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AN EARTH-BASED COORDINATE CLOCK NETWORK

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AN EARTH-BASED COORDINATE CLOCK NETWORK

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This paper investigates some of the possible operational procedures for synchronizing clocks at fixed sites spread around on the earth's surface, to within a 1 nanosecond level of accuracy. Since a common synchronization procedure is by transport of standard clocks in commercial jet airline flights, and most of the effects we shall discuss are fractional corrections to the elapsed time, as a criterion of whether an effect is significant at the 1 nanosecond level we take for comparison purposes an elapsed time T_c = 10 hours. This is a typical time for an intercontinental airplane flight. Analysis of a number of effects which might affect clock synchronization is carried out within the framework of general relativity. These effects include the gravitational fields of the earth, sun, and moon, and orbital motion, rotation, and flattening of the earth. It is shown that the only significant effects are due to the gravitational field and rotation of the earth, and motion of the transported clocks. Operational procedures for construction of a synchronized coordinate clock network using light signals and transported standard clocks are discussed and compared.

Key words: Clocks; coordinate time; general relativity; time

Introduction

The purpose of this paper is to investigate some of the possible operational procedures for synchronizing clocks at various fixed sites spread around on the earth's surface. There are a number of effects which could conceivably play a role in the synchronization of clocks--among these are the gravitational fields of the earth, moon, and sun, and the orbital motions, rotation, and flattening of the earth. Since most of the effects we shall discuss turn out to be fractional corrections to elapsed time (that is, the corrections to the time are proportional to the time), as a criterion of whether an effect is significant at the level of 1 nanosecond (1 ns), we shall take for comparison purposes an elapsed time T_c = 10 hours, which corresponds to a typical time for an intercontinental airplane flight. This "comparison flight" will, where appropriate, be assumed to occur at an altitude of h_c = 12,000 meters and at a speed of 450 meter/sec. The reason for selection of these parameters is that a fairly common synchronization procedure will be by transport of standard clocks in commercial jet airline flights [1].

The analysis will be carried out within the framework of the general theory of relativity; we shall not discuss any modification of procedure which might be necessary in some other gravitational theory such as that due to Brans and Dicke [2].

The basic assumptions and results of the general theory of relativity, which we shall repeatedly use, are as follows (we use the notation and sign conventions of Weber [3]):

1) There exists a metric tensor $g_{\mu\nu}$ (μ,ν = 1, 2, 3) such that the space-time interval ds defined by

$$-ds^2 = g_{\mu\nu}dx^\mu dx^\nu \tag{1}$$

is invariant with respect to arbitrary coordinate transformations.

2) Propagation of light rays along the path element dx^μ is described by the vanishing of ds:

$$0 = g_{\mu\nu}dx^\mu dx^\nu \; . \tag{2}$$

3) The proper time elapsed on a standard clock transported along the space-time path element dx^μ is

$$\frac{1}{c}\,ds = \frac{1}{c}\sqrt{-\,g_{\mu\nu}dx^\mu dx^\nu}. \tag{3}$$

4) Of particular significance to this investigation is the interpretation of the coordinate x^0, which is a global coordinate time having the units of length. It is assumed that at each point of space it is possible to equip an observer with a clock (a coordinate clock) which can be used to measure the coordinate time of any event occurring at that point. The importance of coordinate time may be appreciated by considering, for example, the phenomenon of the redshift of a clock in a static gravitational field. Standard clocks (e.g., atomic clocks) will have different rates depending on the gravitational potential at the position of the clock. In contrast, a coordinate clock will always have the same rate independent of position in the gravitational field.

5) The metric coefficients $g_{\mu\nu}$ may be calculated from the field equations of general relativity. Furthermore, because in the vicinity of the earth's surface all gravitational fields are weak, it will be sufficient to work in the linearized approximation to general relativity in which

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \tag{4}$$

with $\eta_{\mu\nu}$ = (-1, 1, 1, 1) the metric tensor of flat space and

$$h_{\mu\nu} << 1$$

so that squares and products of $h_{\mu\nu}$ are negligible.

2. Summary of Results

We have investigated two principal schemes for obtaining a worldwide network of synchronized coordinate clocks. These are: A) a coordinate clock network measuring the global coordinate time x^0, and B) a coordinate clock network with a discontinuity (the "Hafele-Keating" discontinuity) along some line between the poles.

A) A coordinate clook system measuring global coordinate time x^0

In this scheme, it is assumed that the metric tensor is known theoretically in terms of the earth's mass and rotation. One may then, with some selected point on the geoid (the geoid is the equipotential surface at mean sea level) as reference, transport standard clocks from point to point on the earth's surface. The elapsed global coordinate time Δx^0 during the process may be calculated from the equation

$$\Delta x^0 = \int_{\text{path}} ds \left[1 - \frac{gh}{c^2} + \frac{v^2}{2c^2} + \frac{\omega a_1 v_E \cos\phi}{c^2} \right] \tag{5}$$

where ds is the elapsed proper time as measured on the transported clock; g is the acceleration of gravity; v is the ground velocity of the clock having an eastward component v_E; h is the altitude above the geoid; ω is the rotational angular velocity of the earth; a_1 is the earth's equatorial radius, and ϕ is the geographical latitude. The standard clock readings must thus be corrected for redshift, ground speed, and earth rotation before using them to measure coordinate time.

Note that for a clock at rest on the earth, Equation (5) reduces to

$$\Delta x^0 = \Delta s \left[1 - \frac{gh}{c^2} \right] \tag{6}$$

so that a standard clock at rest on the earth's surface must have its rate corrected for redshift before being used to measure elapsed coordinate time x^0 at that point.

Light signals may be used to perform an equivalent coordinate clock synchronization by using the following formula for the coordinate time elapsed during propagation which involves a correction due to rotation of the earth:

$$\Delta x^0 = \int_{\text{path}} d\sigma \left[1 + \frac{\omega a_1 c_E \cos\phi}{c^2} \right]. \tag{7}$$

In the above expression, $d\sigma$ is the element of proper distance along the path of the light ray, assumed to lie within about 12,000 meters of the geoid, and c_E is the eastward component of the light velocity. The coordinate clock network obtained by this scheme has the following features and disadvantages:

1) Use of transported clocks and light signals as corrected by Equations (5) and (7) will give a mutually consistent network of coordinate clocks.

2) The synchronization procedure gives path-independent results with respect to either transport of clocks or use of light signals.

3) The procedure has the property of transitivity: if clocks A and B are synchronized, then A and C will be synchronized.

4) The coordinate clocks directly measure the global coordinate time x^0.

5) There is no discontinuity in synchronization.

6) A disadvantage is that corrections must be applied to standard clock readings due to redshift (whether the clock is at rest or is transported), ground velocity, and earth rotation.

Equation (5) may also be applied to standard clocks while in motion--for example, while being transported in aircraft.

B) <u>A coordinate clock system with a discontinuity</u>

This scheme has been mentioned by implication in a paper by Schlegel [4]. Here one transports clocks or uses light signals without applying corrections for the earth's rotation. Thus Equation (5) for standard clock transport becomes

$$\Delta x_D^0 = \int_{\text{path}} ds \left[1 - \frac{gh}{c^2} + \frac{v^2}{2c^2}\right] . \tag{8}$$

And for light propagation, Equation (7) becomes

$$\Delta x_D^0 = \int_{\text{path}} d\sigma . \tag{9}$$

This scheme has the following features and advantages:

1) There must be, due to the earth's rotation, a discontinuity in synchronization along some line between the poles given by [5]

$$2\pi\omega a_1^2 \cos^2 \phi/c^2 = 207.4 \cos^2 \phi \text{ ns}. \tag{10}$$

The existence of this discontinuity somewhat cancels the advantage of not having to correct in Equations (8) and (9) for the earth's rotation.

2) The use of clocks or light signals will give a mutually consistent network of synchronized coordinate clocks as long as the discontinuity is not crossed and transport is along a parallel of latitude.

3) The synchronization procedure is in general path-dependent.

4) The procedure is in general non-transitive.

5) The coordinate clocks do not measure the global coordinate time x^0 of relativity theory.

