A METHOD FOR ESTIMATING THE FREQUENCY STABILITY OF AN INDIVIDUAL OSCILLATOR

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Abstract

A method is given for estimating the intensity of random noise frequency modulation of an individual oscillator, using data obtained by comparing it with two or more other oscillators. This method is appropriate even if the oscillators available for comparison are less stable than the oscillator being evaluated, but their frequency fluctuations must be independent. The statistical uncertainty of the results is discussed briefly.

Introduction

In practice, the frequency stability of an oscillator can be measured only by comparing its output with that of one or more other oscillators. The direct result of any such measurement includes not only the instability of the oscillator being examined, but also that of the reference oscillator or ensemble. Thus a laboratory measurement yields the stability of a group of oscillators, whereas one often needs to know the stability of an individual oscillator. In this paper, we present a method of estimating the stability of an individual oscillator by simultaneously comparing its frequency with that of two or more other oscillators.

It is necessary for the success of this method, that the oscillators be independent -- their FM noise should be uncorrelated. This is not a serious limitation: if the frequency stability of an oscillator is important enough to measure, the oscillator has probably been designed and installed with adequate isolation to insure its independence. There is another more serious limitation: although the noise of the ensemble of reference oscillators does not bias the measured result, it does contribute to the uncertainty of the measurement. In extreme cases, the noise of the reference ensemble can be so much greater than that of the oscillator being measured, that no meaningful result can be obtained. Nevertheless in many practical cases, this method does permit measurement of the FM noise level, or at least a probable upper bound for the noise level of an oscillator -- even if it has a lower noise level than any member of the reference ensemble.

The Method

Consider three oscillators, A, B, and C. The frequency stability of oscillator A is to be measured, and oscillators B and C are to serve as the reference ensemble. We can measure the frequency stability of A with respect to B and obtain a variance σ^2_{ab}. Simultaneously we can measure σ^2_{ac}, and σ^2_{bc}. For this paper a two-sample Allan variance was used, and it may be estimated for a finite set of data as follows [1, 2]:

$$\sigma^2 = \frac{1}{2M} \sum_{i=1}^{M} (\Delta y_i)^2 \qquad (1)$$

where Δy is the difference between adjacent frequency measurements, each measured over a nominal sample time τ: $\Delta y_i = y_{i+1} - y_i$, and M is the number of differences. From equation (1), it follows that

$$\sigma^2_{ab} = \sigma^2_a + \sigma^2_b - \frac{1}{M} \sum_{i=1}^{M} (\Delta y_{a_i} \, \Delta y_{b_i}) \qquad (2)$$

The last term, the correlation term, goes to zero as the number of measurements increases; i.e. the oscillators are independent. Neglecting the correlation term, we can write,

$$\begin{aligned} \sigma^2_{ab} &= \sigma^2_a + \sigma^2_b \\ \sigma^2_{ac} &= \sigma^2_a + \sigma^2_c \\ \sigma^2_{bc} &= \sigma^2_b + \sigma^2_c \end{aligned} \qquad (3)$$

Equations (3) can be solved for σ^2_a, a measure of the frequency stability of the individual oscillator A, [3].

$$\sigma^2_a = \tfrac{1}{2}\left(\sigma^2_{ab} + \sigma^2_{ac} - \sigma^2_{bc}\right) \qquad (4)$$

We only need to measure σ^2_{ab}, σ^2_{ac}, and σ^2_{bc} and apply equation (4). Our neglect of the correlation term does not bias the result; it contributes only to the uncertainty of the measurement.

When N reference oscillators are available to measure one oscillator, $\frac{N(N-1)}{2}$ different triads of oscillators can be formed with the oscillator under measurement. From each triad, we can compute a value of σ^2_a and these several estimates can be averaged to obtain a better estimate of σ^2_a. In general, the triads do not all contribute equally to our knowledge of σ^2_a, because some of the N oscillators are more stable than others. Consequently, the estimates of σ^2_a obtained from the several triads should be combined in a weighted average. A reasonable set of weighting factors may be obtained by observing that the uncertainty of measurement using the triad of oscillators, A, B, and C, is proportional to

$$u_{abc} = \sigma_a^2 + \sigma_b^2 + \sigma_c^2 , \tag{5}$$

then choosing the weighting factor to be inversely proportional to u^2 for that triad.

