A UNITED STATES
DEPARTMENT OF
COMMERCE
PUBLICATION

NBS TECHNICAL NOTE 626

The Generation of an Accurate and Uniform Time Scale with Calibrations and Prediction

U.S.
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The Generation of an Accurate and Uniform Time Scale with Calibrations and Prediction

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U.S. DEPARTMENT OF COMMERCE, Peter G. Peterson, Secretary
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Issued November 1972

National Bureau of Standards Technical Note 626

Nat. Bur. Stand.(U.S.), Tech. Note 626, 64 pages (November 1972)
CODEN: NBTNAE

For sale by the Superintendent of Documents, U.S. Government Printing Office, Washington, D. C. 20402
(Order by SD Catalog No. C13.46:626.)

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Abstract

We express a predicted time interval (or frequency) of a single clock as a weighted sum of frequency data obtained by calibrations against a primary standard, and derive a matrix equation for the optimum weighting coefficients (called the optimum filter response function) involving the Allan variances. Two approaches are used. One of the approaches turns out to be a generalization of Barnes' approach described in his 1966 IEEE paper.

We solve the matrix equation to get the optimum filter response functions for white noise frequency modulation (FM), flicker noise FM and linear combinations of them. Other important time dispersive mechanisms exist in practice but are not considered here. We obtain the result that the mean square time prediction error would increase as elapsed time t^2 for the case without intermediate calibrations.

We obtain the result that with a small number of good clocks one can construct a time scale whose accuracy is limited by the accuracy of a primary standard. We show that, over a long time range, linear prediction algorithms based on frequency calibrations with a primary standard give a time scale of much better accuracy and stability than when intermittent calibrations are not used, and that (at least for statistically identical clocks), <u>no advantage is gained</u> by using the time scale itself as a "primary standard" for intermediate calibrations.

Key Words: Accurate and uniform time scale; Allan variance; Dispersion of time scale; Ensemble time (error); Prediction interval; Primary standard and clocks.

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1. INTRODUCTION

An atomic time scale can provide one of the best time scales in terms of uniformity, accuracy, reliability, and reasonable accessibility [1]. In the areas of accuracy and uniformity, it has a clear advantage over astronomical time scales. An atomic time scale can be actually realized with atomic frequency standards and appropriate counters; such devices exhibit undesirable phenomena such as frequency or time jumps, aging, and failures, as well as the signals being contaminated by random noises. All of those phenomena introduce very important problems in the generation of an atomic time scale. Random noises, however, which contaminate the signals from the atomic clocks--commonly white noise frequency modulation (FM) and flicker noise FM in cesium clocks--are included in the process of generating a time scale and result in time dispersion such as random walk and flicker walk. So these random noise processes can be said to contribute basically to the time scale error, and are the only processes treated in this paper.

In order to obtain a much better time scale than that generated by a single clock, a group of clocks is commonly used. There are various algorithms used, such as a weighted sum over clocks or the acceptance or rejection of clocks, to get as good a time scale as possible from the group of clocks. These algorithms involve assumptions about the behavior of each clock. In practice, laboratories adopt their own algorithms to generate local time scales and compare these time scales with each other [2]-[5]. It is common, however, for laboratories to try to obtain a more uniform time scale instead of a more accurate and uniform time scale, basically by taking a weighted sum (usually 0 or 1 weights [6]) over several clocks without using a primary standard. This can result in a significant drift away from the defined atomic frequency and an accumulating time error relative to an ideal time scale [7].

In a recent paper [3], Mungall compares and discusses differences between time scales which do or do not use a primary standard, and points out the problems in the latter time scale. It was also suggested in a recent conference [5] that a study be conducted on how to have concurrently both an accurate and uniform IAT scale (the International Atomic Time scale). In order to accomplish this, a primary standard must be tightly related to the generation of the time scale, namely it must be used to calibrate periodically secondary standards (or clocks) of lower accuracy. Periodic calibration is usually necessary because the primary standard can be operated only for brief periods and because frequent re-evaluations of its accuracy are necessary.

