∞
1.60	3.293+000	1.147+001	3.982+001	1.309+002	4.113+002	1.263+003	3.843+003	1.166+004	3.535+004	∞
1.40	3.265+000	1.118+001	3.721+001	1.130+002	3.183+002	8.636+002	2.304+003	6.103+003	1.613+004	∞
1.20	3.230+000	1.084+001	3.463+001	9.728+001	2.470+002	5.943+002	1.394+003	3.232+003	7.453+003	∞
1.00	3.184+000	1.045+001	3.204+001	8.347+001	1.918+002	4.114+002	8.523+002	1.735+003	3.500+003	∞
0.80	3.123+000	9.989+000	2.938+001	7.118+001	1.487+002	2.862+002	5.269+002	9.466+002	1.678+003	∞
0.60	3.043+000	9.445+000	2.663+001	6.014+001	1.149+002	1.999+002	3.295+002	5.265+002	8.254+002	∞
0.40	2.940+000	8.804+000	2.376+001	5.015+001	8.816+001	1.397+002	2.084+002	2.993+002	4.194+002	∞
0.20	2.810+000	8.065+000	2.080+001	4.111+001	6.690+001	9.750+001	1.331+002	1.741+002	2.214+002	∞
−0.00	2.653+000	7.236+000	1.779+001	3.300+001	5.003+001	6.771+001	8.566+001	1.037+002	1.219+002	∞
−0.20	2.472+000	6.344+000	1.481+001	2.585+001	3.673+001	4.664+001	5.542+001	6.314+001	6.989+001	1.156+002
−0.40	2.273+000	5.430+000	1.199+001	1.971+001	2.642+001	3.177+001	3.592+001	3.910+001	4.153+001	4.922+001
−0.60	2.065+000	4.540+000	9.423+000	1.461+001	1.859+001	2.137+001	2.326+001	2.452+001	2.537+001	2.702+001
−0.80	1.860+000	3.720+000	7.204+000	1.055+001	1.281+001	1.419+001	1.502+001	1.550+001	1.577+001	1.616+001
−1.00	1.667+000	3.000+000	5.375+000	7.429+000	8.653+000	9.312+000	9.652+000	9.825+000	9.912+000	1.000+001
−1.20	1.491+000	2.396+000	3.931+000	5.125+000	5.751+000	6.046+000	6.177+000	6.235+000	6.260+000	6.278+000
−1.40	1.337+000	1.907+000	2.832+000	3.477+000	3.772+000	3.892+000	3.938+000	3.954+000	3.960+000	3.962+000
−1.60	1.205+000	1.522+000	2.021+000	2.329+000	2.449+000	2.490+000	2.502+000	2.505+000	2.505+000	2.504+000
−1.80	1.093+000	1.226+000	1.438+000	1.546+000	1.579+000	1.586+000	1.586+000	1.585+000	1.584+000	1.583+000
−2.00	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000

$B_1(N, r, \mu)$ for $r=0.200$

μ	N: 4	8	16	32	64	128	256	512	1024	∞
2.00	3.333+000	1.200+001	4.533+001	1.760+002	6.933+002	2.752+003	1.097+004	4.378+004	1.749+005	∞
1.80	3.279+000	1.131+001	3.925+001	1.358+002	4.700+002	1.629+003	5.657+003	1.967+004	6.843+004	∞
1.60	3.222+000	1.065+001	3.404+001	1.053+002	3.209+002	9.729+002	2.946+003	8.923+003	2.703+004	∞
1.40	3.162+000	1.002+001	2.955+001	8.204+001	2.208+002	5.868+002	1.552+003	4.096+003	1.081+004	∞
1.20	3.096+000	9.409+000	2.567+001	6.420+001	1.533+002	3.581+002	8.286+002	1.909+003	4.391+003	∞
1.00	3.024+000	8.811+000	2.229+001	5.044+001	1.074+002	2.215+002	4.501+002	9.072+002	1.821+003	∞
0.80	2.942+000	8.219+000	1.933+001	3.977+001	7.593+001	1.392+002	2.496+002	4.420+002	7.771+002	∞
0.60	2.849+000	7.624+000	1.671+001	3.143+001	5.424+001	8.907+001	1.420+002	2.225+002	3.445+002	∞
0.40	2.743+000	7.023+000	1.438+001	2.488+001	3.912+001	5.811+001	8.330+001	1.166+002	1.607+002	∞
0.20	2.622+000	6.414+000	1.229+001	1.969+001	2.847+001	3.870+001	5.054+001	6.421+001	7.994+001	∞
−0.00	2.487+000	5.796+000	1.042+001	1.555+001	2.088+001	2.631+001	3.180+001	3.733+001	4.288+001	∞
−0.20	2.337+000	5.175+000	8.751+000	1.224+001	1.542+001	1.825+001	2.075+001	2.295+001	2.487+001	3.793+001
−0.40	2.176+000	4.558+000	7.262+000	9.581+000	1.143+001	1.287+001	1.399+001	1.484+001	1.550+001	1.757+001
−0.60	2.008+000	3.958+000	5.949+000	7.450+000	8.497+000	9.211+000	9.692+000	1.001+001	1.023+001	1.066+001
−0.80	1.836+000	3.387+000	4.808+000	5.746+000	6.318+000	6.658+000	6.857+000	6.974+000	7.042+000	7.136+000
−1.00	1.667+000	2.857+000	3.833+000	4.395+000	4.692+000	4.845+000	4.922+000	4.961+000	4.980+000	5.000+000
−1.20	1.505+000	2.379+000	3.016+000	3.333+000	3.476+000	3.539+000	3.565+000	3.575+000	3.579+000	3.582+000
−1.40	1.355+000	1.960+000	2.346+000	2.507+000	2.568+000	2.588+000	2.594+000	2.595+000	2.595+000	2.593+000
−1.60	1.221+000	1.601+000	1.805+000	1.873+000	1.890+000	1.892+000	1.890+000	1.889+000	1.888+000	1.886+000
−1.80	1.102+000	1.300+000	1.378+000	1.390+000	1.387+000	1.382+000	1.378+000	1.376+000	1.375+000	1.374+000
−2.00	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000

$B_1(N, r, \mu)$ for $r=0.400$

					N					
μ	4	8	16	32	64	128	256	512	1024	∞
2.00	3.333+000	1.200+001	4.533+001	1.760+002	6.933+002	2.752+003	1.097+004	4.378+004	1.749+005	∞
1.80	3.186+000	1.052+001	3.540+001	1.207+002	4.152+002	1.436+003	4.984+003	1.732+004	6.026+004	∞
1.60	3.047+000	9.247+000	2.780+001	8.345+001	2.511+002	7.574+002	2.289+003	6.929+003	2.099+004	∞
1.40	2.915+000	8.151+000	2.197+001	5.825+001	1.536+002	4.046+002	1.066+003	2.810+003	7.412+003	∞
1.20	2.790+000	7.205+000	1.748+001	4.110+001	9.529+001	2.196+002	5.051+002	1.161+003	2.666+003	∞
1.00	2.670+000	6.384+000	1.401+001	2.936+001	6.011+001	1.216+002	2.447+002	4.909+002	9.832+002	∞
0.80	2.555+000	5.670+000	1.131+001	2.126+001	3.868+001	6.908+001	1.221+002	2.144+002	3.752+002	∞
0.60	2.443+000	5.046+000	9.201+000	1.563+001	2.546+001	4.045+001	6.323+001	9.780+001	1.503+002	∞
0.40	2.334+000	4.499+000	7.544+000	1.167+001	1.721+001	2.457+001	3.433+001	4.725+001	6.433+001	∞
0.20	2.227+000	4.017+000	6.232+000	8.870+000	1.196+001	1.556+001	1.973+001	2.454+001	3.009+001	∞
−0.00	2.121+000	3.589+000	5.187+000	6.856+000	8.572+000	1.032+001	1.209+001	1.387+001	1.566+001	∞
−0.20	2.015+000	3.207+000	4.346+000	5.389+000	6.330+000	7.169+000	7.913+000	8.569+000	9.144+000	1.306+001
−0.40	1.909+000	2.864+000	3.664+000	4.305+000	4.811+000	5.208+000	5.516+000	5.753+000	5.936+000	6.518+000
−0.60	1.802+000	2.555+000	3.105+000	3.489+000	3.755+000	3.937+000	4.060+000	4.144+000	4.200+000	4.313+000
−0.80	1.694+000	2.274+000	2.642+000	2.864+000	2.997+000	3.077+000	3.123+000	3.151+000	3.167+000	3.191+000
−1.00	1.583+000	2.018+000	2.254+000	2.376+000	2.438+000	2.469+000	2.484+000	2.492+000	2.496+000	2.500+000
−1.20	1.471+000	1.782+000	1.926+000	1.987+000	2.011+000	2.021+000	2.024+000	2.025+000	2.026+000	2.025+000
−1.40	1.356+000	1.565+000	1.644+000	1.670+000	1.677+000	1.678+000	1.677+000	1.676+000	1.675+000	1.674+000
−1.60	1.240+000	1.363+000	1.401+000	1.408+000	1.408+000	1.406+000	1.404+000	1.403+000	1.402+000	1.402+000
−1.80	1.121+000	1.175+000	1.188+000	1.188+000	1.186+000	1.184+000	1.183+000	1.182+000	1.182+000	1.182+000
−2.00	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000