6) Calculated corrections must be applied, for ground velocity and redshift, to the readings of transported clocks.

C) <u>Investigation of other effects</u>

Quite a number of effects which influence these synchronization procedures were investigated and found to be insignificant at the 1 ns level. These effects will be discussed in detail later, but it is of interest to mention some of them here. They include variations of the gravitational field strength with latitude and altitude, gravitational anomalies, flattening of the earth due to rotation, mass of the atmosphere, gravitational fields due to the sun and moon and orbital motion of the earth. These latter effects are individually of order 10 ns during a typical flight but cancel very precisely.

3. Synchronization in a Static Gravitational Field

To provide a foundation for later discussion, in this section we shall describe the operational procedure by which one could set up a series of coordinate clocks, measuring the global coordinate time x^0, at rest in space. This procedure is worth some discussion because it is more usual in theoretical work to assume the coordinate clocks exist and then to imagine using them to observe and to compare what happens to standard clocks and other devices.

The real situation is different. We actually have available not coordinate clocks but standard clocks whose rate depends on both position and velocity. Furthermore, we know to high precision the positions and motion of the earth, sun, and moon which are the only significant sources of gravitational fields in the neighborhood of the earth's surface. We may, therefore, regard $g_{\mu\nu}$ as known theoretically and the elapsed time on a moving standard clock as experimentally observable. Equation (3) then allows us to compute the elapsed coordinate time at the position of a transported standard clock and hence to set up a coordinate clock at that position.

We now consider this process in detail. We stress the assumption that there exists a global coordinate time x^0. We shall also assume the metric tensor is static,

$$\frac{\partial g_{\mu\nu}}{\partial x^0} = 0, \tag{11}$$

and time orthogonal,

$$g_{0k} = 0, \qquad (k = 1,2,3). \tag{12}$$

We work entirely within the framework of general relativity, in which the invariant ds^2 is now given by

$$-ds^2 = g_{00}(dx^0)^2 + g_{k\ell}dx^k dx^\ell. \tag{13}$$

where repeated latin indices are summed from 1 to 3. The path of a light ray is characterized by

$$ds^2 = 0 \tag{14}$$

and the distance between two events as measured with standard measuring rods is given by

$$d\sigma = (g_{k\ell}dx^k dx^\ell)^{1/2}. \tag{15}$$

From an operational viewpoint, it is possible at any point of space to place a standard clock at rest. One way of achieving this (at least in principle) has been described as follows [6]: Observers using radar arrange to move along the coordinate world lines x^k = const., (k = 1, 2, 3). They do this by adjusting their velocities until each finds that the radar echoes from his neighbors require the same round-trip time at each repetition. Equivalently, each returning echo must show zero Doppler shift.

Let Δs be the invariant proper time between beats of a standard clock placed at rest. This will be the same number no matter where the clock is placed. Then since the clock is at rest, the coordinate time Δx^0 between beats is given by

$$\Delta x^0 = \Delta s \Big/ \sqrt{-g_{00}} \, . \tag{16}$$

This result can be used to determine the rate of a coordinate clock, which measures the coordinate time x^0 , placed at the position of the standard clock. In this expression we may imagine that g_{00} is a function of position determined from general relativity in terms of the distribution of mass. Thus since standard clocks and numerical values of g_{00} are available, Equation (16) provides an operational procedure for setting the rates of coordinate clocks.

The next problem is to initialize the coordinate clocks at different points in space so that the coordinate clock readings provide us with the desired global coordinate times. We shall consider two alternate (but equivalent) procedures for accomplishing this: A) use of light signals, and B) slow transport of standard clocks.

A) Synchronization (Initialization) using light signals

In Figure 1, the lines A and B represent the world lines of two coordinate clocks at rest at positions x^k , $x^k + dx^k$ which we wish to synchronize.

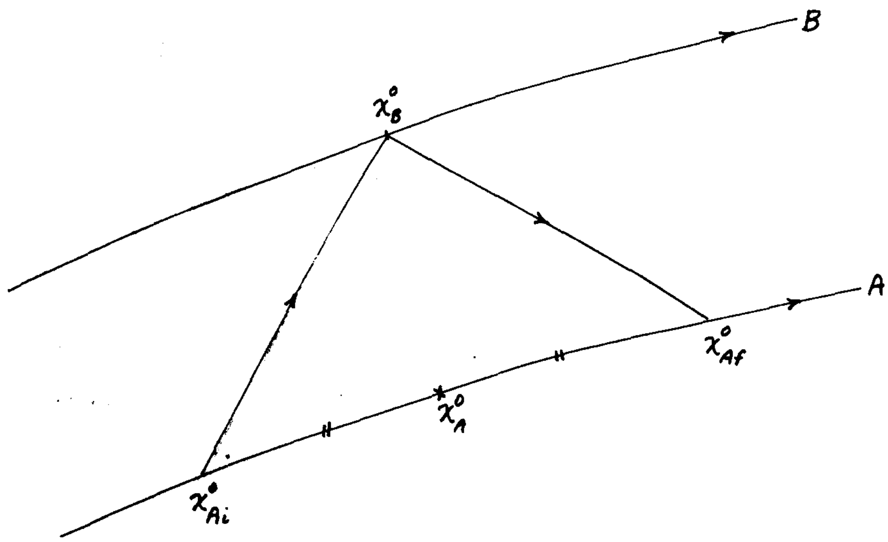

Figure 1

A light signal leaves A at coordinate time $x^0_{A_i}$, coincides with B at x^0_B , and returns to A at $x^0_{A_f}$. The coordinate times $x^0_{A_i}$ and $x^0_{A_f}$ are measured and the coordinate clock at A halfway between the events $x^0_{A_i}$ and $x^0_{A_f}$ reads

$$x^0_A = 1/2\left(x^0_{A_i} + x^0_{A_f}\right) \tag{17}$$

The event at x^0_A should then have the same coordinate time as that at x^0_B , so the coordinate clock at B (or at A) is zeroed so that

$$x^0_B = x^0_A = 1/2\left(x^0_{A_i} + x^0_{A_f}\right) . \tag{18}$$

To see the basis for this procedure starting from the expression for the metric, we have for the coordinate time for light to go from A to B ,

$$g_{00}\left(dx^0_{go}\right)^2 + d\sigma^2 = 0 \tag{19}$$

and to come back from B to A ,

$$g_{00}\left(dx^0_{come}\right)^2 + d\sigma^2 = 0 \tag{20}$$

where $d\sigma$ is the standard distance from A to B. Thus,

$$dx^0_{go} = dx^0_{come} = d\sigma \Big/ \sqrt{-g_{00}} \tag{21}$$

and

$$x^0_B = x^0_{A_i} + d\sigma \Big/ \sqrt{-g_{00}}$$

$$x^0_{A_f} = x^0_B + d\sigma \Big/ \sqrt{-g_{00}}$$

because the metric is static. Thus,

$$x^0_A = 1/2\left(x^0_{A_f} + x^0_{A_i}\right) = 1/2\left(x^0_B + d\sigma \Big/ \sqrt{-g_{00}} + x^0_B - d\sigma \Big/ \sqrt{-g_{00}}\right) = x^0_B , \tag{22}$$

and using Equation (18) gives the desired synchronization procedure.

Equivalently, the above procedure can be used to synchronize a series of coordinate clocks along a path taken by a light ray by writing

$$dx^0 = d\sigma \Big/ \sqrt{-g_{00}} \tag{23}$$

and integrating along the path

$$\Delta x^0 = \int_{\text{path}} d\sigma / \sqrt{-g_{00}} \; . \tag{24}$$

This procedure gives a resulting synchronization which is independent of the path along which the light ray travels; otherwise it would contradict the hypothesis that there exists a global coordinate time. It is also independent of time, since the positions of the coordinate clocks and the numerical values of $g_{\mu\nu}$, are time-independent.