In this paper, we predict the time change for each clock between calibration intervals by filtering calibrated frequency data and derive a matrix equation for the optimum filter response functions which make the mean square predicted time error a minimum. We apply these results to the generation of a time scale from several clocks. We find that we can predict the correct time with quite a small error using a small number of clocks (even one), and thus generate a time scale whose accuracy is limited by the accuracy of the primary standard--the best case.

The dispersion of the time scale thus calculated is in a statistical sense essentially proportional to the square root of the elapsed time, while the time scale without intermediate calibrations may disperse proportionally to the elapsed time for the flicker noise FM case which is usually the dominant noise process after some time (more than 10 days, for example) has elapsed. So in a short time range, the latter time scale, which may use a large number of clocks, may be better in uniformity than the former; but in a very long range, the former may become better and better in accuracy and uniformity.

An approach in which the time of a secondary standard is predicted with a primary standard was presented in Barnes' 1966 IEEE paper [8]. One of two approaches described in this paper is a generalization of Barnes' approach to include any number of calibrations at arbitrary positions.

We also discuss the application of the linear prediction method to the case without a primary standard, trying to obtain better uniformity in a time scale than that due to accuracy of the primary standard for both short and long time range (where accuracy here is used for only the random uncorrelated contributions of the primary standard), and show that (at least for statistically identical clocks), no advantage is gained by using the time scale itself as a "primary standard" for intermediate calibrations.

2. ALGORITHMS TO OBTAIN THE OPTIMUM FILTER RESPONSE FUNCTIONS

Figure 1(a) shows an ideal case in which the time of a clock is moving away in a linear manner from the horizontal line--an ideal or defined time scale. If this slope is determined, the whole movement of the clock times can be perfectly predicted, so we can construct an ideal time scale using this clock by subtracting a linear term from its time. But signals from actual clocks are contaminated by noise--commonly by white noise FM (spectral density of frequency fluctuation $S_y(f) \propto f^0$) or flicker noise FM ($S_y(f) \propto f^{-1}$)or both (referred to symbolically as f^0 and f^{-1} FM noise). In this case we cannot predict perfectly the movement of the clock time, thus leaving some error of prediction as shown in figure 1(b). But by making the predicted error as small as possible, one may construct a time scale close to an ideal time scale. By taking a weighted sum of prediction errors for a group of clocks, we may obtain a smaller error in the time scale as shown with the dotted curve in figure 1(c). We call this weighted error the ensemble time error, or (for simplicity) the ensemble time. The ensemble time thus calculated can be used as a time reference (taking the place of an ideal time scale) with respect to which the time of any clock can be determined.

We will describe two approaches, or two prediction methods, using frequency data for a single clock obtained by calibrations against a perfect primary frequency standard. Deterministic frequency offset (or time drift) and frequency drift, however, are not considered here. Frequency or time jumps and any kinds of clock aging, which are also important phenomena to be considered for the generation of a time scale, are not dealt with either. With no loss of generality, we will assume that the noise processes are stationary through all the calculations. We must determine how best to predict the time slope or frequency.

A. Approach using calibrated frequency data which are located before a prediction interval

The first approach assumes the calibration intervals precede the prediction interval as shown in figure 2. Let the duration of the prediction interval be τ and the duration of the calibration interval be $h\tau$. Here, $h\tau$ and the $\ell\tau$ are not necessarily integral multiples of τ.

Over a frequency calibration with a primary standard, the average fractional frequency offset on the ℓ^{th} calibration interval due only to noise is written as

$$\bar{y}_\ell = \frac{1}{h\tau}\int_{t_\ell - h\tau}^{t_\ell} y(t)\,dt = \frac{x(t_\ell) - x(t_\ell - h\tau)}{h\tau} \tag{1}$$

$$\equiv (x_\ell - x_{\ell+h})/h\tau$$

where x is the clock time referred to an ideal time scale. Let us define the predicted average fractional frequency over the prediction interval as a summation of eq (1):

$$\hat{y}_0 = \sum_{\ell=\ell_1}^{\ell_L} a_\ell \bar{y}_\ell , \qquad \sum_{\ell=\ell_I}^{\ell_L} a_\ell = 1 \tag{2}$$

where the a_ℓ are weighting factors for $\bar{y}_\ell$. That is, the numbers a_ℓ weight the various measurements made in time on the single clock being considered. In a very real sense the a_ℓ define a response function in the time domain for a prediction filter.