$B_1(N, r, \mu)$ for $r=0.800$

					N					
μ	4	8	16	32	64	128	256	512	1024	∞
2.00	3.333+000	1.200+001	4.533+001	1.760+002	6.933+002	2.752+003	1.097+004	4.378+004	1.749+005	∞
1.80	3.032+000	9.694+000	3.213+001	1.089+002	3.739+002	1.292+003	4.483+003	1.558+004	5.421+004	∞
1.60	2.765+000	7.876+000	2.297+001	6.806+001	2.038+002	6.135+002	1.853+003	5.608+003	1.698+004	∞
1.40	2.529+000	6.441+000	1.658+001	4.305+001	1.125+002	2.953+002	7.770+002	2.047+003	5.398+003	∞
1.20	2.319+000	5.306+000	1.210+001	2.762+001	6.317+001	1.447+002	3.319+002	7.617+002	1.749+003	∞
1.00	2.134+000	4.405+000	8.950+000	1.804+001	3.622+001	7.259+001	1.453+002	2.908+002	5.817+002	∞
0.80	1.969+000	3.688+000	6.718+000	1.203+001	2.132+001	3.753+001	6.579+001	1.150+002	2.008+002	∞
0.60	1.823+000	3.117+000	5.128+000	8.225+000	1.296+001	2.018+001	3.116+001	4.782+001	7.310+001	∞
0.40	1.693+000	2.660+000	3.986+000	5.783+000	8.193+000	1.140+001	1.567+001	2.130+001	2.876+001	∞
0.20	1.578+000	2.293+000	3.160+000	4.196+000	5.416+000	6.840+000	8.492+000	1.040+001	1.260+001	∞
−0.00	1.477+000	1.998+000	2.558+000	3.149+000	3.761+000	4.389+000	5.027+000	5.671+000	6.318+000	∞
−0.20	1.387+000	1.760+000	2.115+000	2.446+000	2.749+000	3.023+000	3.267+000	3.483+000	3.673+000	4.968+000
−0.40	1.308+000	1.568+000	1.786+000	1.966+000	2.112+000	2.229+000	2.321+000	2.392+000	2.447+000	2.624+000
−0.60	1.238+000	1.413+000	1.541+000	1.634+000	1.700+000	1.748+000	1.780+000	1.803+000	1.818+000	1.850+000
−0.80	1.177+000	1.287+000	1.357+000	1.400+000	1.427+000	1.444+000	1.454+000	1.461+000	1.464+000	1.470+000
−1.00	1.125+000	1.187+000	1.219+000	1.234+000	1.242+000	1.246+000	1.248+000	1.249+000	1.250+000	1.250+000
−1.20	1.081+000	1.109+000	1.117+000	1.118+000	1.117+000	1.116+000	1.115+000	1.114+000	1.114+000	1.113+000
−1.40	1.045+000	1.050+000	1.045+000	1.039+000	1.035+000	1.032+000	1.030+000	1.029+000	1.028+000	1.028+000
−1.60	1.017+000	1.010+000	1.000+000	9.922−001	9.872−001	9.843−001	9.827−001	9.819−001	9.814−001	9.809−001
−1.80	1.001+000	9.914−001	9.826−001	9.768−001	9.734−001	9.715−001	9.704−001	9.699−001	9.697−001	9.694−001
−2.00	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000

$B_1(N, r, \mu)$ for $r=1.00$

	N									
μ	4	8	16	32	64	128	256	512	1024	∞
2.00	3.330+000	1.200+001	4.533+001	1.760+002	6.933+002	2.752+003	1.097+004	4.378+004	1.749+005	∞
1.80	2.988+000	9.490+000	3.138+001	1.063+002	3.646+002	1.260+003	4.372+003	1.519+004	5.286+004	∞
1.60	2.688+000	7.555+000	2.191+001	6.479+001	1.938+002	5.833+002	1.762+003	5.331+003	1.615+004	∞
1.40	2.426+000	6.059+000	1.546+001	3.999+001	1.044+002	2.738+002	7.202+002	1.897+003	5.003+003	∞
1.20	2.198+000	4.900+000	1.104+001	2.506+001	5.717+001	1.308+002	2.999+002	6.881+002	1.580+003	∞
1.00	2.000+000	4.000+000	8.000+000	1.600+001	3.200+001	6.400+001	1.280+002	2.560+002	5.120+002	∞
0.80	1.827+000	3.299+000	5.894+000	1.045+001	1.841+001	3.230+001	5.652+001	9.872+001	1.722+002	∞
0.60	1.677+000	2.750+000	4.424+000	7.006+000	1.096+001	1.698+001	2.614+001	4.005+001	6.114+001	∞
0.40	1.546+000	2.320+000	3.391+000	4.846+000	6.801+000	9.407+000	1.287+001	1.744+001	2.350+001	∞
0.20	1.432+000	1.982+000	2.658+000	3.471+000	4.432+000	5.555+000	6.858+000	8.363+000	1.010+001	∞
0.00	1.333+000	1.714+000	2.133+000	2.581+000	3.048+000	3.528+000	4.016+000	4.509+000	5.005+000	∞
−0.20	1.247+000	1.502+000	1.754+000	1.994+000	2.216+000	2.418+000	2.599+000	2.759+000	2.900+000	3.863+000
−0.40	1,172+000	1.333+000	1.476+000	1.599+000	1.700+000	1.782+000	1.847+000	1.898+000	1.938+000	2.065+000
−0.60	1.107+000	1.197+000	1.271+000	1.327+000	1.370+000	1.401+000	1.422+000	1.438+000	1.448+000	1.470+000
−0.80	1.050+000	1.088+000	1.117+000	1.137+000	1.150+000	1.160+000	1.165+000	1.169+000	1.171+000	1.175+000
−1.00	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000
−1.20	9.569−001	9.284−001	9.105−001	8.997−001	8.933−001	8.897−001	8.877−001	8.866−001	8.860−001	8.854−001
−1.40	9.193−001	8.700−001	8.410−001	8.245−001	8.154−001	8.105−001	8.079−001	8.065−001	8.058−001	8.051−001
−1.60	8.866−001	8.221−001	7.864−001	7.672−001	7.570−001	7.517−001	7.490−001	7.476−001	7.468−001	7.461−001
−1.80	8.581−001	7.827−001	7.431−001	7.226−001	7.122−001	7.068−001	7.042−001	7.028−001	7.021−001	7.014−001
−2.00	8.333−001	7.500−001	7.083−001	6.875−001	6.771−001	6.719−001	6.693−001	6.680−001	6.673−001	6.667−001