B) Slow transport of standard clocks

We next establish the fact that if a standard clock is moved sufficiently slowly over a small distance past a stationary standard clock and then brought slowly back into coincidence with the stationary clock, the two clocks will undergo the same elapsed proper time. To prove this, we observe that the stationary clock will undergo an elapsed proper time

$$\Delta s = \sqrt{-g_{00}} \, \Delta x^0 \; . \tag{25}$$

The moving clock will undergo the elapsed proper time

$$\begin{aligned} \Delta s &= \int_{\text{path}} \sqrt{-g_{00}} \, dx^0 \left[1 + \; d\sigma^2 \; g_{00} (dx^0)^2 \right]^{1/2} \\ &\cong \sqrt{-g_{00}} \, \Delta x^0 - 1/2 \int \frac{dx^0}{\sqrt{-g_{00}}} \, \frac{v^2}{c^2} \; . \end{aligned} \tag{26}$$

If the velocity is sufficiently small the second term is negligible. If v is replaced by its maximum value, v_{max} , the condition that the second term in Equation (26) be negligible in comparison with the contribution from h_{00} , the deviation of g_{00} from -1 , is

$$\frac{v^2{}_{max}}{c^2} << \left| 1 + g_{00} \right| \tag{27}$$

Then in the first term of Equation (26) for infinitesimally short paths, g_{00} may be evaluated at any position along the path. This argument establishes the statement in the first paragraph of this section.

Next consider the process of synchronization by means of slow transport of standard clocks; in Figure 2, the solid lines with arrows represent light signals used to synchronize coordinate clocks at A and B repeatedly. We choose

$$x_A^{0''} - x_A^{0'} = x_A^{0'} - x_A^0 \tag{28}$$

and then set the coordinate clocks at point B , an infinitesimal distance away from A , so that

$$x_B^0 = x_A^0 , \qquad x_B^{0'} = x_A^{0'} , \qquad x_B^{0''} = x_A^{0''} \tag{29}$$

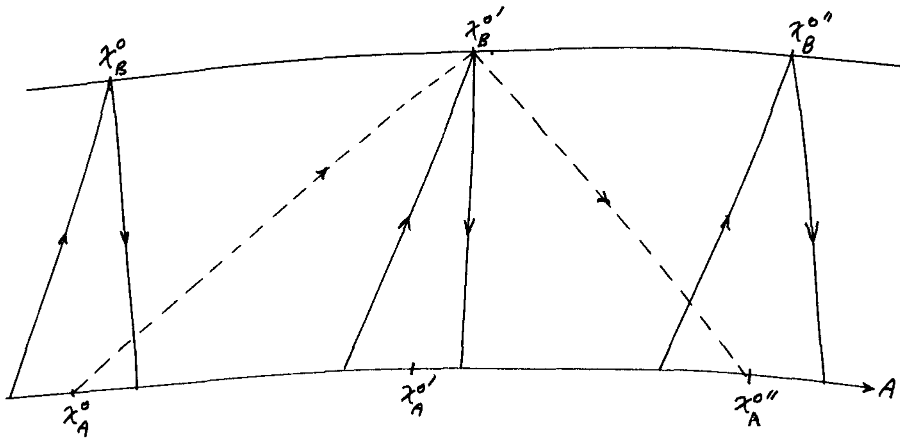

Figure 2

Now starting at coordinate time x_A^0 , cause a standard clock to be transported slowly and uniformly from A to B and back again (dotted line in Figure 2), arriving at B at $x_B^{0'}$ (measured by the coordinate clock at B), and at A at $x_A^{0''}$. The coordinate time required to go from A to B is

$$dx^0 = x_B^{0'} - x_A^0 = x_A^{0'} - x_A^0 = 1/2(x_A^{0''} - x_A^0) . \tag{30}$$

Note that $x_B^{0'} - x_A^0$ is a combination of measurements made by coordinate clocks at different locations. In terms of proper time elapsed on a standard clock at A ,

$$dx^0 = 1/2 \frac{s''_A - s_A}{\sqrt{-g_{00}(A)}} . \tag{31}$$

If the elapsed proper time on the moving standard clock is Δs , then we have seen already that $\Delta s = s''_A - s_A$. Thus,

$$dx^0 = 1/2 \ \Delta s \Big/ \sqrt{-g_{00}(A)} . \tag{32}$$

By symmetry, $1/2 \ \Delta s$ is the proper time elapsed on the slowly moving standard clock in going from A to B , which we shall denote by ds . Thus,

$$dx^0 = ds \Big/ \sqrt{-g_{00}} \tag{33}$$

where to first order in infinitesimal quantities, g_{00} can be evaluated at any point from A to B .

To extend this procedure to clocks which are arbitrarily distant from each other, imagine carrying a standard clock very slowly from one point to another along a path for which g_{00} is known. Then

$$\Delta x^0 = \int_{\text{path}} \frac{ds}{\sqrt{g_{00}}} . \tag{34}$$

The following argument establishes the equivalence of the two methods of coordinate clock synchronization discussed above. In Figure 3, let coordinate clocks at A and B , a finite distance apart, be synchronized by means of light signals and let the dotted line represent the path of a standard clock carried from A to B . Synchronization signals are

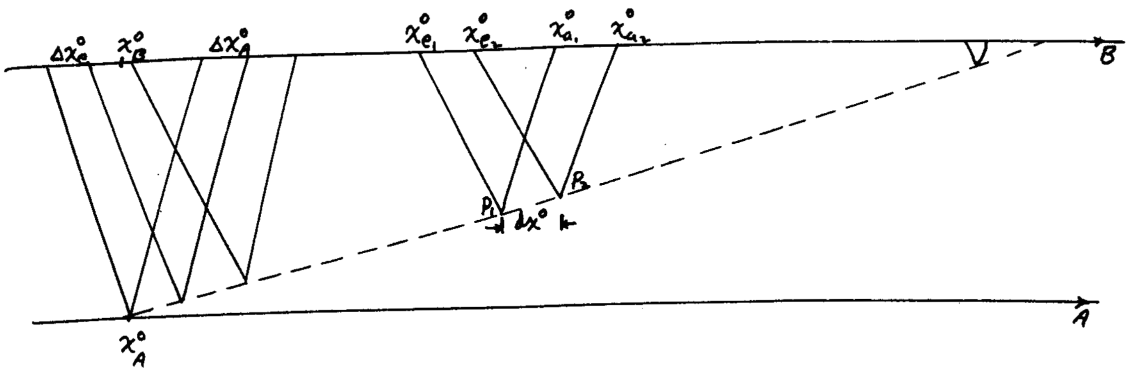

Figure 3

sent out from B at intervals Δx_e^0 and arrive back after reflection from the slowly moving clock at intervals Δx_a^0 . At some point along the path, bounded by reflections of signals which were emitted from B at coordinate times $x_{e_1}^0$ and $x_{e_2}^0$ and which arrive back at B at coordinate times $x_{a_1}^0$ and $x_{a_2}^0$, the coordinate time interval between the reflections at P_1 and P_2 will be given by

$$\begin{aligned} dx^o &= 1/2 \left[x_{e_2}^0 + x_{a_2}^0 - x_{e_1}^0 - x_{a_1}^0\right] \\ &= 1/2 \left[x_{e_2}^0 - x_{e_1}^0 + x_{a_2}^0 - x_{a_1}^0\right] \\ &= 1/2 \left[\Delta x_e^0 + \Delta x_a^0\right] \end{aligned} \tag{35}$$

where the right side of this expression contains only quantities measured by the coordinate clock at B . However, we have already shown that

$$dx^0 = ds \sqrt{-g_{00}} \tag{36}$$

where ds is the increment of proper time on the transported clock. Thus, integrating along the path,

$$\Delta x^0 = \int_{\text{path}} \frac{ds}{\sqrt{-g_{00}}} = 1/2 \sum (\Delta x_e^0 + \Delta x_a^0)$$

$$= 1/2 \left[x_a^0 \text{ (last) } - x_a^0 \text{ (first) } + x_e^0 \text{ (last) } - x_e^0 \text{ (first)} \right], \tag{37}$$

where the arguments "last" and "first" refer to the last and first synchronization signals sent out from B . Then since x_a^0 (last) = x_e^0 (last) = x_{arr}^0 , the arrival time of the transported clock at B , we have

$$\Delta x^0 = x_{arr}^0 - 1/2(x_a^0 \text{ (first) } + x_e^0 \text{ (first)})$$

$$= x_{arr}^0 - x_B^0 = x_{arr}^0 - x_A^0$$

$$= \int_{\text{path}} \frac{ds}{\sqrt{-g_{00}}} \tag{38}$$

Thus by either method, a consistent set of mutually synchronized coordinate clocks is obtained. This synchronization scheme has the following properties:

1) Reflexivity. If A is synchronized with B , B is synchronized with A. This follows from time-independence of the metric and the fact that the integral of $d\sigma / \sqrt{-g_{00}}$ is independent of direction along a path.