We define the predicted time $\hat{x}_0$ at the end of the prediction interval as $\hat{x}_0 = x_1 + \tau \hat{y}_0$. Then the error is $\epsilon_0 = \hat{x}_0 - x_0$. Using (2), we can obtain then

$$\epsilon_0 = \tau \hat{y}_0 - (x_0 - x_1) = \frac{1}{h} \sum_{\ell} a_{\ell} \Delta x_{\ell, h} - \Delta x_{0, 1} \tag{3}$$

where $\Delta x_{\ell, h} = x_{\ell} - x_{\ell + h}$ and $\Delta x_{0, 1} = x_0 - x_1$.

The mean square prediction error is written as (4):

$$\begin{aligned} \langle \epsilon_0^2 \rangle = & \frac{1}{h^2} \Big[\sum_{\ell} a_{\ell} U_x(h\tau) + 2 \sum\sum_{\ell < \ell'} a_{\ell} a_{\ell'} \{ -U_x(k\tau) + \frac{1}{2} U_x\big((k + h)\tau\big) \\ & + \frac{1}{2} U_x\big((k - h)\tau\big) \} \Big] + \frac{1}{h^2} \sum_{\ell} a_{\ell} \{ U_x(\ell\tau) + U_x\big((\ell + h - 1)\tau\big) \\ & - U_x\big((\ell + h)\tau\big) - U_x\big((\ell - 1)\tau\big) \} + U_x(\tau) \end{aligned} \tag{4}$$

where $\langle \epsilon_0^2 \rangle$ denotes the ensemble average of ϵ_0^2, and $k \equiv \ell' - \ell$ and $U_x(k\tau) \equiv 2[R_x(0) - R_x(k\tau)]$ (see reference [9]). $R_x(\tau)$ is the autocorrelation of time.

Using the relation between the U-function and the Allan variance with zero dead time [9],

$$U_x(k\tau) = -k(k - 1)\, \tau^2 \langle \sigma_y^2(k, \tau) \rangle + k^2 U_x(\tau), \tag{5}$$

we can obtain

$$\langle \epsilon_0^2 \rangle = -\frac{2}{h^2}\tau^2 \sum\sum_{\ell < \ell'} a_\ell a_{\ell'} C(k) + \frac{2}{h}\tau^2 \sum_\ell a_\ell Q(\ell) - \frac{h-1}{h}\tau^2 \langle \sigma_y^2(h, \tau)\rangle \sum_\ell a_\ell^2 \tag{6}$$

where C(k) and $Q(\ell)$ are functions of the Allan variances given by

$$C(k) = -k(k-1)\langle \sigma_y^2(k, \tau)\rangle + \frac{1}{2}(k+h)(k+h-1)\langle \sigma_y^2(k+h, \tau)\rangle + \frac{1}{2}(k-h)(k-h-1)\langle \sigma_y^2(k-h, \tau)\rangle \tag{7}$$

and

$$Q(\ell) = \frac{1}{2}\{-\ell(\ell-1)\langle \sigma_y^2(\ell, \tau)\rangle - (\ell+h-1)(\ell+h-2)\langle \sigma_y^2(\ell+h-1, \tau)\rangle + (\ell+h)(\ell+h-1)\langle \sigma_y^2(\ell+h, \tau)\rangle + (\ell-1)(\ell-2)\langle \sigma_y^2(\ell-1, \tau)\rangle\}.$$

Equation (6) is the expression for the mean square of the predicted time error in terms of the Allan variances, frequency stability measures in the time domain.