$B_1(N, r, \mu)$ for $r=1.01$

	N									
μ	4	8	16	32	64	128	256	512	1024	∞
2.00	3.333+000	1.200+001	4.533+001	1.760+002	6.933+002	2.752+003	1.097+004	4.378+004	1.749+005	∞
1.80	2.986+000	9.482+000	3.135+001	1.062+002	3.643+002	1.259+003	4.367+003	1.518+004	5.280+004	∞
1.60	2.685+000	7.542+000	2.187+001	6.466+001	1.934+002	5.822+002	1.758+003	5.321+003	1.611+004	∞
1.40	2.422+000	6.045+000	1.542+001	3.988+001	1.041+002	2.730+002	7.180+002	1.892+003	4.988+003	∞
1.20	2.194+000	4.885+000	1.100+001	2.497+001	5.695+001	1.303+002	2.987+002	6.854+002	1.573+003	∞
1.00	1.995+000	3.985+000	7.966+000	1.593+001	3.185+001	6.369+001	1.274+002	2.547+002	5.095+002	∞
0.80	1.822+000	3.285+000	5.864+000	1.039+001	1.830+001	3.212+001	5.619+001	9.814+001	1.712+002	∞
0.60	1.672+000	2.738+000	4.400+000	6.963+000	1.089+001	1.687+001	2.597+001	3.978+001	6.073+001	∞
0.40	1.541+000	2.309+000	3.371+000	4.814+000	6.754+000	9.340+000	1.277+001	1.731+001	2.332+001	∞
0.20	1.428+000	1.972+000	2.642+000	3.447+000	4.400+000	5.512+000	6.804+000	8.296+000	1.002+001	∞
−0.00	1.329+000	1.706+000	2.120+000	2.563+000	3.025+000	3.500+000	3.984+000	4.472+000	4.963+000	∞
−0.20	1.243+000	1.495+000	1.743+000	1.980+000	2.200+000	2.400+000	2.579+000	2.737+000	2.877+000	3.829+000
−0.40	1.168+000	1.327+000	1.468+000	1.589+000	1.689+000	1.770+000	1.835+000	1.885+000	1.924+000	2.050+000
−0.60	1.104+000	1.193+000	1.265+000	1.321+000	1.362+000	1.393+000	1.414+000	1.429+000	1.440+000	1.461+000
−0.80	1.048+000	1.085+000	1.113+000	1.133+000	1.147+000	1.155+000	1.161+000	1.165+000	1.167+000	1.170+000
−1.00	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000
−1.20	9.599−001	9.331−001	9.160−001	9.056−001	8.995−001	8.960−001	8.940−001	8.929−001	8.924−001	8.917−001
−1.40	9.286−001	8.840−001	8.573−001	8.419−001	8.333−001	8.287−001	8.262−001	8.249−001	8.242−001	8.235−001
−1.60	9.098−001	8.566−001	8.264−001	8.098−001	8.009−001	7.963−001	7.938−001	7.926−001	7.920−001	7.913−001
−1.80	9.156−001	8.682−001	8.425−001	8.288−001	8.217−001	8.181−001	8.162−001	8.153−001	8.148−001	8.143−001
−2.00	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000

$B_1(N, r, \mu)$ for $r=1.10$

	N									
μ	4	8	16	32	64	128	256	512	1024	∞
2.00	3.333+000	1.200+001	4.533+001	1.760+002	6.933+002	2.752+003	1.097+004	4.378+004	1.749+005	∞
1.80	2.972+000	9.418+000	3.111+001	1.053+002	3.614+002	1.249+003	4.333+003	1.506+004	5.238+004	∞
1.60	2.660+000	7.443+000	2.154+001	6.366+001	1.904+002	5.730+002	1.731+003	5.237+003	1.586+004	∞
1.40	2.391+000	5.929+000	1.508+001	3.897+001	1.016+002	2.666+002	7.011+002	1.847+003	4.870+003	∞
1.20	2.158+000	4.766+000	1.069+001	2.422+001	5.521+001	1.263+002	2.894+002	6.640+002	1.524+003	∞
1.00	1.957+000	3.870+000	7.696+000	1.535+001	3.065+001	6.126+001	1.225+002	2.449+002	4.898+002	∞
0.80	1.783+000	3.177+000	5.637+000	9.954+000	1.750+001	3.068+001	5.365+001	9.366+001	1.634+002	∞
0.60	1.633+000	2.640+000	4.213+000	6.639+000	1.035+001	1.602+001	2.463+001	3.771+001	5.754+001	∞
0.40	1.504+000	2.223+000	3.219+000	4.575+000	6.398+000	8.828+000	1.205+001	1.632+001	2.197+001	∞
0.20	1.393+000	1.898+000	2.521+000	3.272+000	4.160+000	5.199+000	6.404+000	7.797+000	9.402+000	∞
−0.00	1.298+000	1.643+000	2.026+000	2.435+000	2.863+000	3.304+000	3.752+000	4.205+000	4.661+000	∞
−0.20	1.217+000	1.444+000	1.671+000	1.889+000	2.091+000	2.275+000	2.440+000	2.586+000	2.714+000	3.593+000
−0.40	1.147+000	1.288+000	1.416+000	1.526+000	1.617+000	1.692+000	1.751+000	1.797+000	1.832+000	1.948+000
−0.60	1.088+000	1.166+000	1.230+000	1.281+000	1.319+000	1.346+000	1.366+000	1.379+000	1.389+000	1.408+000
−0.80	1.040+000	1.072+000	1.096+000	1.114+000	1.126+000	1.134+000	1.139+000	1.142+000	1.144+000	1.147+000
−1.00	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000
−1.20	9.696−001	9.482−001	9.341−001	9.254−001	9.201−001	9.171−001	9.154−001	9.145−001	9.140−001	9.134−001
−1.40	9.493−001	9.157−001	8.948−001	8.825−001	8.755−001	8.717−001	8.696−001	8.685−001	8.680−001	8.674−001
−1.60	9.415−001	9.048−001	8.831−001	8.709−001	8.642−001	8.607−.001	8.588−001	8.578−001	8.574−001	8.569−001
−1.80	9.527−001	9.243−001	9.082−001	8.994−001	8.948−001	8.924−001	8.911−001	8.905−001	8.902−001	8.898−001
−2.00	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000

$B_1(N, r, \mu)$ for $r=2.00$

	N									
μ	4	8	16	32	64	128	256	512	1024	∞
2.00	3.333+000	1.200+001	4.533+001	1.760+002	6.933+002	2.752+003	1.097+004	4.378+004	1.749+005	∞
1.80	2.908+000	9.132+000	3.006+001	1.017+002	3.487+002	1.205+003	4.179+003	1.453+004	5.053+004	∞
1.60	2.552+000	7.012+000	2.014+001	5.935+001	1.773+002	5.334+002	1.611+003	4.874+003	1.476+004	∞
1.40	2.255+000	5.439+000	1.366+001	3.512+001	9.142+001	2.395+002	6.299+002	1.659+003	4.374+003	∞
1.20	2.007+000	4.271+000	9.409+000	2.114+001	4.800+001	1.096+002	2.510+002	5.756+002	1.321+003	∞
1.00	1.800+000	3.400+000	6.600+000	1.300+001	2.580+001	5.140+001	1.026+002	2.050+002	4.098+002	∞
0.80	1.628+000	2.750+000	4.733+000	8.215+000	1.431+001	2.494+001	4.347+001	7.576+001	1.320+002	∞
0.60	1.486+000	2.264+000	3.485+000	5.371+000	8.262+000	1.267+001	1.937+001	2.955+001	4.498+001	∞
0.40	1.369+000	1.901+000	2.644+000	3.659+000	5.025+000	6.847+000	9.267+000	1.247+001	1.670+001	∞
0.20	1.273+000	1.629+000	2.075+000	2.615+000	3.256+000	4.005+000	4.876+000	5.882+000	7.042+000	∞
−0.00	1.195+000	1.427+000	1.688+000	1.971+000	2.267+000	2.573+000	2.884+000	3.198+000	3.515+000	∞
−0.20	1.133+000	1.277+000	1.425+000	1.568+000	1.702+000	1.824+000	1.934+000	2.031+000	2.117+000	2.704+000
−0.40	1.084+000	1.168+000	1.246+000	1.315+000	1.373+000	1.420+000	1.457+000	1.487+000	1.509+000	1.583+000
−0.60	1.046+000	1.090+000	1.126+000	1.156+000	1.178+000	1.195+000	1.207+000	1.215+000	1.221+000	1.233+000
−0.80	1.019+000	1.035+000	1.048+000	1.058+000	1.064+000	1.069+000	1.072+000	1.074+000	1.075+000	1.077+000
−1.00	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000
−1.20	9.886−001	9.799−001	9.738−001	9.699−001	9.675−001	9.661−001	9.653−001	9.648−001	9.646−001	9.643−001
−1.40	9.837−001	9.719−001	9.641−001	9.593−001	9.565−001	9.550−001	9.541−001	9.537−001	9.534−001	9.532−001
−1.60	9.845−001	9.737−001	9.669−001	9.630−001	9.607−001	9.595−001	9.589−001	9.586−001	9.584−001	9.582−001
−1.80	9.901−001	9.836−001	9.796−001	9.774−001	9.761−001	9.755−001	9.752−001	9.750−001	9.749−001	9.748−001
−2.00	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000