A Program of Computation

As an example we consider eight cesium-beam clocks. The time difference between each pair of clocks is recorded every day for 800 days. We wish to calculate an Allan variance of the frequency of clock 1, for a sample time of $\tau = 1$ day. With the seven reference clocks, we form 21 triads, each including clock 1. Using the time readings taken on days 1, 2, and 3, we compute a two sample frequency variance for clock 1 with each of the 21 triads. Because weighting factors are not yet available, we must be content to combine the results from the 21 triads in an unweighted average. We then do the same using data from days 2, 3, and 4, proceeding in this manner until the data from all 800 days has been utilized. We then average all our results to give a first, unweighted, estimate of the stability of clock 1. By the same procedure, we obtain unweighted estimates of the stability of each of the seven reference clocks. With these preliminary values of σ, we use equation (5) to obtain weighting factors for the 21 triads. Finally, we repeat the entire calculation for clock 1, this time using a weighted average.

The computation could be simplified by using the individual values of σ, which go into the weighted average, to compute their own weighting factors. Unfortunately that procedure gives greatest weight to the smallest values of σ, giving a biased result. To minimize this bias, the σ's used in equation (5) should each be an average obtained from many measurements.

Results

To demonstrate the effectiveness of this triangulation method, we have used a computer to generate a mixture of white and flicker noise FM, simulating the behavior of seven commercial cesium-beam clocks, and one of the newer high-performance commercial cesium clocks [3, 4]. Unlike laboratory data, this computer-generated data represents the behavior of a single clock, as measured with respect to an ideal, i.e. noiseless reference clock. For comparison, we computed an Allan variance of the frequency of clock 1, the high-performance clock whose noise is less than any of the others. Then we compared the time difference data for these eight clocks, in pairs, putting it into a form which is often obtained in the laboratory. Using only this data on the time differences of pairs of clocks, we carried out the program of computation described above. The triangulation method successfully determined the stability of the individual clock 1 using data obtained by comparing it with the ensemble of seven clocks, all of which were less stable than clock 1. Figure 1 shows the square root of an Allan variance for clock 1, by direct computation, and by the triangulation method. Triangulation has clearly come very close to giving the stability of clock 1 with respect to an ideal, noiseless reference clock, using only the type of data actually available from a laboratory experiment.

We find that the triangulation method gives consistent results when applied to real atomic clocks: We measured the FM noise levels for each of a group of six cesium clocks and one NASA prototype hydrogen maser clock, using the triangulation method. We also combined the cesium clocks to form a time scale, against which the stability of the maser was measured. Direct measurement of the combined noise levels of the maser and the time scale gives very nearly the same result obtained by adding the noise levels of the maser and of the time scale, as shown in Figure 2. The time-scale noise level was calculated from the noise levels of its cesium clocks, as determined by triangulation. Some similar results have been obtained by D. B. Percival at the USNO [5].

Statistical Uncertainty

The scatter of repeated estimates of an Allan variance of an individual oscillator by triangulation arises partly because of the random noise of the oscillator being measured, but also, in part, because of the noise of the reference ensemble. Ideally, the noise of the reference ensemble would not contaminate the results if the correlation term in equation (2) could be evaluated. If it were retained, equation (4) would take the form:

$$\sigma_a^2 = \frac{1}{2}\left(\sigma_{ab}^2 + \sigma_{ac}^2 - \sigma_{bc}^2\right) + \frac{1}{M}\sum_{i=1}^{M}\left(\Delta y_{a_i}\Delta y_{b_i} + \Delta y_{a_i}\Delta y_{c_i} - \Delta y_{b_i}\Delta y_{c_i}\right) \tag{6}$$

If the oscillators are truly independent, the sum will clearly approach zero for many measurements (large M), and it will be positive as often as negative. Thus it does not bias the resulting value of an Allan variance, but it does contribute to the scatter of repeated estimates.

If the noise level of the oscillator being measured is low enough, and the scatter high enough, equation (4) may occasionally give a negative value for the variance. Yet we know, from the definition of variance, that it cannot be negative. To interpret such a result, observe that the triangulation procedure gives only an estimate of the variance. The true value lies within some bounds of uncertainty about the value given by equation (4). The actual bounds themselves could be the object of further study.