In order to obtain the minimum value of eq (6), let us consider a function

$$F \equiv \langle \epsilon_0^2 \rangle + \lambda(\sum_\ell a_\ell - 1)$$

where λ is a Lagrange undetermined multiplier. Differentiating F with respect to a_ℓ with $\ell = \ell_p$ or $\ell = \ell_1$ and canceling out λ, we get a matrix equation for determining the optimum filter response functions;

$$\{C(\ell_p - \ell_1) - H\}\, a_{\ell_1} + \{H - C(\ell_p - \ell_1)\}\, a_{\ell_p} + \sum_{\substack{\ell \neq \ell_1 \\ \ell \neq \ell_p}} a_\ell \{C(\ell - \ell_p) - C(\ell - \ell_1)\} = h\{Q(\ell_p) - Q(\ell_1)\}$$

$$(\ell_p = \ell_2, \ell_3, \ldots, \ell_L) \qquad (8)$$

and $\sum_{\ell=\ell_1}^{\ell_L} a_\ell = 1$ where $H \equiv h(h-1)\langle \sigma_y^2(h, \tau)\rangle$.

In eqs (6) to (8), h and the ℓ need not necessarily be integers as we will see later; the Allan variances in such a case may be interpolated. The Allan variances for the various noise models are calculated and given in reference [10].

First, consider the white noise FM case where the Allan variance takes a simple form because of its independence of k:

$$\langle \sigma_y^2(k, \tau)\rangle = \sigma_y^2(\tau), \quad C(k) = h^2 \sigma_y^2(\tau), \quad Q(\ell) = h\sigma_y^2(\tau),$$

where $\sigma_y^2(\tau) = \langle \sigma_y^2(2, \tau)\rangle$. From the matrix equation, we get constant factors for all $\ell, a_\ell = I/L$, so the normalized mean square error will be:

$$\frac{\langle \epsilon_0^2\rangle}{\tau^2 \sigma_y^2(\tau)} = \frac{1}{h}\sum_\ell a_\ell^2 + 1 = \frac{1}{Lh} + 1 \qquad (9)$$

$$\rightarrow 1 \quad \text{as} \quad Lh \gg 1$$

$$\rightarrow 2 \quad \text{as} \quad = 1$$

$$\rightarrow \frac{1}{Lh} \quad \text{as} \quad << 1 .$$

Notice that the optimum a_ℓ does not include the clock parameter $\sigma_y^2(\tau)$, but only the parameter L, the number of calibrations used for prediction. For large Lh, eq (9) approaches 1, the smallest value.

For flicker noise FM, the Allan variance takes a more complicated form, but $C(k)$ or $Q(\ell)$ can be written in a product form of the Allan variance and other parameters.

$$\langle \sigma_y^2(k, \tau) \rangle = \frac{\sigma_y^2(\tau)}{c(1)} \cdot \frac{k\, \ell n\,(k)}{k - 1} ,$$

so

$$C(k) = \sigma_y^2(\tau)\, c(k)/c(1), \quad Q(\ell) = \sigma_y^2(\tau)\, q(\ell)/c(1)$$

where

$$\begin{aligned} c(1) &= 2\ \ell n\ 2 \\ c(k) &= -k^2 \ell n(k) + \frac{1}{2}(k+h)^2 \ell n(k+h) + \frac{1}{2}(k-h)^2 \ell n(k-h) \\ q(\ell) &= \frac{1}{2}\{-\ell^2 \ell n(\ell) - (\ell + h - 1)^2 \ell n(\ell + h - 1) \\ &\qquad + (\ell + h)^2 \ell n(\ell + h) + (\ell - 1)^2 \ell n(\ell - 1)\} . \end{aligned} \tag{10}$$

The mean square error and the matrix equation will be given by

$$\frac{\langle \epsilon_0^2 \rangle}{\tau^2 \sigma_y^2(\tau)/c(1)} = -\frac{2}{h^2} \sum\sum_{\ell < \ell'} a_\ell a_{\ell'}\, c(k) + \frac{2}{h} \sum_\ell a_\ell q(\ell) - \sum_\ell a_\ell^2\, \ell n(h) \tag{11}$$

and

$$\{c(\ell_p - \ell_1) - H\} a_{\ell_1} + \{H - c(\ell_p - \ell_1)\} a_{\ell_p} + \sum_{\substack{\ell \neq \ell_1 \\ \ell \neq \ell_p}} a_\ell \{c(\ell_p - \ell) - c(\ell - \ell_1)\} = h\{q(\ell_p) - q(\ell_1)\}$$

$$(\ell_p = \ell_1, \ell_2, \ldots, \ell_L) \tag{12}$$

respectively. Notice that the matrix equation does not contain the Allan variance, so the a_ℓ depend only on the calibration parameters.