$B_1(N, r, \mu)$ for $r=4.00$

μ	N 4	8	16	32	64	128	256	512	1024	∞
2.00	3.333+000	1.200+001	4.533+001	1.760+002	6.933+002	2.752+003	1.097+004	4.378+004	1.749+005	∞
1.80	2.879+000	9.006+000	2.960+001	1.000+002	3.431+002	1.186+003	4.112+003	1.429+004	4.972+004	∞
1.60	2.504+000	6.819+000	1.952+001	5.745+001	1.715+002	5.160+002	1.558+003	4.714+003	1.428+004	∞
1.40	2.194+000	5.219+000	1.303+001	3.340+001	8.686+001	2.275+002	5.981+002	1.575+003	4.154+003	∞
1.20	1.938+000	4.045+000	8.826+000	1.974+001	4.472+001	1.020+002	2.336+002	5.356+002	1.229+003	∞
1.00	1.727+000	3.182+000	6.091+000	1.191+001	2.355+001	4.682+001	9.336+001	1.865+002	3.726+002	∞
0.80	1.555+000	2.548+000	4.303+000	7.385+000	1.278+001	2.219+001	3.860+001	6.718+001	1.170+002	∞
0.60	1.415+000	2.082+000	3.129+000	4.748+000	7.229+000	1.101+001	1.676+001	2.550+001	3.875+001	∞
0.40	1.303+000	1.742+000	2.356+000	3.196+000	4.326+000	5.834+000	7.837+000	1.049+001	1.399+001	∞
0.20	1.214+000	1.495+000	1.848+000	2.275+000	2.783+000	3.377+000	4.067+000	4.865+000	5.784+000	∞
−0.00	1.144+000	1.318+000	1.514+000	1.726+000	1.949+000	2.179+000	2.414+000	2.651+000	2.890+000	∞
−0.20	1.092+000	1.194+000	1.298+000	1.399+000	1.494+000	1.580+000	1.658+000	1.727+000	1.788+000	2,204+000
−0.40	1.054+000	1.109+000	1.160+000	1.205+000	1.243+000	1.274+000	1.299+000	1.318+000	1.333+000	1.382+000
−0.60	1.027+000	1.053+000	1.075+000	1.093+000	1.107+000	1.117+000	1.124+000	1.129+000	1.132+000	1.140+000
−0.80	1.010+000	1.019+000	1.026+000	1.031+000	1.035+000	1.037+000	1.039+000	1.040+000	1.041+000	1.042+000
−1.00	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000
−1.20	9.953−001	9.916−001	9.890−001	9.873−001	9.862−001	9.856−001	9.853−001	9.851−001	9.850−001	9.849−001
−1.40	9.941−001	9.898−001	9.869−001	9.851−001	9.840−001	9.834−001	9.831−001	9.829−001	9.829−001	9.828−001
−1.60	9.952−001	9.918−001	9.896−001	9.884−001	9.876−001	9.873−001	9.870−001	9.869−001	9.869−001	9.868−001
−1.80	9.974−001	9.957−001	9.946−001	9.940−001	9.936−001	9.935−001	9.934−001	9.933−001	9.933−001	9.933−001
−2.00	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000

$B_1(N, r, \mu)$ for $r=8.00$

μ	N 4	8	16	32	64	128	256	512	1024	∞
2.00	3.333+000	1.200+001	4.533+001	1.760+002	6.933+002	2.752+003	1.097+004	4.378+004	1.749+005	∞
1.80	2.870+000	8.963+000	2.945+001	9.951+001	3.413+002	1.179+003	4.090+003	1.421+004	4.945+004	∞
1.60	2.486+000	6.751+000	1.930+001	5.678+001	1.695+002	5.099+002	1.540+003	4.658+003	1.411+004	∞
1.40	2.170+000	5.135+000	1.279+001	3.276+001	8.515+001	2.230+002	5.862+002	1.544+003	4.071+003	∞
1.20	1.910+000	3.953+000	8.590+000	1.917+001	4.341+001	9.898+001	2.265+002	5.195+002	1.192+003	∞
1.00	1.696+000	3.087+000	5.870+000	1.143+001	2.257+001	4.483+001	8.935+001	1.784+002	3.565+002	∞
0.80	1.521+000	2.453+000	4.101+000	6.996+000	1.206+001	2.090+001	3.630+001	6.315+001	1.099+002	∞
0.60	1.380+000	1.991+000	2.950+000	4.433+000	6.706+000	1.017+001	1.544+001	2.345+001	3.559+001	∞
0.40	1.268+000	1.657+000	2.202+000	2.946+000	3.949+000	5.286+000	7.062+000	9.414+000	1.252+001	∞
0.20	1.181+000	1.420+000	1.720+000	2.083+000	2.514+000	3.019+000	3.606+000	4.284+000	5.066+000	∞
−0.00	1.116+000	1.255+000	1.413+000	1.584+000	1.763+000	1.969+000	2.137+000	2.328+000	2.520+000	∞
−0.20	1.069+000	1.145+000	1.223+000	1.299+000	1.371+000	1.436+000	1.494+000	1.546+000	1.592+000	1.905+000
−0.40	1.037+000	1.075+000	1.110+000	1.142+000	1.168+000	1.189+000	1.207+000	1.220+000	1.231+000	1.264+000
−0.60	1.017+000	1.033+000	1.047+000	1.058+000	1.067+000	1.073+000	1.078+000	1.081+000	1.083+000	1.088+000
−0.80	1.006+000	1.011+000	1.014+000	1.017+000	1.020+000	1.021+000	1.022+000	1.022+000	1.023+000	1.023+000
−1.00	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000
−1.20	9.980−001	9.963−001	9.952−001	9.945−001	9.940−001	9.938−001	9.936−001	9.935−001	9.935−001	9.934−001
−1.40	9.978−001	9.961−001	9.950−001	9.944−001	9.940−001	9.937−001	9.936−001	9.936−001	9.935−001	9.935−001
−1.60	9.984−001	9.973−001	9.966−001	9.962−001	9.960−001	9.958−001	9.958−001	9.957−001	9.957−001	9.957−001
−1.80	9.993−001	9.988−001	9.985−001	9.983−001	9.982−001	9.981−001	9.981−001	9.981−001	9.981−001	9.981−001
−2.00	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000