2) Transitivity. If A is synchronized with B , and B is synchronized with C , then A is synchronized with C.

3) Time-independence, because the metric is static.

4) Path-independence, otherwise a global coordinate time would not exist, contrary to hypothesis.

4. Effect of Gravitational Fields of the Sun and Moon

Since it has been suggested in the literature[1, 4] that the gravitational potentials of the sun and moon play a role in processes of clock synchronization, the purpose of this section is to estimate the magnitude of such effects. In the neighborhood of the earth, the gravitational potentials of the sun and moon are small, so it is clear that we can work in a linear approximation in which squares and products of $h_{\mu\nu}$ can be neglected. Consider the term $h_{\mu\nu}$ in a heliocentric coordinate system, and consider first of all the effect of the (static) gravitational potential of the sun on the earth-moon system. We can think of the center of mass of the earth-moon system as freely falling towards the sun. Consequently, at this point we may erect a coordinate system whose origin moves along with the center of mass of the earth-moon system and whose orthonormal axes are obtained by parallel transport of some initially chosen orthonormal tetrad of basis vectors. (We ignore for the moment the gravitational fields of the earth and moon.) Such a coordinate system is termed a normal fermi system and has been discussed by Manasse and Misner[7]. In normal fermi coordinates x^{μ}, the metric takes the form (see[7] for derivation)

$$g_{00} = -1 + R_{0\ell 0m} x^{\ell} x^{m} \tag{39}$$

$$g_{0i} = 2/3\ R_{0\ell im}\ x^{\ell} x^{m} \tag{40}$$

$$g_{ij} = \delta_{ij} + 1/3\ R_{i\ell jm}\ x^{\ell} x^{m} \tag{41}$$

where $R_{i\ell jm}$ is the Riemann-Christoffel curvature tensor, evaluated at the origin of the normal fermi system, along the freely falling orbit. The distances x^{m} are proper distances measured from the origin.

The components of the curvature tensor can, however, be evaluated in the heliocentric system of coordinates. Then in transforming them to normal fermi coordinates, the transformation coefficients will be of the form

$$\frac{\partial x^{\mu}}{\partial x^{\nu'}} = \delta^{\mu}_{\nu} + \text{terms of order } \sqrt{GM_s / c^2 R_s} \tag{42}$$

where R_s is the radius of the orbit and M_s is the sun's mass. Since we are retaining only first-order terms, the transformation coefficients are Kronecker delta-functions, and the curvature tensor is an invariant.

To see this in more detail, the typical transformation formula for a coordinate will be of the form

$$x = x' + R_s \cos(\omega t')$$

where ω is the angular velocity of the earth in its orbit. Then

$$\frac{\partial x}{\partial x^{o'}} = \frac{R_s \omega}{c} = \frac{R_s}{c}\ \frac{v}{R_s} = \frac{v}{c}\ .$$

But since the velocity is determined by

$$\frac{GM_s}{R_s^2} = \frac{v^2}{R_s}$$

we obtain

$$\frac{v}{c} \cong \sqrt{\frac{GM_s}{c^2 R_s}}\ .$$

Furthermore, in a linear approximation the curvature tensor may be expressed as

$$R_{\alpha\beta\gamma\delta} = h_{\alpha\delta,\beta\gamma} - h_{\beta\delta,\alpha\gamma} - h_{\alpha\gamma,\beta\delta} + h_{\beta\gamma,\alpha\delta} \tag{43}$$

which thus involves second derivatives of the metric tensor.

The largest terms in h_{00} in normal fermi coordinates will thus be of the form

$$-h'_{00,\ell m} x^{\ell} x^{m} \tag{44}$$

where h'_{00} is calculated in heliocentric coordinates in which [3]

$$h'_{00} = \frac{2GM_s}{c^2 r} \tag{45}$$

where r is the distance from the sun to the observation point, and M_s is the sun's mass. Since $x^1 \sim a_1$, the earth's radius, we have

$$h_{00} \cong \frac{GM_s a_1^2}{c^2 R_s^3} \tag{46}$$

Then using

$$M_s = 1.970 \times 10^{33} \text{ gm},$$

$$R_s = 1.495 \times 10^{10} \text{ cm}$$

$$a_1 = 6.3781 \times 10^{8} \text{ cm}$$

we get

$$h_{00} \cong 4 \times 10^{-17} \tag{47}$$

which is negligibly small when computing elapsed times. This term is not negligible when computing forces on particles--it is responsible for tidal forces.

In a similar manner, all other contributions from the sun's mass to $h_{\mu\nu}$ in normal fermi coordinates may be shown to be negligibly small. This conclusion is valid only in the neighborhood of the earth.

These conclusions may be explained qualitatively as follows. A standard clock at a radius slightly greater than that of the freely falling orbit (which for simplicity is assumed circular) will be blue-shifted in the sun's gravitational field since it is further from the origin. The fractional amount by which it is blue-shifted is $g_s \Delta r/c^2$ where $g_s = GM_s/R_s^2$ is the gravitational field strength due to the sun at the earth's orbit. Thus the fractional blue-shift is $GM_s \, \Delta r/c^2 R_s^2$.

However, since the clock is at a greater radius than one at the origin, its velocity in a heliocentric system will be greater and it will be redshifted more due to time dilation. Let v_0 be the orbital velocity of the earth and $v = v_0 \, (1 + \Delta r/R_s)$ be the velocity of the clock at radius $R_s + \Delta r$. This clock would be essentially at rest in the local inertial frame. The fractional redshift compared to a clock at the origin will be

$$\sqrt{1 - \frac{v_0^2}{c^2}} - \sqrt{1 - \frac{v^2}{c^2}} \cong \frac{v_0^2 \Delta\theta}{c^2 R_s}$$

But since the orbit is determined by the condition

$$\frac{GM_s}{R_s^2} = \frac{v_0^2}{R_s}$$

the redshift cancels the blue-shift to terms linear in the distance from the origin. Hoffman[3] has previously discussed the effect of this cancellation on clock rates.

Thus to high accuracy, in a system of local inertial coordinates erected at the center of mass of the earth-moon system, the effect of the sun's gravitational field can be neglected. The above qualitative argument shows this is because the center of mass is in free fall toward the sun.

We can now imagine that starting from a heliocentric system of coordinates, a coordinate transformation is made to normal fermi coordinates whose origin is at the center of mass of the earth-moon system. In this system, over distances of the order of the earth's radius, the effect of the sun may be neglected while calculating elapsed times. The actual distance from the center of the earth to the origin of these normal fermi coordinates is

$$R_e = R_m \frac{M_m}{M_m + M_e} = 4.64 \times 10^8 \text{ cm}$$

where R_m is the earth-moon distance, and M_m and M_e are the masses of the moon and earth, respectively. This distance is somewhat less than the earth's radius.

We can now ask what the effect of the moon's gravitational field on the metric tensor is. In this coordinate system, the earth is in free fall toward the moon. Hence by another coordinate transformation to a second set of normal fermi coordinates whose origin is at the center of the mass of the earth, it is evident that the effect of the moon can be similarly transformed away.

In any case, even if the second coordinate transformation is not made, the effect of the moon can be seen to be negligible by means of the following argument. Due to the free fall of the earth and moon toward their mutual center of mass, the earth executes a slow orbital motion about the center of mass. The velocity of a point on the earth's surface due to this motion is less than

$$\omega_0(a_1 + R_e) \cong 2.9 \times 10^3 \text{ cm/sec}$$

where

$$\omega_0 = 2.6 \times 10^{-6} \text{ sec}^{-1}$$

is the angular velocity of the earth-moon axis. The fractional redshift due to time dilation of a standard clock due to this motion may be expected to be of order

$$\frac{v^2}{c^2} \cong 10^{-14}$$

and during a comparison flight of duration T_c, would contribute less than .36 ns to the elapsed time, which is negligible. In actuality, there will be an additional blue-shift contribution due to the moon's gravitational field which will nearly cancel exactly the time-dilation effect. Therefore, we conclude that the gravitational fields of the sun and the moon may be neglected, for our purposes, in the vicinity of the earth's surface.

