$$\overline{b_n^2} = \frac{4e^2}{T^2}\overline{\left(\sum_{m=1}^{M} \sin\frac{2n\pi}{T}t_m\right)^2}. \qquad (34)$$

Since the average values of the cross terms are zero, this reduces to:

$$\overline{b_n}^2 = \frac{4e^2}{T^2}\frac{M\omega_m}{2\pi I_{dc}}\int_0^{2\pi/\omega_m} \sin^2\omega_m t\left[I_{dc} + AJ_2(\Delta\phi)\cos 2\omega_m t\right] dt$$

$$= \frac{2e}{T}\left[I_{dc} - \frac{A\omega_m}{4\pi}\int_0^{2\pi/\omega_m} J_2(\Delta\phi)\cos^2 2\omega_m t\, dt\right]$$

$$= \frac{2e}{T}\left[I_{dc} - \frac{1}{2}\Delta J_2(\Delta\phi)\right]$$

$$= 2e\left[C + \frac{A}{2}(1 + J_0(\Delta\phi) - J_2(\Delta\phi))\right]B_n. \qquad (35)$$

If we take the limiting bandwidth to be the low-pass bandwidth after phase sensitive detection, as we do in the text, the expression on the right must be multiplied by two to allow for the folding of the band when its center frequency is translated to zero.

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A Statistical Model of Flicker Noise

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***Abstract*—By the method of fractional order of integration, it is shown that it is possible to generate flicker noise from "white" noise. A formal expression for the relation of flicker noise to white noise is given. An approximate method, amenable to the use of digital computers, is also given for the generation of flicker noise modulated numbers from random, independent numbers.**

INTRODUCTION

FLICKER noise is a noise characterized by a *power* spectral density which varies inversely proportional to frequency. This type of noise has been found in many devices including semiconductors and quartz crystal oscillators. The presence of this type of noise modulating the frequency of high-quality oscillators has been of particular interest to the authors, and it is in this application that the present paper is directed. The principles used, however, are applicable to flicker noise in general but find convenient expression in terms of the phase and frequency of an oscillator.

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Many books in noise theory [1] establish the relation,

$$S_{\dot f}(\omega) = |\omega|^2 S_f(\omega) \qquad (1)$$

between the power spectral density $S_f(\omega)$ of a function $f(t)$ and the power spectral density $S_{\dot f}(\omega)$ of the derivative of $f(t)$. In the work which follows it will be assumed that if $\dot f(t)$ has a power spectral density $S_{\dot f}(\omega)$, then $f(t)$ also has a power spectral density $S_f(\omega)$. Equation (1) may be inverted to the form

$$S_f(\omega) = \frac{1}{|\omega|^2} S_{\dot f}(\omega). \qquad (2)$$

This integration $[\dot f(t)$ to $f(t)]$ cannot, however, guarantee the stationarity of $f(t)$ and, hence, it is not obvious that the power spectral density of $f(t)$ can be defined in an unambiguous fashion even though $\dot f(t)$ may be completely well behaved. Indeed the formal justification for the existence of a flicker noise power spectral density is lacking.

Statistical Model

Consider an ergodic ensemble of functions

$$f_n(t), \qquad n = 1, 2, 3, \cdots$$

such that the ensemble average is given by

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} f_n(t) \equiv \overline{f(t)} = 0, \tag{3}$$

where the bar indicates ensemble average. Since the ensemble is assumed ergodic, the autocovariance function $R_f(\tau)$ is given by

$$R_f(\tau) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} f_n(t+\tau) f_n(t)\, dt = \overline{f(t+\tau) f(t)},$$

and it will be further assumed that

$$R_f(\tau) = A\delta(\tau), \tag{4}$$

that is, white noise. The power spectral density of $f_n(t)$ is then given by

$$S_f(\omega) = k.$$

If now one assumes that there exists some function $g(t)$ such that

$$\frac{d}{dt} g(t) \equiv \dot{g}(t) = f(t)$$

according to (2) and the assumptions mentioned before,

$$S_g(\omega) = \frac{k}{|\omega|^2}.$$

Similarly, if $\dot{h}(t) = g(t)$, then

$$S_h(\omega) = \frac{k}{|\omega|^4}.$$

That is, each time the function is integrated, a factor of $|\omega|^{-2}$ is applied to the power spectral density. If we define $S_f^{(m)}(\omega)$ to be the power spectral density of the m-fold integral of $f(t)$, we may write

$$S_f^{(m)}(\omega) = \frac{k}{|\omega|^{2m}}.$$

In order to describe flicker noise frequency modulation, we are interested in the case where

$$2m = 1$$

or in other words the $\frac{1}{2}$th order integral of $f(t)$. In this model, $S_f^{(1/2)}(\omega)$ will be taken to represent the power spectral density of the instantaneous frequency. Since the instantaneous frequency of an oscillator is the derivative of the phase $\phi(t)$, one may write

$$S_\phi(\omega) = S_f^{(3/2)}(\omega)$$

for our present model of phase fluctuations.