$B_1(N, r, \mu)$ for $r = 16.00$

μ	N = 4	8	16	32	64	128	256	512	1024	∞
2.00	3.333+000	1.200+001	4.533+001	1.760+002	6.933+002	2.752+003	1.097+004	4.378+004	1.749+005	∞
1.80	2.866+000	8.949+000	2.940+001	9.934+001	3.406+002	1.177+003	4.083+003	1.419+004	4.936+004	∞
1.60	2.480+000	6.727+000	1.923+001	5.654+001	1.688+002	5.077+002	1.533+003	4.639+003	1.405+004	∞
1.40	2.161+000	5.104+000	1.270+001	3.251+001	8.450+001	2.213+002	5.817+002	1.532+003	4.039+003	∞
1.20	1.898+000	3.914+000	8.491+000	1.894+001	4.285+001	9.769+001	2.236+002	5.127+002	1.177+003	∞
1.00	1.681+000	3.043+000	5.766+000	1.121+001	2.211+001	4.389+001	8.747+001	1.746+002	3.489+002	∞
0.80	1.504+000	2.404+000	3.997+000	6.794+000	1.169+001	2.023+001	3.512+001	6.106+001	1.062+002	∞
0.60	1.360+000	1.939+000	2.848+000	4.254+000	6.409+000	9.696+000	1.469+001	2.228+001	3.379+001	∞
0.40	1.247+000	1.606+000	2.108+000	2.794+000	3.717+000	4.950+000	6.587+000	8.754+000	1.162+001	∞
0.20	1.160+000	1.372+000	1.637+000	1.958+000	2.340+000	2.787+000	3.307+000	3.907+000	4.599+000	∞
−0.00	1.097+000	1.214+000	1.346+000	1.489+000	1.639+000	1.794+000	1.952+000	2.112+000	2.273+000	∞
−0.20	1.054+000	1.113+000	1.174+000	1.233+000	1.289+000	1.339+000	1.385+000	1.425+000	1.461+000	1.705+000
−0.40	1.026+000	1.053+000	1.078+000	1.101+000	1.119+000	1.135+000	1.147+000	1.157+000	1.164+000	1.188+000
−0.60	1.011+000	1.021+000	1.030+000	1.037+000	1.043+000	1.047+000	1.050+000	1.052+000	1.053+000	1.056+000
−0.80	1.003+000	1.006+000	1.008+000	1.010+000	1.011+000	1.012+000	1.012+000	1.013+000	1.013+000	1.013+000
−1.00	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000
−1.20	9.991−001	9.984−001	9.979−001	9.976−001	9.974−001	9.973−001	9.972−001	9.972−001	9.972−001	9.971−001
−1.40	9.992−001	9.985−001	9.981−001	9.979−001	9.977−001	9.976−001	9.976−001	9.976−001	9.975−001	9.975−001
−1.60	9.995−001	9.991−001	9.989−001	9.987−001	9.987−001	9.986−001	9.986−001	9.986−001	9.986−001	9.986−001
−1.80	9.998−001	9.997−001	9.996−001	9.995−001	9.995−001	9.995−001	9.995−001	9.995−001	9.995−001	9.995−001
−2.00	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000

$B_1(N, r, \mu)$ for $r = 32.00$

μ	N = 4	8	16	32	64	128	256	512	1024	∞
2.00	3.333+000	1.200+001	4.533+001	1.760+002	6.933+002	2.752+003	1.097+004	4.378+004	1.749+005	∞
1.80	2.865+000	8.945+000	2.938+001	9.928+001	3.405+002	1.176+003	4.081+003	1.418+004	4.934+004	∞
1.60	2.478+000	6.719+000	1.920+001	5.646+001	1.685+002	5.070+002	1.531+003	4.632+003	1.403+004	∞
1.40	2.158+000	5.091+000	1.266+001	3.242+001	8.425+001	2.206+002	5.799+002	1.527+003	4.027+003	∞
1.20	1.893+000	3.898+000	8.448+000	1.883+001	4.261+001	9.714+001	2.223+002	5.097+002	1.170+003	∞
1.00	1.674+000	3.021+000	5.716+000	1.111+001	2.188+001	4.344+001	8.656+001	1.728+002	3.453+002	∞
0.80	1.494+000	2.378+000	3.940+000	6.685+000	1.149+001	1.986+001	3.447+001	5.992+001	1.042+002	∞
0.60	1.348+000	1.908+000	2.787+000	4.147+000	6.231+000	9.408+000	1.424+001	2.158+001	3.270+001	∞
0.40	1.233+000	1.572+000	2.046+000	2.693+000	3.565+000	4.729+000	6.275+000	8.320+000	1.102+001	∞
0.20	1.146+000	1.338+000	1.579+000	1.871+000	2.218+000	2.625+000	3.097+000	3.643+000	4.272+000	∞
−0.00	1.083+000	1.184+000	1.297+000	1.420+000	1.550+000	1.683+000	1.819+000	1.957+000	2.095+000	∞
−0.20	1.043+000	1.090+000	1.139+000	1.186+000	1.230+000	1.271+000	1.307+000	1.339+000	1.368+000	1.563+000
−0.40	1.019+000	1.039+000	1.057+000	1.073+000	1.087+000	1.098+000	1.107+000	1.114+000	1.119+000	1.136+000
−0.60	1.007+000	1.014+000	1.019+000	1.024+000	1.028+000	1.030+000	1.032+000	1.033+000	1.034+000	1.036+000
−0.80	1.002+000	1.003+000	1.005+000	1.006+000	1.006+000	1.007+000	1.007+000	1.007+000	1.007+000	1.008+000
−1.00	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000
−1.20	9.996−001	9.993−001	9.991−001	9.990−001	9.989−001	9.988−001	9.988−001	9.988−001	9.988−001	9.988−001
−1.40	9.997−001	9.994−001	9.993−001	9.992−001	9.991−001	9.991−001	9.991−001	9.991−001	9.991−001	9.991−001
−1.60	9.998−001	9.997−001	9.996−001	9.996−001	9.996−001	9.995−001	9.995−001	9.995−001	9.995−001	9.995−001
−1.80	9.999−001	9.999−001	9.999−001	9.999−001	9.999−001	9.998−001	9.998−001	9.998−001	9.998−001	9.998−001
−2.00	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000

$B_1(N, r, \mu)$ for $r = 64.00$

	N									
μ	4	8	16	32	64	128	256	512	1024	∞
2.00	3.333+000	1.200+001	4.533+001	1.760+002	6.933+002	2.752+003	1.097+004	4.378+004	1.749+005	∞
1.80	2.865+000	8.943+000	2.938+001	9.927+001	3.404+002	1.176+003	4.080+003	1.418+004	4.933+004	∞
1.60	2.477+000	6.716+000	1.919+001	5.644+001	1.685+002	5.067+002	1.530+003	4.630+003	1.402+004	∞
1.40	2.156+000	5.087+000	1.265+001	3.238+001	8.415+001	2.204+002	5.793+002	1.526+003	4.022+003	∞
1.20	1.890+000	3.891+000	8.429+000	1.879+001	4.251+001	9.690+001	2.218+002	5.085+002	1.167+003	∞
1.00	1.670+000	3.010+000	5.691+000	1.105+001	2.177+001	4.322+001	8.611+001	1.719+002	3.435+002	∞
0.80	1.489+000	2.363+000	3.909+000	6.624+000	1.137+001	1.966+001	3.411+001	5.929+001	1.031+002	∞
0.60	1.341+000	1.889+000	2.749+000	4.080+000	6.119+000	9.229+000	1.396+001	2.114+001	3.203+001	∞
0.40	1.224+000	1.548+000	2.003+000	2.624+000	3.461+000	4.578+000	6.060+000	8.023+000	1.062+001	∞
0.20	1.135+000	1.313+000	1.536+000	1.808+000	2.129+000	2.506+000	2.944+000	3.450+000	4.033+000	∞
−0.00	1.073+000	1.161+000	1.261+000	1.369+000	1.482+000	1.600+000	1.719+000	1.840+000	1.961+000	∞
−0.20	1.035+000	1.073+000	1.112+000	1.151+000	1.187+000	1.220+000	1.249+000	1.275+000	1.298+000	1.456+000
−0.40	1.014+000	1.028+000	1.042+000	1.054+000	1.064+000	1.072+000	1.078+000	1.083+000	1.087+000	1.100+000
−0.60	1.005+000	1.009+000	1.013+000	1.016+000	1.018+000	1.020+000	1.021+000	1.022+000	1.022+000	1.024+000
−0.80	1.001+000	1.002+000	1.003+000	1.003+000	1.004+000	1.004+000	1.004+000	1.004+000	1.004+000	1.004+000
−1.00	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000
−1.20	9.998−001	9.997−001	9.996−001	9.995−001	9.995−001	9.995−001	9.995−001	9.995−001	9.995−001	9.995−001
−1.40	9.999−001	9.998−001	9.997−001	9.997−001	9.997−001	9.997−001	9.997−001	9.996−001	9.996−001	9.996−001
−1.60	9.999−001	9.999−001	9.999−001	9.999−001	9.999−001	9.999−001	9.998−001	9.998−001	9.998−001	9.998−001
−1.80	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000
−2.00	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000