5. Clock Synchronization on the Rotating Earth

We have shown that because the earth is freely falling in the gravitational fields of the sun and the moon, there exists a locally minkowskian inertial frame whose origin is at the center of the earth and in which these fields can be ignored. The only remaining gravitational field is that due to the (rotating) earth itself.

In local inertial (normal fermi) coordinates with origin at the center of mass of the earth, the metric tensor now takes the form

$$g_{00} = -1 - 2W(\vec{r})/c^2 \tag{48}$$

$$g_{i0} = 0 \tag{49}$$

$$g_{ij} = \delta_{ij} \, (1 - 2W(\vec{r})/c^2) \tag{50}$$

where $W(\vec{r})$ is the potential due solely to the earth's gravitational field. We now transform to rotating coordinates (x',y',z',t') by means of

$$x' = x \cos \omega t + y \sin \omega t \tag{51}$$

$$y' = y \cos \omega t - x \sin \omega t \tag{52}$$

$$z' = z \tag{53}$$

$$t' = t \qquad \text{or} \qquad x^{0'} = x^0 \; . \tag{54}$$

where the angular velocity of rotation of the earth is

$$\omega = 7.2921 \times 10^{-5} \text{ sec}^{-1} \tag{55}$$

The metric then becomes in linear approximation

$$g_{00} = -1 - \frac{2}{c^2} \left[W(\vec{r}) - 1/2\ \omega^2(x'^2 + y'^2)\right] \tag{56}$$

$$g_{01} = g_{10} = -y'\omega/c \tag{57}$$

$$g_{02} = g_{20} = x'\omega/c \tag{58}$$

$$g_{ij} = \delta_{ij}\ (1 - 2W(\vec{r})/c^2) \tag{59}$$

Thus keeping only linear terms

$$-ds^2 = -\left[1 + \frac{2}{c^2}\ (W(\vec{r}) - 1/2\ \omega^2(x'^2 + y'^2))\right] (dx^0)^2 + 2(\vec{\omega} \times \vec{r}') \cdot d\vec{r}'\ dx^{0'}/c$$

$$+ \left[1 - 2W(\vec{r})/c^2\right] \left[(dx')^2 + (dy')^2 + (dz')^2\right]. \tag{60}$$

The neglected terms are, for propagation of light, at most

$$\frac{2}{c^2}\ \frac{GM_e}{a_1}\ \frac{\omega a_1}{c}\ (\Delta x^0)^2$$

and over a path length 10,000 km are completely negligible. For objects moving with speed v, the neglected terms would be of order v/c smaller and hence negligible.

Henceforth we drop primes on the coordinates and take the metric to be:

$$-ds^2 = -\left[1 + \frac{2}{c^2}\ (W(\vec{r}) - 1/2\ \omega^2(x^2 + y^2))\right] (dx^0)^2$$

$$+ 2\vec{\omega} \times \vec{r} \cdot d\vec{r} dx^0/c + \left(1 - \frac{2}{c^2}\ W(\vec{r})\right) \left((dx^2 + dy^2 + dy^2\right) \tag{61}$$

The potential due to the earth's flattened distribution of mass may, with accuracy sufficient for our purposes, be taken as [9, 10]:

$$W(\vec{r}) = -\frac{M_e G}{r} \left[1 + \left(\frac{a_1}{r}\right)^2 C_2 P_2\ (\cos\theta)\right] \tag{62}$$

where θ is the polar angle measured from the rotation axis and

$$M_e G = 3.986003 \times 10^{20}\ \text{cm}^3/\text{sec}^2 \tag{63}$$

$$C_2 = -\ 1.08270 \times 10^{-3} \tag{64}$$

$$a_1 = 6.378139 \times 10^8\ \text{cm} \tag{65}$$

$$P_2\ (\cos\theta) = 1/2 \left[3 \cos^2\theta - 1\right] \tag{66}$$

To express this potential in more convenient form, in terms of geographical latitude ϕ , and altitude h above mean sea level, we use the following expansions

$$r \cong a_1 - a_1 f \sin^2 \phi + h \tag{67}$$

where f is the flattening of the earth, and

$$\cos^2 \theta \cong \sin^2 \phi \,(1 - 4f \cos^2 \phi) \cong \sin^2 \phi . \tag{68}$$

The derivations of Equations (67) and (68) are given in the Appendix. Then if we put

$$V(\vec{r}) = W(\vec{r}) - 1/2\, \omega^2 \,(x^2 + y^2) \tag{69}$$

we have

$$V(\vec{r}) = - \frac{M_e G}{r} \left[1 - \frac{C_2 a_1^2}{2r^2} \,(1 - 3 \cos^2 \theta) \right] - 1/2\, \omega^2 r^2 \,(1 - \cos^2 \theta). \tag{70}$$

The geoid is the equipotential at altitude $h = 0$, or $r = a_1(1-f \sin^2 \phi)$. The value of this potential is, to sufficient accuracy,

$$V_o = - \frac{M_e G}{r} \left[1 - 1/2\, C_2 (1 - 3 \cos^2 \theta) \right] - 1/2\, \omega^2 a_1^2 \,(1 - \cos^2 \theta) \tag{71}$$

and this may be solved for r to give

$$\begin{aligned} r &= - \frac{M_e G}{V_0} \left[1 - 1/2\, C_2 \,(1 - 3 \sin^2 \phi) \right] - 1/2\, \omega^2 a_1^3 \,(1 - \sin^2 \phi)/V_0 \\ &\cong - \frac{M_e G}{V_0} \left[1 - \frac{C_2}{2} + \frac{\omega^2 a_1^3}{2 M_e G} \right] \left[1 - \left(\frac{\omega^2 a_1^3}{2 M_e G} - \frac{3C_2}{2} \right) \sin^2 \phi \right] . \end{aligned} \tag{72}$$

Identifying the above expression for the radius of the geoid as a function of angle with (see Equation (67))

$$r = a_1 \,(1 - f \sin^2 \phi) \tag{73}$$

we obtain the following expression for the flattening f :

$$f = \frac{\omega^2 a_1^3}{2 M_e G} - \frac{3C_2}{2} \cong \frac{1}{298.08} \tag{74}$$

To obtain an expression for the acceleration of gravity, we calculate the expansion of $V(\vec{r})$ to the first order in h:

$$g = + \frac{\partial V}{\partial r}\bigg|\, r = a_1(1 - f \sin^2 \phi)$$

$$= \frac{M_e G}{a_1^2} - \frac{3}{2}\frac{M_e G C_2}{a_1^2} - \omega^2 a_1 + \sin^2 \phi \left[\frac{2M_e Gf}{a_1^2} + \frac{9}{2}\;\frac{M_e GC_2}{a_1^2} + \omega^2 a_1\right]$$

$$= 978.027 + 5.192 \sin^2\phi \quad \text{cm/sec}^2 \;. \tag{75}$$

We thus obtain for g_{00}:

$$g_{00} = - \left(1 + \frac{2}{c^2}\,(V_0 + g(\phi)h)\right). \tag{76}$$

The term $2V_0/c^2$ is of order

$$\frac{2M_e G}{c^2 a_1} \sim 1.4 \times 10^{-9}.$$

For a standard clock at rest at mean sea level (where $h = 0$, the geoid in this model),

$$ds = \left(1 - \frac{M_e G}{c^2 a_1}\right) dx^0 \;. \tag{77}$$

And during a flight of duration T_c , the term in V_c would contribute a correction

$$\frac{M_e G T_c}{c^2 a_1} = 50{,}000 \text{ ns}. \tag{78}$$

which is certainly a large effect but which is the <u>same</u> for all clocks, since all available standard clocks are in fact placed near the geoid. Therefore, if we redefine the rates of all coordinate clocks so as to absorb this term, we may write

$$g_{00} = - \left[1 + \frac{2g(\phi)h}{c^2}\right] . \tag{79}$$

The metric is then to a sufficient accuracy

$$- ds^2 = -\left(1 + \frac{2g(\phi)h}{c^2}\right) (dx^o)^2 + \left(1 - \frac{2W(\vec{r})}{c^2}\right) (dx^2 + dy^2 + dy^2)$$

$$+ 2\vec{\omega} \times \vec{r} \cdot d\vec{r}\; dx^o/c. \tag{80}$$

It should be noted that the expression $g(\phi)$ appearing in Equations (75) and (80) includes rotation.