In the triangulation method, statistical uncertainty comes from two sources. First, the random character of the noise of the oscillator itself insures that it can only be estimated during a finite period of measurement. This source of uncertainty has been discussed by Lesage and Audoin [6]. Second, the noise of the reference ensemble enters, through our neglect of the correlation terms. For the same data used to prepare Figure 1, we have estimated the statistical uncertainty as follows: the computer was instructed to divide 800 days data into independent (non-overlapping) segments each being just large enough to permit the computation of one value of σ_a^2. These several estimates of σ_a^2 were used to compute the standard deviation of the mean. The standard deviation was calculated by the usual formula, with one modification. When the computer averaged the results of the $\frac{N(N-1)}{2}$ triads, it divided by $\frac{N(N-1)}{2}$ whereas, it is estimated that there are only $N-1$ degrees of freedom. To obtain the standard deviation, it should have divided by $N-1$. We permitted the computer to use $\frac{N(N-1)}{2}$ in calculating σ_a^2, then increased the standard deviation by a factor of $\sqrt{\frac{N}{2}}$. This appears to be a reasonable approximation which we have not tested for other values of N. It is probably not theoretically exact because it does not account for our use of a weighted average. However, the intention in computing this value was to enable us to compare similar measures of spread for the values of σ_a^2 obtained by triangulation and those values of σ_a^2 obtained directly from the simulated data. The standard deviation obtained from this direct calculation arises from the noise of oscillator A alone. We can subtract it from the standard deviation obtained in the triangulation calculation. The difference is the additional statistical uncertainty which results from the use of the triangulation method; physically it is due to the noise of the reference ensemble. Figure 3 shows the magnitude of this additional uncertainty in the measurement of the stability of each of our eight, computer-simulated clocks. It is interesting to note that for a sample time of one day, σ_1, for the high-performance clock is approximately ten times smaller than the σ's of the other seven clocks. Nevertheless, all of the clocks, including clock 1, show about the same statistical uncertainty of measurement due to the use of the triangulation method. That result is to be expected, because the reference ensemble is approximately as good when measuring any one of the eight clocks. The fact that it proves so, in Figure 3, lends confidence to our belief that our method of calculating the statistical uncertainty, is correct.

Acknowledgements

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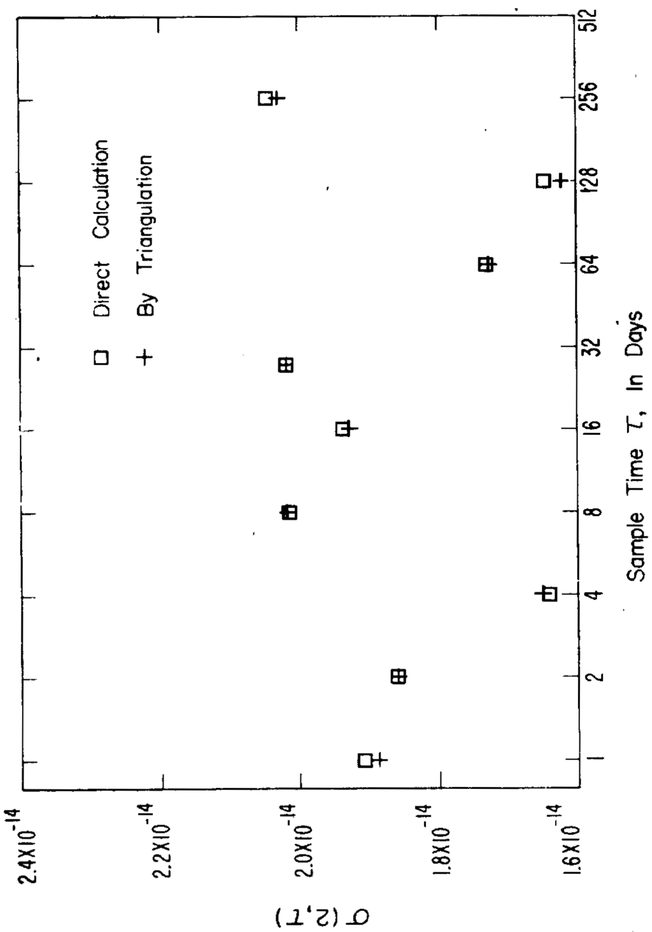

Figure 1. An Allan deviation of the frequency of a simulated cesium clock by triangulation and by direct calculation.