$B_1(N, r, \mu)$ for $r = 128.00$

	N									
μ	4	8	16	32	64	128	256	512	1024	∞
2.00	3.333+000	1.200+001	4.533+001	1.760+002	6.933+002	2.752+003	1.097+004	4.378+004	1.749+005	∞
1.80	2.865+000	8.943+000	2.938+001	9.926+001	3.404+002	1.176+003	4.080+003	1.418+004	4.932+004	∞
1.60	2.477+000	6.715+000	1.919+001	5.643+001	1.684+002	5.067+002	1.530+003	4.629+003	1.402+004	∞
1.40	2.156+000	5.085+000	1.265+001	3.237+001	8.411+001	2.203+002	5.790+002	1.525+003	4.021+003	∞
1.20	1.889+000	3.887+000	8.421+000	1.877+001	4.246+001	9.680+001	2.215+002	5.079+002	1.166+003	∞
1.00	1.668+000	3.005+000	5.679+000	1.103+001	2.172+001	4.311+001	8.589+001	1.714+002	3.426+002	∞
0.80	1.486+000	2.354+000	3.891+000	6.589+000	1.131+001	1.955+001	3.391+001	5.893+001	1.025+002	∞
0.60	1.336+000	1.877+000	2.725+000	4.037+000	6.048+000	9.115+000	1.378+001	2.086+001	3.160+001	∞
0.40	1.217+000	1.532+000	1.973+000	2.576+000	3.388+000	4.471+000	5.909+000	7.813+000	1.033+001	∞
0.20	1.127+000	1.294+000	1.504+000	1.759+000	2.062+000	2.416+000	2.827+000	3.303+000	3.851+000	∞
−0.00	1.065+000	1.144+000	1.232+000	1.329+000	1.430+000	1.534+000	1.640+000	1.748+000	1.856+000	∞
−0.20	1.029+000	1.060+000	1.092+000	1.124+000	1.153+000	1.181+000	1.205+000	1.226+000	1.245+000	1.375+000
−0.40	1.010+000	1.021+000	1.031+000	1.040+000	1.047+000	1.053+000	1.058+000	1.062+000	1.065+000	1.074+000
−0.60	1.003+000	1.006+000	1.008+000	1.010+000	1.012+000	1.013+000	1.014+000	1.014+000	1.015+000	1.015+000
−0.80	1.001+000	1.001+000	1.002+000	1.002+000	1.002+000	1.002+000	1.002+000	1.002+000	1.002+000	1.002+000
−1.00	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000
−1.20	9.999−001	9.999−001	9.998−001	9.998−001	9.998−001	9.998−001	9.998−001	9.998−001	9.998−001	9.998−001
−1.40	1.000+000	9.999−001	9.999−001	9.999−001	9.999−001	9.999−001	9.999−001	9.999−001	9.999−001	9.999−001
−1.60	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000	9.999−001	9.999−001	9.999−001	9.999−001
−1.80	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000
−2.00	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000

$B_1(N, r, \mu)$ for $r = 256.00$

μ	N: 4	8	16	32	64	128	256	512	1024	∞
2.00	3.333+000	1.200+001	4.533+001	1.760+002	6.933+002	2.752+003	1.097+004	4.378+004	1.749+005	∞
1.80	2.865+000	8.943+000	2.938+001	9.926+001	3.404+002	1.176+003	4.080+003	1.418+004	4.932+004	∞
1.60	2.477+000	6.715+000	1.919+001	5.642+001	1.684+002	5.066+002	1.530+003	4.629+003	1.402+004	∞
1.40	2.156+000	5.084+000	1.264+001	3.236+001	8.410+001	2.202+002	5.789+002	1.525+003	4.020+003	∞
1.20	1.889+000	3.886+000	8.418+000	1.876+001	4.244+001	9.675+001	2.214+002	5.077+002	1.165+003	∞
1.00	1.668+000	3.003+000	5.673+000	1.101+001	2.169+001	4.305+001	8.578+001	1.712+002	3.421+002	∞
0.80	1.484+000	2.350+000	3.881+000	6.570+000	1.127+001	1.948+001	3.380+001	5.873+001	1.022+002	∞
0.60	1.333+000	1.869+000	2.709+000	4.010+000	6.003+000	9.042+000	1.366+001	2.068+001	3.132+001	∞
0.40	1.212+000	1.520+000	1.952+000	2.541+000	3.335+000	4.394+000	5.800+000	7.662+000	1.012+001	∞
0.20	1.121+000	1.280+000	1.479+000	1.722+000	2.009+000	2.346+000	2.737+000	3.189+000	3.710+000	∞
−0.00	1.059+000	1.130+000	1.210+000	1.296+000	1.388+000	1.482+000	1.577+000	1.674+000	1.772+000	∞
−0.20	1.024+000	1.050+000	1.077+000	1.103+000	1.127+000	1.150+000	1.170+000	1.188+000	1.204+000	1.311+000
−0.40	1.008+000	1.016+000	1.023+000	1.029+000	1.035+000	1.039+000	1.043+000	1.046+000	1.048+000	1.055+000
−0.60	1.002+000	1.004+000	1.005+000	1.007+000	1.008+000	1.008+000	1.009+000	1.009+000	1.010+000	1.010+000
−0.80	1.000+000	1.001+000	1.001+000	1.001+000	1.001+000	1.001+000	1.001+000	1.001+000	1.001+000	1.001+000
−1.00	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000
−1.20	1.000+000	9.999−001	9.999−001	9.999−001	9.999−001	9.999−001	9.999−001	9.999−001	9.999−001	9.999−001
−1.40	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000	9.999−001	9.999−001	9.999−001
−1.60	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000
−1.80	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000
−2.00	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000

$B_1(N, r, \mu)$ for $r = 512.00$

μ	N: 4	8	16	32	64	128	256	512	1024	∞
2.00	3.333+000	1.200+001	4.533+001	1.760+002	6.933+002	2.752+003	1.097+004	4.378+004	1.749+005	∞
1.80	2.865+000	8.943+000	2.938+001	9.926+001	3.404+002	1.176+003	4.080+003	1.418+004	4.932+004	∞
1.60	2.477+000	6.714+000	1.919+001	5.642+001	1.684+002	5.066+002	1.530+003	4.628+003	1.402+004	∞
1.40	2.156+000	5.084+000	1.264+001	3.236+001	8.410+001	2.202+002	5.789+002	1.525+003	4.020+003	∞
1.20	1.889+000	3.885+000	8.416+000	1.876+001	4.243+001	9.673+001	2.214+002	5.076+002	1.165+003	∞
1.00	1.667+000	3.001+000	5.670+000	1.101+001	2.168+001	4.303+001	8.572+001	1.711+002	3.419+002	∞
0.80	1.483+000	2.347+000	3.875+000	6.558+000	1.125+001	1.945+001	3.373+001	5.862+001	1.020+002	∞
0.60	1.331+000	1.864+000	2.699+000	3.992+000	5.973+000	8.994+000	1.359+001	2.056+001	3.114+001	∞
0.40	1.209+000	1.512+000	1.936+000	2.516+000	3.296+000	4.338+000	5.721+000	7.553+000	9.973+000	∞
0.20	1.116+000	1.268+000	1.460+000	1.692+000	1.967+000	2.290+000	2.665+000	3.098+000	3.598+000	∞
−0.00	1.054+000	1.118+000	1.191+000	1.270+000	1.353+000	1.438+000	1.526+000	1.614+000	1.703+000	∞
−0.20	1.020+000	1.042+000	1.064+000	1.086+000	1.107+000	1.125+000	1.142+000	1.157+000	1.170+000	1.261+000
−0.40	1.006+000	1.012+000	1.017+000	1.022+000	1.026+000	1.030+000	1.032+000	1.034+000	1.036+000	1.041+000
−0.60	1.001+000	1.003+000	1.004+000	1.004+000	1.005+000	1.006+000	1.006+000	1.006+000	1.006+000	1.007+000
−0.80	1.000+000	1.000+000	1.001+000	1.001+000	1.001+000	1.001+000	1.001+000	1.001+000	1.001+000	1.001+000
−1.00	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000
−1.20	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000
−1.40	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000
−1.60	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000
−1.80	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000
−2.00	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000