The concept of fractional order of integration has long been developed and may be found in several references [2]. Thus we may consider an ensemble of oscillators whose phase $\phi_n(t)$ is given by the relation [3]

$$\phi_n(t) = \frac{1}{\Gamma(\lambda)} \int_0^t (t-u)^{\lambda-1} f_n(u)\, du \tag{5}$$

for $\lambda = \frac{3}{2}$, or

$$\phi_n(t) = \frac{1}{\Gamma(\frac{3}{2})} \int_0^t \sqrt{t-u}\, f_n(u)\, du.$$

One may now obtain several relations which are of use later:

$$\overline{\phi(t)} = \frac{2}{\sqrt{\pi}} \int_0^t \sqrt{t-u}\, \overline{f_n(u)}\, du = 0$$

from (3). Similarly

$$\begin{aligned}\overline{\phi(t)\phi(t+\tau)} &= \frac{4A}{\pi} \int_0^t \sqrt{(t-u)^2 + \tau(t-u)}\, du \\ &= \frac{A}{\pi} \Bigg\{ (2t+\tau)\sqrt{t^2 + t\tau} \\ &\quad - \frac{\tau^2}{2} \ln \left[\frac{2t + \tau + 2\sqrt{t^2 + t\tau}}{\tau} \right] \Bigg\}\end{aligned} \tag{6}$$

from (4) and the δ-function behavior of $\overline{f(u)f(u')}$. Also

$$\begin{aligned}\overline{[\phi(t)]^2} &= \frac{4A}{\pi} \int_0^t (t-u)\, du \\ &= \frac{2At^2}{\pi}.\end{aligned} \tag{7}$$

It should be noted that both (6) and (7) depend on t [i.e., $\phi_n(t)$ is nonstationary]. In order to further establish the connection of (5) with flicker noise, it is convenient to evaluate the quantity

$$\overline{(\Delta^2\phi)^2} \equiv \overline{[\phi(t+2\tau) - 2\phi(t+\tau) + \phi(t)]^2}.$$

That is the ensemble average of the square of the second difference of the phase [4]. One may write

$$\begin{aligned}\overline{(\Delta^2\phi)^2} &= \overline{[\phi(t+2\tau)]^2} + 4\overline{[\phi(t+\tau)]^2} + \overline{[\phi(t)]^2} \\ &\quad - 4\overline{[\phi(t+2\tau)\phi(t+\tau)]} - 4\overline{[\phi(t+\tau)\phi(t)]} \\ &\quad + 2\overline{[\phi(t+2\tau)\phi(t)]}.\end{aligned} \tag{8}$$

At this point the algebra becomes excessive. It is, however, of value to recognize that flicker noise is normally observed on equipment which has been operating for long periods of time. Thus, it is reasonable to consider the asymptotic behavior of (8) as $t/\tau \to \infty$. If one considers first the terms not involving logarithms, it is possible to expand these terms in τ^2 times a descending series in $\rho \equiv t/\tau$. It is found that the coefficients of ρ^j for $j \geq -1$ vanish identically and thus only the logarithmic terms contribute to the asymptotic value. Thus (8) becomes

$$\overline{(\Delta^2\phi)^2} \approx \frac{2A\tau^2}{\pi}\cdot\ln\left\{\frac{[2\rho+1+2\sqrt{\rho^2+\rho}][2\rho+3+2\sqrt{(\rho+1)^2+\rho+1}]}{[\rho+1+\sqrt{\rho^2+2\rho}]^2}\right\} \tag{9}$$

which reduces to

$$\overline{(\Delta^2\phi)^2} \approx \left(\frac{4A}{\pi}\right)\tau^2 \ln 2 \tag{10}$$

for large t/τ. Comparison of this result with another treatment of flicker noise [4] indicates complete agreement for the dependence on τ.

The Generation of "Flicker Noise Numbers"

While it might be possible to have an analog computer evaluate (5), it is of value to generate a series of numbers which behave analogously to (10) for discrete τ. That is if ψ_j [the analog to $\phi(t)$] is a variable defined over the range of the integer j, then

$$\langle(\psi_{j+2N} - 2\psi_{j+N} + \psi_j)^2\rangle = kN^2,$$

where k is a constant and the brackets indicate an average over the entire range of j (N is the discrete analog of τ).

One can show [5] that if a_j is also a discrete variable, the mth-fold finite integral of a_j is given by

$$(\Delta^{-1})^m a_j = \frac{1}{\Gamma(m)}\sum_{i=1}^{j}(j+1-i)^{[m-1]}a_i, \tag{11}$$

where the brackets on the exponent are defined to mean

$$\chi^{[l]} \equiv \chi(\chi+1)(\chi+2)\cdots(\chi+l-1). \tag{12}$$

Equation (11) is thus the discrete analog to (5). Unfortunately (12) does not have an obvious generalization to fractional exponents.

This problem was approached in an experimental fashion. A set of numbers ψ_j were generated from a set of random, independent numbers a_i obtained from reference [6]. The ψ_j were related to the a_i according to the equation

$$\psi_j = \sum_{i=1}^{j}\sqrt{j+1-i}\,a_i \tag{13}$$

and computed on a digital computer. Using programs described elsewhere [4], [7], an attempt to classify the statistics of the set ψ_j was made. If one assumes that

$$\sqrt{\langle(\psi_{j+2N} - 2\psi_{j+N} + \psi_j)^2\rangle} = \alpha N^\mu \tag{14}$$

the values of μ obtained were 0.83 and 0.84. From (10) one sees that for flicker noise, μ should be the integer one

A modification of (13) to the form

$$\psi_j = \sum_{i=1}^{j}(j+1-i)^{2/3}a_i \tag{15}$$

yielded data which conformed to (14) with a μ-value of one as closely as the experimental procedure allowed ($\mu = 1.00 \pm 0.05$). Thus (15) seems to generate flicker noise as precisely as our techniques of analysis can determine.

Conclusions

It has been shown that a half-order integral of white noise displays the properties of flicker noise. The existence of a formal expression relating flicker noise to white noise suggests the possibility of recognizing additional sources and physical mechanisms for the generation of flicker noise. It is possible to generate numbers with a digital computer [using (15)] which present properties similar to flicker noise, which is also of value. Thus it is possible to employ computer simulation of equipment perturbed by flicker noise processes.

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$$\phi_n(t) = \frac{1}{\Gamma(\lambda)}\int_0^t f_n(u)(t-u)^{\lambda-1}du$$

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