With the aid of Equation (80), we are now in a position to analyze various processes of clock synchronization. Consider first a series of standard clocks at rest at altitudes $h = 0$ (on the geoid). The proper time elapsed will be given by:

$$ds = dx^0 \ . \tag{81}$$

Thus we have shown that by distributing a system of standard clocks about on the geoid, at rest, all such clocks will beat at the same rate, equal to the rate at which the global coordinate clocks will beat. This result was discussed by Cocke [11]. We still have the problem of initializing these clocks.

A) Synchronization of coordinate clocks by slow transport of standard clocks

Let a standard clock be set in motion and carried from one point to another along a path near the geoid, so that Equation (80) holds. It is easy to see from the metric, Equation (80), that the proper time elapsed on the standard clock is path-dependent. In particular, the term $2\vec{\omega}x\vec{r}\cdot d\vec{r}dx^0/c$ is responsible for the well-known 207.4 ns east-west asymmetry [1] in equatorial around-the-globe flights. Nevertheless, since all the coefficients in the above expression for the metric are known when the altitude, course, and speed of plane carrying the clock are known, the metric can be inverted to calculate the elapsed coordinate time in terms of the elapsed proper time on the transported clock:

$$\Delta x^0 = \int_{path} ds \left[1 - \frac{g(\phi)h}{c^2} + 1/2\,\frac{v^2}{c^2} + \frac{\vec{\omega}\ x\ \vec{r}\cdot\vec{v}}{c}\right]. \tag{82}$$

Hence by selecting one location as a reference, all coordinate clocks (whether on the geoid or not) may be synchronized by slow transport of standard clocks. In Equation (82), v is the ground speed (measured on the flattened earth) and $\vec{r}$ is the vector from the center of mass of the earth to the position of the moving clock measured in the rotating coordinate system. The coordinate clock synchronization scheme based on the above equation, because of the existence in general relativity theory of a global coordinate time, is both path-independent and transitive.

With the aid of equation (82), we can estimate the magnitude of a number of effects.

a) Effect of latitude variations in $g(\phi)$ on contributions from redshift

This term, during a comparison flight, would be of order $5.2\sin^2\phi h_c T_c/c^2$. For a flight of duration 10 hours and altitude 12,000 meters, the contribution will be of order 0.25 ns and is, therefore, negligible.

b) Effect of altitude variations in $g(\phi)$ on contributions from redshift

Although a term proportional to h^2 is not explicitly included in our expansion of $V(\vec{r})$, Helmert's Equation [12] for the acceleration of gravity is, to the accuracy of terms considered here,

$$g(\phi) = 978 + 5.2\sin^2\phi - 3.086 \times 10^{-6} h \ \text{cm/sec}^2 \tag{83}$$

(with h in cm). This gives a potential term in g_{00} of order

$$1.5 \times 10^{-6}\ h^2/c^2$$

And for a flight of duration T_c and altitude $h_c = 1.2 \times 10^6$ cm , the correction due to altitude variations in g is approximately 0.09 ns , which is negligible.

c) Effect of gravitational anomalies on redshift

According to Caputo [9] , undulations of the geoid from the calculated ellipsoidal shape are 50 meters or less. A height variation of $\Delta h = 50$ meters would give rise to corrections to the elapsed time of order $g\Delta hT_c/c^2 \cong 0.2$ ns , which is negligibly small.

d) Errors in altitude measurement

The calculation in paragraph c above shows that if an accuracy level of 1 ns is desired, h must be measured during transport to an accuracy of better than 250 meters. This is not a very severe restriction.

Next let us examine the term which depends explicitly on the rotation of the earth. Using Equation (67) for r , the altitude contribution would be of order

$$\frac{\omega h T_c v}{2c^2} \cong 0.08 \text{ ns.} \tag{84}$$

where we have assumed a maximum ground speed of $v = 1{,}000$ mi/hr $\cong 4.5 \times 10^4$ cm/sec. This contribution is, therefore, negligible.

e) Effect of flattening of earth

The flattening contribution in Equation (67) would affect the contribution from the rotational term in the amount

$$\frac{\omega a_1\ f\ T_c v}{c^2} \cong 0.28 \text{ ns.}$$

which is negligible.

With these results, the rotational term can be simplified as follows:

$$\vec{\omega} \times \vec{r} \cdot \vec{v} = \omega a_1\ v_E \cos\phi \tag{85}$$

where v_E is the eastward component of ground velocity and ϕ is the latitude. (v_E is negative if the clock is transported westward.) Thus the final expression for elapsed coordinate time is

$$\Delta x^0 = \int_{\text{path}} ds \left[1 - \frac{gh}{c^2} + \frac{v^2}{2c^2} + \frac{\omega a_1\ v_E \cos\ \phi}{c^2}\right] \tag{86}$$

In developing Equation (85), flattening of the earth was neglected. Referring to the Appendix, the relation between polar angle and latitude is

$$\cos\theta = \sin\phi\ (1 - 2f\cos^2\phi) \tag{87}$$

so the error introduced into Equation (85) is of order f . This error would contribute about the same amount as that calculated in section e above, .28 ns , and is negligible.

B) Infinitesimally slow transport on the geoid

In the limit as $v \to 0$, and if $h = 0$, we have

$$\Delta x^0 = \int ds\ (1 + a_1 v_E \cos\phi/c^2) \tag{88}$$

$$= \Delta s + \frac{\omega a_1}{c^2}\int ds\ v_E \cos\phi . \tag{89}$$

We consider several examples.

1) Transport along a meridian: then $v_E = 0$ and $\Delta x^0 = \Delta s$. We could use this result to synchronize coordinate clocks over the entire surface of the earth by selecting one location as reference and transporting standard clocks slowly along a meridian to one of the poles and then back along another meridian to an arbitrary position on the earth's surface. The proper time elapsed on the transported clock will read coordinate time.

2) Transport along a parallel of latitude: then $v = v_E$ and $\int ds\ v_E \cos\phi \cong cL \cos\phi$ where L is the distance traversed eastwards. The coordinate time elapsed will be

$$\Delta x^0 = \Delta s + \frac{\omega a_1}{c} L \cos\phi. \tag{90}$$

Note that the correction term is independent of velocity, provided only that the velocity is small. This result implies that the proper time elapsed on a standard clock carried eastward around the globe will be $2\pi\omega a_1^2 \cos^2\phi/c^2 = 207.4 \cos^2\phi$ ns less than that on a standard clock which remains at rest, while a standard clock carried slowly westward in the geoid would lead a standard clock, which remained at rest, by $207.4 \cos^2\phi$ ns.

C) Global coordinate clock synchronization using light signals

The propagation of a light ray is described by $ds^2 = 0$. This gives

$$0 = (dx^0)^2 \left(1 + \frac{2g(\phi)h}{c^2} - \frac{2\omega a_1 \cos\phi\ c_E\ (dx^0)^2}{c^2}\right)$$

$$- \left(1 - \frac{2}{c^2} W(\vec{r})\right) \left((dx)^2 + (dy)^2 + (dz)^2\right). \tag{91}$$

where c_E is the eastward component of velocity of the light ray. For motions in the geoid, and in linear approximation, $W(\vec{r})$ can be replaced by its approximate value

$$W(\vec{r}) = -\frac{M_e G}{r} = -\frac{M_e G}{a_1} (1 + f \sin^2\phi - h) \cong -\frac{M_e G}{a_1} . \tag{92}$$

The fractional errors in coordinate time of propagation of a light signal introduced are as follows:

A) The altitude terms in the coefficient of $dx^2 + dy^2 + dz^2$ are of order

$$\frac{2M_e Gh}{c^2 a_1^2} \cong 3 \times 10^{-12}$$

which for a light signal traversing a path of length 10,000 km on the earth's surface, would amount to only about 10^{-13} sec, which is negligible.