$B_1(N, r, \mu)$ for $r = 1024.00$

	N								
μ	4	8	16	32	64	128	256	512	1024	∞
2.00	3.333+000	1.200+001	4.533+001	1.760+002	6.933+002	2.752+003	1.097+004	4.378+004	1.749+005	∞
1.80	2.865+000	8.943+000	2.938+001	9.926+001	3.404+002	1.176+003	4.080+003	1.418+004	4.932+004	∞
1.60	2.477+000	6.714+000	1.919+001	5.642+001	1.684+002	5.066+002	1.530+003	4.628+003	1.402+004	∞
1.40	2.156+000	5.084+000	1.264+001	3.236+001	8.409+001	2.202+002	5.789+002	1.525+003	4.020+003	∞
1.20	1.889+000	3.885+000	8.416+000	1.876+001	4.243+001	9.672+001	2.213+002	5.075+002	1.165+003	∞
1.00	1.667+000	3.001+000	5.668+000	1.100+001	2.167+001	4.301+001	8.569+001	1.711+002	3.418+002	∞
0.80	1.482+000	2.345+000	3.872+000	6.552+000	1.124+001	1.942+001	3.369+001	5.855+001	1.018+002	∞
0.60	1.330+000	1.860+000	2.693+000	3.980+000	5.953+000	8.963+000	1.354+001	2.049+001	3.102+001	∞
0.40	1.206+000	1.505+000	1.924+000	2.497+000	3.268+000	4.297+000	5.663+000	7.472+000	9.862+000	∞
0.20	1.112+000	1.259+000	1.444+000	1.668+000	1.934+000	2.245+000	2.607+000	3.025+000	3.507+000	∞
−0.00	1.049+000	1.108+000	1.175+000	1.248+000	1.324+000	1.402+000	1.483+000	1.564+000	1.645+000	∞
−0.20	1.017+000	1.035+000	1.054+000	1.073+000	1.090+000	1.106+000	1.120+000	1.132+000	1.144+000	1.220+000
−0.40	1.004+000	1.009+000	1.013+000	1.017+000	1.020+000	1.022+000	1.024+000	1.026+000	1.027+000	1.031+000
−0.60	1.001+000	1.002+000	1.002+000	1.003+000	1.003+000	1.004+000	1.004+000	1.004+000	1.004+000	1.004+000
−0.80	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000
−1.00	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000
−1.20	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000
−1.40	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000
−1.60	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000
−1.80	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000
−2.00	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000

$B_1(N, r, \mu)$ for $r = 2048.00$

	N									
μ	4	8	16	32	64	128	256	512	1024	∞
2.00	3.333+000	1.200+001	4.533+001	1.760+002	6.933+002	2.752+003	1.097+004	4.378+004	1.749+005	∞
1.80	2.865+000	8.943+000	2.938+001	9.926+001	3.404+002	1.176+003	4.080+003	1.418+004	4.932+004	∞
1.60	2.477+000	6.714+000	1.919+001	5.642+001	1.684+002	5.066+002	1.530+003	4.628+003	1.402+004	∞
1.40	2.156+000	5.084+000	1.264+001	3.236+001	8.409+001	2.202+002	5.789+002	1.525+003	4.020+003	∞
1.20	1.889+000	3.885+000	8.415+000	1.875+001	4.243+001	9.672+001	2.213+002	5.075+002	1.165+003	∞
1.00	1.667+000	3.000+000	5.667+000	1.100+001	2.167+001	4.301+001	8.568+001	1.710+002	3.417+002	∞
0.80	1.482+000	2.345+000	3.870+000	6.548+000	1.123+001	1.941+001	3.367+001	5.851+001	1.018+002	∞
0.60	1.329+000	1.858+000	2.688+000	3.972+000	5.941+000	8.942+000	1.351+001	2.044+001	3.095+001	∞
0.40	1.204+000	1.501+000	1.916+000	2.483+000	3.247+000	4.266+000	5.620+000	7.412+000	9.780+000	∞
0.20	1.108+000	1.251+000	1.431+000	1.648+000	1.906+000	2.209+000	2.560+000	2.966+000	3.434+000	∞
−0.00	1.045+000	1.100+000	1.162+000	1.229+000	1.299+000	1.372+000	1.446+000	1.521+000	1.596+000	∞
−0.20	1.014+000	1.030+000	1.046+000	1.061+000	1.076+000	1.089+000	1.101+000	1.112+000	1.122+000	1.186+000
−0.40	1.003+000	1.007+000	1.010+000	1.012+000	1.015+000	1.017+000	1.018+000	1.019+000	1.020+000	1.023+000
−0.60	1.001+000	1.001+000	1.002+000	1.002+000	1.002+000	1.002+000	1.003+000	1.003+000	1.003+000	1.003+000
−0.80	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000
−1.00	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000
−1.20	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000
−1.40	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000
−1.60	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000
−1.80	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000
−2.00	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000

$B_1(N, r, \mu)$ for $r=\infty$

μ	N: 4	8	16	32	64	128	256	512	1024	∞
2.00	3.333+000	1.200+001	4.533+001	1.760+002	6.933+002	2.752+003	1.097+004	4.378+004	1.749+005	∞
1.80	2.865+000	8.943+000	2.938+001	9.926+001	3.404+002	1.176+003	4.080+003	1.418+004	4.932+004	∞
1.60	2.477+000	6.714+000	1.919+001	5.642+001	1.684+002	5.066+002	1.530+003	4.628+003	1.402+004	∞
1.40	2.156+000	5.084+000	1.264+001	3.236+001	8.409+001	2.202+002	5.789+002	1.525+003	4.020+003	∞
1.20	1.889+000	3.885+000	8.415+000	1.875+001	4.243+001	9.672+001	2.213+002	5.075+002	1.165+003	∞
1.00	1.667+000	3.000+000	5.667+000	1.100+001	2.167+001	4.300+001	8.567+001	1.710+002	3.417+002	∞
0.80	1.482+000	2.343+000	3.867+000	6.543+000	1.123+001	1.940+001	3.364+001	5.846+001	1.017+002	∞
0.60	1.327+000	1.854+000	2.680+000	3.958+000	5.916+000	8.903+000	1.344+001	2.034+001	3.080+001	∞
0.40	1.198+000	1.487+000	1.890+000	2.441+000	3.184+000	4.174+000	5.489+000	7.231+000	9.533+000	∞
0.20	1.091+000	1.210+000	1.360+000	1.541+000	1.757+000	2.009+000	2.303+000	2.642+000	3.033+000	∞
−0.00	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000
−0.20	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000
−0.40	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000
−0.60	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000
−0.80	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000
−1.00	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000
−1.20	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000
−1.40	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000
−1.60	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000
−1.80	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000
−2.00	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000