For $f^0 + f^{-1}$ FM noise, the Allan variance is given by the sum of the Allan variance of each process, assuming the processes are independent;

$$\langle \sigma_y^2(k, \tau) \rangle = \langle \sigma_y^2(k, \tau) \rangle_{f^0} + \langle \sigma_y^2(k, \tau) \rangle_{f^{-1}} .$$

So the mean square error is given by

$$\frac{\langle \epsilon_0^2 \rangle}{\tau^2 \sigma_y^2(\tau)_{f^{-1}}} = r\left[\frac{1}{h} \sum_\ell a_\ell^2 + 1\right] + \frac{1}{c(1)} \left[- \frac{2}{h^2} \sum_{\ell < \ell'}\sum a_\ell a_{\ell'} c(k) + \frac{2}{h} \sum_\ell a_\ell q(\ell) - \sum_\ell a_\ell^2 \ell n(h)\right] \tag{13}$$

and the matrix equation by

$$\{c(\ell_p - \ell_1) - H\} a_{\ell_1} + \{H - c(\ell_p - \ell_1)\} a_{\ell_p}$$

$$+ \sum_{\substack{\ell \neq \ell_1 \\ \ell \neq \ell_p}} a_\ell \{c(\ell - \ell_p) - c(\ell - \ell_1)\} = h\{q(\ell_p) - q(\ell_1)\}$$

$$(\ell_p = \ell_1, \ell_2, \ldots, \ell_L) \qquad (14)$$

where $r = r(\tau) \equiv \sigma_y^2(\tau)_{f^0} / \sigma_y^2(\tau)_{f^{-1}}$ and $H \equiv h^2 \ell n(h) - r\, hc(1)$.

Figure 3 is a computed result for a prediction with only one calibration interval. The horizontal axis shows the time distance of the calibration interval from the prediction interval. The vertical axis shows the mean square prediction error normalized by that with calibration position equal to 1. The prediction error of the white noise FM case does not depend on the position of the calibration, while the prediction error of the flicker noise FM case is getting worse with the distance. So it is better with flicker noise FM to locate a calibration interval as close to the prediction interval as possible.

Figure 4 is the case where from two to five calibration intervals are used for a prediction. The last calibration is always located just preceding the prediction interval. But positions of others are changeable, which are shown on the horizontal axis as "spacing." The vertical axis shows the mean square prediction error normalized by τ^2 times the Allan variance.

As expected, with white noise FM, the prediction error does not depend on the position of calibration, only on how many calibrations are used for a prediction (actually on the total calibration time). With flicker noise FM, it depends on the position, as well as the number of calibrations used. In the two cases for flicker noise FM, there are optimum spacings which are shown in the diagram with black points where the mean square error becomes minimum.

Figure 5 shows the optimum filter response functions with L equal to 5. One is for a two-day (τ = 1 day) spacing between calibrations and the other for contiguous calibrations. For white noise FM, a_ℓ factors are always constant, while for flicker noise FM, the calibration interval nearest the prediction interval is always given the largest value, showing that it may be enough to use only one calibration interval nearest to the prediction interval. With contiguous calibration, the a_ℓ are oscillating, perhaps due to an aliasing phenomenon.

B. Approach using calibrated frequency data which are located in a prediction interval

In this approach using calibrated frequency data, the calibration intervals are located within the prediction interval as shown in figure 6. M and ℓ are not necessarily intergers. The calculation procedure is almost the same as that described for approach A.