B) The term arising from neglect of flattening contributes for a light path of this length

$$\frac{2M_e Gf}{c^2 a_1} \times \frac{10^9}{c} \cong .05 \text{ ns.}$$

which is also negligible. The term arising from neglect of the angularly-dependent part of $W(\vec{r})$ can from Equations (62) and (74) be seen to be of similar order of magnitude. The remaining term is constant, and is

$$\frac{2M_e G}{c^2 a_1} \cong 1.4 \times 10^{-9} .$$

Solving Equation (91) for dx^0, in linear approximation, we have

$$dx^0 = \sqrt{dx^2 + dy^2 + dy^2} \left[1 + \frac{M_e G}{c^2 a_1} - \frac{g(\phi)h}{c^2} + \frac{\omega a_1 \; c_E \cos \phi}{c^2}\right]. \tag{93}$$

The first term in brackets in Equation (93) is of order 10^{-9}, but it is the same for all propagation paths within about 12,000 meters of the earth's surface and we may incorporate it into measurement of proper distance. The proper distance $d\sigma$ between two points on the geoid will be obtained by integrating

$$d\sigma = \left(1 + \frac{M_e G}{c^2 a_1}\right)\left(dx^2 + dy^2 + dz^2\right)^{1/2}. \tag{94}$$

Furthermore, the term gh/c^2 in Equation (93) is of order 10^{-12} which is negligibly small. Then approximately,

$$dx^0 = d\sigma \left[1 + \frac{\omega a_1 \; c_E \cos \phi}{c^2}\right] \tag{95}$$

The only significant effect is due to rotation of the earth. In the rotational term in Equation (95), altitude and flattening effects may be safely neglected over 10,000-meter path lengths at the 1 ns level of accuracy.

Then in using one-way light signals to synchronize the coordinate clocks placed on the geoid, one may select a reference clock and send out light signals within 12,000 meters of the earth's surface; the coordinate time elapsed is

$$\Delta x^0 = \int d\sigma + \frac{\omega a_1}{c^2} \int_{\text{path}} c_E \cos\phi \, d\sigma. \tag{96}$$

during the propagation. It should be noted that for 1 ns accuracy, the proper distance along the path in the first term, $\int d\sigma$, must be known to an accuracy of 30 cm or better.

We now consider some examples.

1) Light signals sent along a meridian: For this case, $c_E = 0$ and the elapsed coordinate time is just

$$\Delta x^0 = \int_{\text{path.}} d\sigma .$$

2) Signals sent along a parallel of latitude: For this case,

$$\Delta x^0 = \int_{\text{path}} d\sigma + \frac{\omega a_1}{c} \cos\phi \int_{\text{path}} d\sigma .$$

Thus if we adopt the global coordinate time x^0 as a basis for a synchronized coordinate clock nerwork on the surface of the earth, then light propagation times must be corrected for the earth's rotation according to Equation (95). The resulting synchronization will agree with that based on transport of standard clocks as expressed by Equation (86) and will be path-independent and transitive.

D) <u>The "Hafele-Keating" discontinuity</u>

It has been suggested [4] that one way to achieve a synchronized coordinate time net on the surface of the rotating earth is to select a meridian or other line between the poles along which one allows for a 207.1 $\cos^2\phi$ ns discontinuity in synchronization of the coordinate clock network. The network could be synchronized by slow transport of clocks or by light signals <u>without correction for earth rotation</u> as long as the discontinuity is not crossed and transport is along parallels of latitude. A clock traversing the discontinuity from west to east would have to be advanced by $2\pi a_1^2 \omega \cos^2\phi/c^2$, while one traversing the discontinuity from east to west would have to be retarded by the same amount.

Suppose we adopt on the earth's surface a coordinate time net x_D^0 with a discontinuity related to the global coordinate time x^0 by

$$x_D^0 = x^0 - \frac{\omega a_1 \cos\phi \, L}{c} \tag{97}$$

where L is the distance eastward from some reference meridian

$$-\pi a_1 \cos\phi < L < \pi a_1 \cos\phi \tag{98}$$

of the clock in question and where, for convenience, we have chosen the discontinuity at the meridian on the opposite side of the globe from the reference meridian on which $L = 0$.

Then from Equation (89), use of slowly transported standard clocks without correction for earth rotation will give the required time x^o_D of Equation (97):

$$\Delta s = \Delta x^o - \frac{\omega a_1}{c^2} \int ds v_E \cos \phi$$

only if the transport is carried out along a parallel of latitude. This is because of the appearance of the factor $\cos \phi$ in the integrand. Similarly from Equation (96), use of light signals without correction for rotation will give an equivalent synchronization procedure:

$$\int d\sigma = \Delta x^o - \frac{\omega a_1}{c^2} \int d\sigma \, c_E \cos \phi$$

only if the light path is along a parallel of latitude, since again a factor $\cos \phi$ appears in the integrand.

Hence we have proved that use of light signals and standard clocks, uncorrected for rotation, can be used to construct equivalent synchronized coordinate clock nets with a discontinuity for only a limited class of paths. This scheme has several disadvantages:

1) The effects of redshift and time dilation on transported clocks must still be included.

2) Only a limited class of paths can be used; otherwise the synchronization procedure is path-dependent and intransitive.

3) The time readings obtained do not represent the global coordinate time, although the transformation to global coordinate time (Equation (97)) is fairly simple.

4) The discontinuity may be a nuisance for clocks which must traverse it.

E) Use of satellites to construct a coordinate time network

The use of satellites to carry reference clocks and to send synchronization pulses to earth-based clocks could be analyzed by use of Equation (93). The term involving $g(\phi)h/c^2$ for satellite heights up to 1,200 km is still negligibly small. Altitude corrections to the rotational term are, however, of order

$$\frac{\omega h}{c} \cong 3 \times 10^{-7}$$

which in a propagation time of $t \sim 10^9/c \cong 3 \times 10^{-2}$ sec would be of order 10 ns and would be appreciable. Thus the rotational term must be written as in the following expression for the elapsed global coordinate time along the path of the pulse.

$$\Delta x^o = \int_{path} d\sigma \left[1 + \frac{\vec{\omega} \times \vec{r} \cdot \vec{c}}{c^2}\right] \tag{99}$$

where $\vec{r}$ is the radius vector from the earth's center to the path of the pulse of velocity $\vec{c}$.

The above equation, Equation (99), could be used in principle to synchronize any receiving station using reference pulses from a satellite which were sent out at regular coordinate time intervals. To obtain a clock on a satellite which measures coordinate time x^o requires more analysis.

Returning to the expressions (48, 49, 50) for the metric tensor before introducing the transformation to rotating coordinates, we write approximately $W(\vec{r}) = -M_e G/r$, so

$$-ds^2 = -\left(1 - \frac{2M_e G}{c^2 r}\right)(dx^o)^2 + \left(1 + \frac{2GM_e}{c^2 r}\right)(dx^2 + dy^2 + dy^2) \tag{100}$$

and for a satellite moving in a circular orbit at low speed,

$$ds^2 = \left(1 - \frac{2GM_e}{c^2 r} - \frac{v^2}{c^2}\right)(dx^o)^2 \tag{101}$$

Thus in terms of elapsed proper time on a standard clock carried in the satellite,

$$(dx^o)^2 = ds^2 \left[1 + \frac{2GM_e}{c^2 r} + \frac{v^2}{c^2}\right] \tag{102}$$

and since the orbit is determined by

$$\frac{GM_e}{r^2} = \frac{v^2}{r}$$

we have

$$dx^o = ds \left[1 + \frac{3GM_e}{2c^2 r}\right] . \tag{103}$$

And from this equation or its generalization for an elliptical orbit, equal intervals of coordinate time on the satellite can be determined by calculation [13] assuming that the positions $\vec{r}(x^o)$ of the satellite can be determined and the integral of Equation (99) performed for clock synchronization. Since in 1 ns light travels 30 cm, positional accuracies of the satellite and of the receiving stations must be better than ± 30 cm, for 1 ns synchronization accuracy.