$B_2(r, \mu)$

μ	r: 0.001	0.003	0.010	0.030	0.100	0.200	0.400	0.800
2.00	1.000−006	9.000−006	1.000−004	9.000−004	1.000−002	4.000−002	1.600−001	6.400−001
1.80	1.072−006	9.645−006	1.072−004	9.642−004	1.070−002	4.263−002	1.686−001	6.525−001
1.60	1.152−006	1.037−005	1.152−004	1.036−003	1.147−002	4.547−002	1.776−001	6.652−001
1.40	1.245−006	1.120−005	1.244−004	1.118−003	1.233−002	4.860−002	1.871−001	6.781−001
1.20	1.356−006	1.221−005	1.355−004	1.216−003	1.333−002	5.207−002	1.972−001	6.910−001
1.00	1.500−006	1.349−005	1.495−004	1.337−003	1.450−002	5.600−002	2.080−001	7.040−001
0.80	1.698−006	1.524−005	1.683−004	1.493−003	1.593−002	6.052−002	2.196−001	7.170−001
0.60	2.001−006	1.788−005	1.955−004	1.708−003	1.773−002	6.583−002	2.321−001	7.299−001
0.40	2.530−006	2.228−005	2.381−004	2.020−003	2.006−002	7.219−002	2.457−001	7.426−001
0.20	3.594−006	3.048−005	3.100−004	2.494−003	2.316−002	7.996−002	2.607−001	7.549−001
−0.00	6.065−006	4.745−005	4.404−004	3.250−003	2.742−002	8.962−002	2.773−001	7.667−001
−0.20	1.260−005	8.606−005	6.921−004	4.507−003	3.340−002	1.018−001	2.959−001	7.775−001
−0.40	3.174−005	1.809−004	1.204−003	6.664−003	4.195−002	1.175−001	3.168−001	7.869−001
−0.60	9.231−005	4.280−004	2.288−003	1.047−002	5.438−002	1.379−001	3.407−001	7.944−001
−0.80	2.949−004	1.101−003	4.662−003	1.735−002	7.271−002	1.646−001	3.682−001	7.992−001
−1.00	1.000−003	3.000−003	1.000−002	3.000−002	1.000−001	2.000−001	4.000−001	8.000−001
−1.20	3.525−003	8.489−003	2.225−002	5.362−002	1.410−001	2.472−001	4.371−001	7.954−001
−1.40	1.276−002	2.467−002	5.081−002	9.828−002	2.032−001	3.104−001	4.808−001	7.832−001
−1.60	4.708−002	7.306−002	1.183−001	1.836−001	2.979−001	3.956−001	5.324−001	7.606−001
−1.80	1.762−001	2.195−001	2.793−001	3.479−001	4.431−001	5.107−001	5.936−001	7.236−001
−2.00	6.667−001	6.667−001	6.667−001	6.667−001	6.667−001	6.667−001	6.667−001	6.667−001

$B_2(r, \mu)$

μ	r = 1.00	1.01	1.10	2.00	4.00	8.00	16.00	32.00
2.00	1.000+000	1.020+000	1.210+000	4.000+000	1.600+001	6.400+001	2.560+002	1.024+003
1.80	1.000+000	1.019+000	1.198+000	3.642+000	1.289+001	4.513+001	1.574+002	5.485+002
1.60	1.000+000	1.018+000	1.186+000	3.316+000	1.039+001	3.188+001	9.706+001	2.947+002
1.40	1.000+000	1.017+000	1.174+000	3.018+000	8.389+000	2.259+001	6.008+001	1.590+002
1.20	1.000+000	1.016+000	1.162+000	2.747+000	6.784+000	1.607+001	3.741+001	8.644+001
1.00	1.000+000	1.015+000	1.150+000	2.500+000	5.500+000	1.150+001	2.350+001	4.750+001
0.80	1.000+000	1.014+000	1.138+000	2.275+000	4.475+000	8.297+000	1.495+001	2.653+001
0.60	1.000+000	1.013+000	1.126+000	2.071+000	3.658+000	6.051+000	9.673+000	1.516+001
0.40	1.000+000	1.012+000	1.114+000	1.886+000	3.007+000	4.473+000	6.404+000	8.951+000
0.20	1.000+000	1.011+000	1.102+000	1.718+000	2.489+000	3.364+000	4.365+000	5.514+000
−0.00	1.000+000	1.010+000	1.089+000	1.566+000	2.078+000	2.581+000	3.082+000	3.582+000
−0.20	1.000+000	1.009+000	1.075+000	1.429+000	1.752+000	2.027+000	2.265+000	2.472+000
−0.40	1.000+000	1.007+000	1.060+000	1.304+000	1.494+000	1.633+000	1.738+000	1.817+000
−0.60	1.000+000	1.006+000	1.043+000	1.192+000	1.290+000	1.351+000	1.392+000	1.418+000
−0.80	1.000+000	1.004+000	1.024+000	1.091+000	1.128+000	1.148+000	1.159+000	1.166+000
−1.00	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000
−1.20	1.000+000	9.929−001	9.693−001	9.181−001	8.990−001	8.912−001	8.879−001	8.865−001
−1.40	1.000+000	9.776−001	9.281−001	8.446−001	8.192−001	8.103−001	8.071−001	8.058−001
−1.60	1.000+000	9.429−001	8.708−001	7.787−001	7.561−001	7.494−001	7.472−001	7.465−001
−1.80	1.000+000	8.614−001	7.883−001	7.196−001	7.062−001	7.028−001	7.018−001	7.015−001
−2.00	1.000+000	6.667−001	6.667−001	6.667−001	6.667−001	6.667−001	6.667−001	6.667−001

$B_2(r, \mu)$

μ	r = 64.00	128.00	256.00	512.00	1024.00	2048.00	∞
2.00	4.096+003	1.638+004	6.554+004	2.621+005	1.049+006	4.194+006	∞
1.80	1.910+003	6.653+003	2.317+004	8.067+004	2.809+005	9.783+005	∞
1.60	8.937+002	2.710+003	8.215+003	2.490+004	7.550+004	2.288+005	∞
1.40	4.201+002	1.109+003	2.928+003	7.727+003	2.039+004	5.381+004	∞
1.20	1.991+002	4.579+002	1.052+003	2.418+003	5.555+003	1.277+004	∞
1.00	9.550+001	1.915+002	3.835+002	7.675+002	1.536+003	3.071+003	∞
0.80	4.669+001	8.179+001	1.429+002	2.493+002	4.346+002	7.570+002	∞
0.60	2.348+001	3.609+001	5.521+001	8.418+001	1.281+002	1.946+002	∞
0.40	1.231+001	1.674+001	2.260+001	3.031+001	4.050+001	5.395+001	∞
0.20	6.834+000	8.351+000	1.009+001	1.209+001	1.439+001	1.704+001	∞
−0.00	4.082+000	4.582+000	5.082+000	5.582+000	6.082+000	6.582+000	∞
−0.20	2.652+000	2.809+000	2.945+000	3.064+000	3.167+000	3.258+000	3.863+000
−0.40	1.877+000	1.923+000	1.957+000	1.983+000	2.003+000	2.018+000	2.065+000
−0.60	1.436+000	1.447+000	1.455+000	1.460+000	1.463+000	1.465+000	1.470+000
−0.80	1.170+000	1.172+000	1.173+000	1.174+000	1.174+000	1.174+000	1.175+000
−1.00	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000	1.000+000
−1.20	8.859−001	8.856−001	8.855−001	8.854−001	8.854−001	8.854−001	8.854−001
−1.40	8.053−001	8.052−001	8.051−001	8.851−001	8.051−001	8.051−001	8.051−001
−1.60	7.462−001	7.462−001	7.461−001	7.461−001	7.461−001	7.461−001	7.461−001
−1.80	7.015−001	7.014−001	7.014−001	7.014−001	7.014−001	7.014−001	7.014−001
−2.00	6.667−001	6.667−001	6.667−001	6.667−001	6.667−001	6.667−001	6.667−001