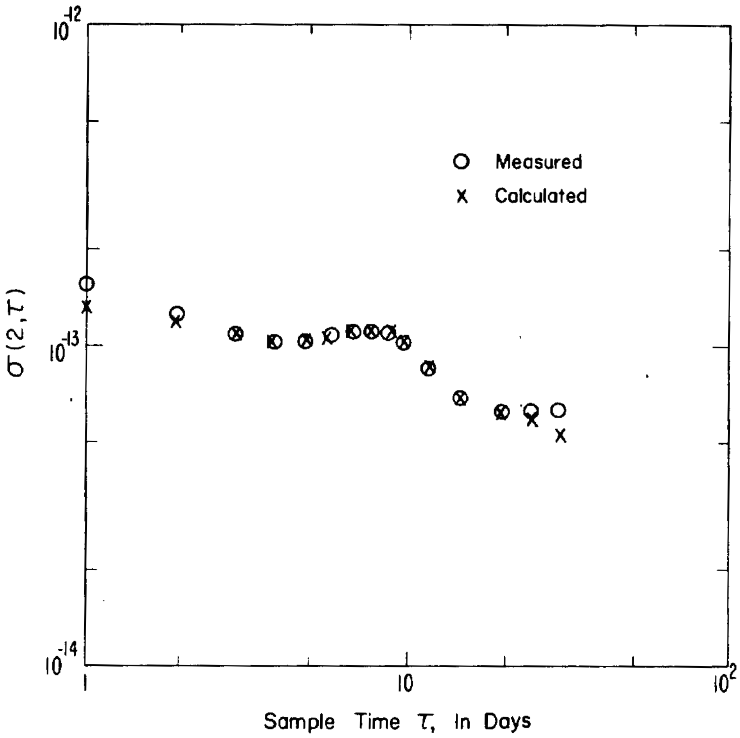

Figure 2. An Allan deviation of the relative frequency of a hydrogen maser clock and an ensemble of cesium clocks as calculated by triangulation and as measured.

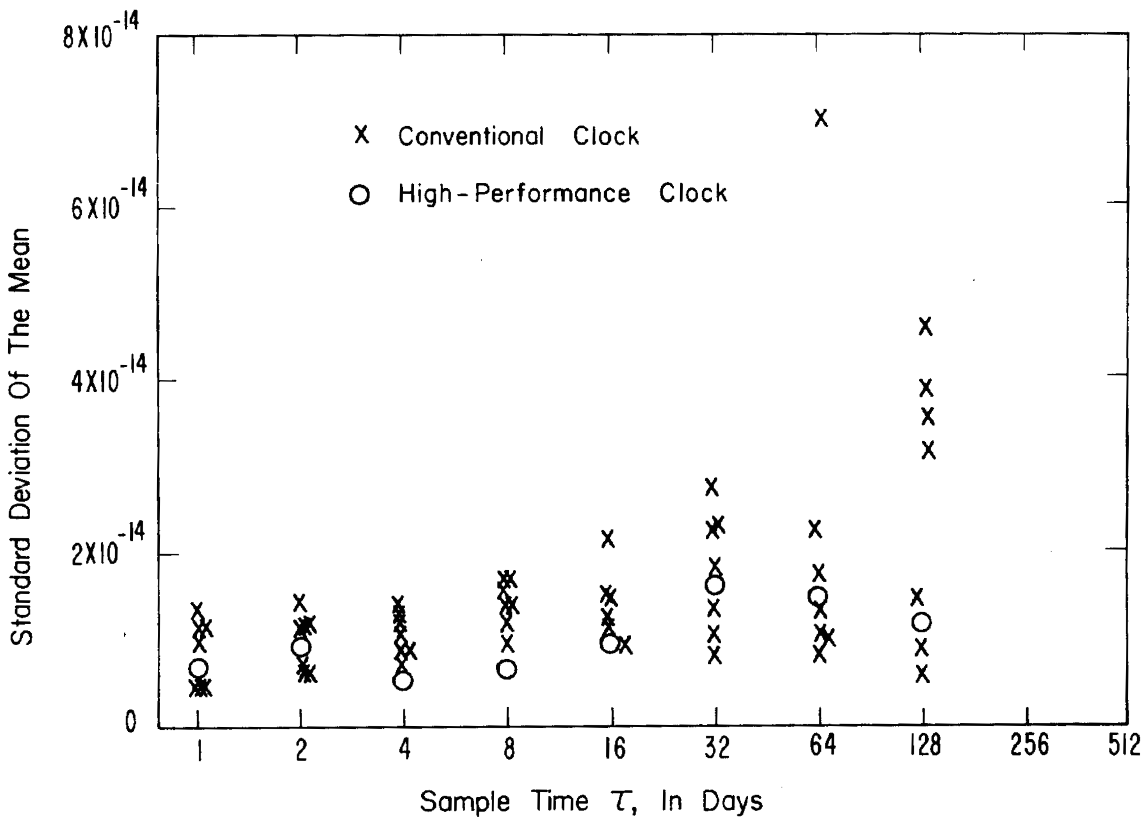

Figure 3. Statistical uncertainty introduced by a triangulation calculation for eight simulated cesium clocks.