In addition, variations in propagation time due to dispersion of the electromagnetic waves in the ionosphere must be considered. These variations may be of the order of a few nanoseconds [14].

6. Miscellaneous Effects

A) Mass of the atmosphere

The mass of the atmosphere is about 10^{-6} that of the earth. As the mass varies, the gravitational potential will vary, but neglect of the variation due to the atmosphere's mass could at most introduce an error of 10^{-6} times the redshift due to the earth's mass and is, therefore, negligible.

B) Effect of flattening of the earth on discontinuity

The "Hafele-Keating" discontinuity in its exact form is

$$\frac{\omega r \sin \theta L}{c^2} = \frac{2\pi\omega(r \sin \theta)^2}{c^2} .$$

From the Appendix, at zero altitude, we have

$$(r \sin \theta)^2 = \frac{a_1^2 \cos^2\phi}{\cos^2\phi + (1-f)^2 \sin^2 \phi} . \tag{104}$$

Thus, keeping only corrections of first order in f, the discontinuity is

$$2\pi\omega a_1^2 \cos^2\phi \; (1 + 2f \sin^2\phi)/c^2 = 207.4 \cos^2\phi \; (1 + 2f \sin^2\phi) \text{ ns.} \tag{105}$$

Since the maximum value of $\sin^2\phi \cos^2\phi$ is 1/4, the correction from flattening is less than

$$2f(207.4)/4 \cong 0.35 \text{ ns.}$$

and is, therefore, negligible.

C) Retardation

One may ask whether the finite propagation time of gravitational signals from the moon affects the gravitational field in which the earth is freely falling and in which normal fermi coordinates are finally erected at the center of the earth's orbit. Consider the terms $h_{\mu\nu}$ in a coordinate system whose origin is at the center of mass of the earth-moon system. Weber [3] has given explicit expressions for $h_{\mu\nu}$ in the linear approximation. These expressions for a source mass m at $\vec{r}'$, moving with velocity $\vec{v}$, and with observation point $\vec{r}$, are as follows. Defining

$$\phi_{\mu\nu} = h_{\mu\nu} - 1/2 \; \delta_{\mu\nu} h^{\alpha}_{\alpha} \tag{106}$$

we have

$$\phi_{00} = \frac{4Gm/c^2 |\vec{r} - \vec{r}'|}{\sqrt{1 - v'^2/c^2}\left(1 - \frac{v' \cdot (\vec{r} - \vec{r}')}{c|\vec{r} - \vec{r}'|}\right)} \tag{107}$$

$$\phi_{0i} = \frac{-\,4Gmv_i'/c^3|\vec{r} - \vec{r}\,'|}{\sqrt{1 - v'^2/c^2}\left(1 - \dfrac{\vec{v}\,' \cdot (\vec{r} - \vec{r}\,')}{c|\vec{r} - \vec{r}\,'|}\right)} \tag{108}$$

$$\phi_{ij} = \frac{4GMv_i'v_j'/c^4|\vec{r} - \vec{r}\,'|}{\sqrt{1 - v'^2/c^2}\left(1 - \dfrac{\vec{v}\,' \cdot (\vec{r} - \vec{r}\,')}{c|\vec{r} - \vec{r}\,'|}\right)} \,. \tag{109}$$

If we apply the above expressions to the case of the moon, the leading term in ϕ_{00} will be proportional to the ordinary newtonian potential, $4Gm/c^2|\vec{r} - \vec{r}\,'|$. The leading correction term is due to retardation and will be of order

$$\frac{4Gm}{c^2|\vec{r} - \vec{r}\,'|}\ \frac{v}{c}$$

The velocity of the moon is such that $v/c \sim 3 \times 10^{-6}$. Estimating the magnitude of this retardation correction in the vicinity of the earth where $|\vec{r} - \vec{r}\,'| = R_m = 3.84 \times 10^{10}$ cm , we obtain

$$\frac{4Gm}{c^2|\vec{r} - \vec{r}\,'|}\ \frac{v}{c} \cong 10^{-18}$$

This retardation term contributes a term of order 10^{-18} to h_{00} . In a typical flight lasting 10 hours , such a term would contribute no more than about 4×10^{-14} sec to the elapsed coordinate time and may, therefore, be neglected. Thus, we conclude that retardation corrections to the metric from motion of the moon may be neglected.

References and Footnotes

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[5] Schlegel's expression for this discontinuity contains an error.

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[13] Note that this equation does not take account of the redefinition of coordinate-clock rates mentioned in connection with Equation (77).

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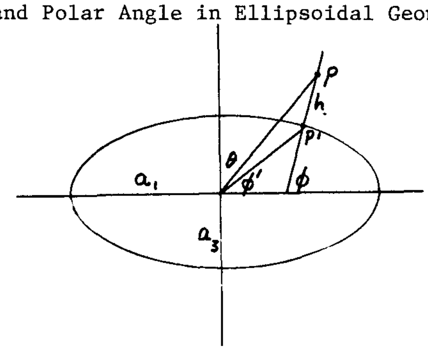
APPENDIX

Latitude and Polar Angle in Ellipsoidal Geometry



In this paper, the rotating earth is considered an ellipsoid of revolution with equatorial radius a_1 and polar radius a_3, where

$$a_3 = a_1(1 - f) \tag{A1}$$

with $f \cong 1/300$ the flattening of the earth. In this appendix, we wish to derive relations between the polar angle Θ measured from the rotation axis and the geographic latitude ϕ, measured along a plumb line which is normal to the ellipsoid. The problem is, given the position of an observer in terms of altitude h and latitude ϕ, to express r and θ in terms of h and ϕ. Let X,Y be the cartesian coordaintes of the point P. Then

$$X = r \sin \theta , \qquad Y = r \cos \theta . \tag{A2}$$

We next express the coordinates (x,y) of the point P' on the surface which is the projection of P along a plumb line, in terms of ϕ. Write

$$r_{p'} \sin \phi' = y = S \sin \phi \tag{A3}$$

$$r_{p'} \cos \phi' = x = C \cos \phi \tag{A4}$$

Equations (A3) and (A4) define new quantities S and C. We have

$$\tan \phi' = y/x \tag{A5}$$

and from analytic geometry,

$$\tan \phi = y a_1^2 / x a_3^2 , \tag{A6}$$

so

$$\frac{C}{S} = \frac{\tan \phi}{\tan \phi'} = \frac{a_1^2}{a_3^2} = \frac{1}{(1-f)^2} \tag{A7}$$

and hence

$$S = (1-f)^2 C \ . \tag{A8}$$

Since the equation of the ellipsoid is

$$\frac{x^2}{a_1^2} + \frac{y^2}{a_3^2} = 1 \tag{A9}$$

we obtain

$$\frac{C^2 \cos^2\phi}{a_1^2} + \frac{(1-f)^4 C^2 \sin^2\phi}{a_3^2} = 1 \tag{A10}$$

or

$$C = \frac{a_1}{\sqrt{\cos^2\phi + (1-f)^2 \sin^2 \phi}} \ . \tag{A11}$$

Now we may write the coordinates of P as

$$X = h \cos \phi + \frac{a_1 \cos \phi}{\sqrt{\cos^2\phi + (1-f)^2 \sin^2\phi}} \tag{A12}$$

$$Y = h \sin \phi + \frac{(1-f)^2 a_1 \sin \phi}{\sqrt{\cos^2\phi + (1-f)^2 \sin^2\phi}} \tag{A13}$$

Then

$$r^2 = X^2 + Y^2 \tag{A14}$$

and after some algebra, we find that retaining only first-order corrections to a_1 in flattening and altitude,

$$r \cong a_1 - a_1 f \sin^2 \phi + h \tag{A15}$$

Also, neglecting altitude corrections which can be shown to be of order $(h/a_1)^2$ and hence negligible, we find for the polar angle

$$\cos^2\theta = \frac{Y^2}{r^2} \cong \sin^2\phi \; (1-4f \; \cos^2\phi). \tag{A16}$$

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