The average fractional frequency offset due to noise over the calibration intervals is given by

$$\bar{y}_\ell = (x_\ell - x_{\ell-1})/\tau \qquad (\ell = \ell_1,\ \ell_2,\ \ldots,\ \ell_L)\ . \tag{15}$$

We define an estimated average fractional frequency offset $\hat{y}_{0M}$ during the 0 - M interval by a summation of (15), and a predicted time interval $(\hat{x}_M - x_0)$ as follows:

$$\hat{y}_{0M} = \sum_{\ell=\ell_1}^{\ell_L} a_\ell \bar{y}_\ell\ , \quad \sum_\ell a_\ell = 1,\quad \hat{x}_M - x_0 = M\tau\hat{y}_{0M}. \tag{16}$$

The predicted time error is

$$\epsilon_{0M} = \hat{x}_M - x_M = M\tau\hat{y}_{0M} - (x_M - x_0)$$
$$= M\sum_{\ell} a_{\ell}\Delta x_{\ell} - \Delta x_{0M} \tag{17}$$

where $\Delta x_{\ell} = \tau\bar{y}_{\ell} = x_{\ell} - x_{\ell-1}$ and $\Delta x_{0M} = x_M - x_0$.

Then the mean square of the predicted time error is finally given by

$$\langle\epsilon_{0M}^2\rangle = -2M^2\tau^2\sum\sum_{\ell<\ell'} a_{\ell}a_{\ell'}C(k) + 2M\tau^2\sum_{\ell} a_{\ell}Q(\ell) - \tau^2M(M-1)\langle\sigma_y^2(M,\tau)\rangle \tag{18}$$

where $k = \ell' - \ell$, and

$$C(k) = -k(k-1)\langle\sigma_y^2(k,\tau)\rangle + \frac{1}{2}(k+1)k\langle\sigma_y^2(k+1,\tau)\rangle + \frac{1}{2}(k-1)(k-2)\langle\sigma_y^2(k-1,\tau)\rangle$$

and (18′)

$$Q(\ell) = \frac{1}{2}\{-(M-\ell)(M-\ell-1)\langle\sigma_y^2(M-\ell,\tau)\rangle + (M-\ell+1)(M-\ell)\langle\sigma_y^2(M-\ell+1,\tau)\rangle - (\ell-1)(\ell-2)\langle\sigma_y^2(\ell-1,\tau)\rangle + \ell(\ell-1)\langle\sigma_y^2(\ell,\tau)\rangle\} .$$

From eq (18), the matrix equation for the optimum a_ℓ is

$$C(\ell_p - \ell_1)\, a_{\ell_1} - C(\ell_p - \ell_1)\, a_{\ell_p} - \sum_{\substack{\ell \neq \ell_1 \\ \ell \neq \ell_p}} a_\ell \{C(\ell - \ell_p) - C(\ell - \ell_1)\} = \frac{1}{M}\{Q(\ell_p) - Q(\ell_1)\}$$

$$(\ell_p = \ell_2,\ \ell_3,\ \ldots,\ \ell_L) \qquad (19)$$

and $\sum_\ell a_\ell = 1$.

For white noise FM, the optimum response functions again become constant. Then the normalized mean square error will be given by

$$\frac{\langle \epsilon_{0M}^2 \rangle}{\tau^2 \sigma_y^2(\tau)} = M^2 \sum_\ell a_\ell^2 - M = \frac{M - L}{L} \qquad (20)$$

$$\begin{aligned} &\to M^2/L && \text{as} \quad M \gg L \\ &\to 0 && \text{as} \quad M = L \\ &\to M(M-1) && \text{as} \quad L = 1 \end{aligned}$$

where $a_\ell = 1/L$.

For flicker noise FM, the mean square error is given by eq (21) and the matrix equation for the optimum a_ℓ by eq (22).

$$\frac{\langle \epsilon_{0M}^2 \rangle}{\tau^2 \sigma_y^2(\tau)/c(1)} = -2M^2 \sum\sum_{\ell < \ell'} a_\ell a_{\ell'}\, c(k) + 2M \sum_\ell a_\ell q(\ell) - M^2 \ell n(M) . \qquad (21)$$

$$c(\ell_p - \ell_1) a_{\ell_1} - c(\ell_p - \ell_1) a_{\ell_p}$$

$$+ \sum_{\substack{\ell \neq \ell_1 \\ \ell \neq \ell_p}} a_\ell \{c(\ell - \ell_p) - c(\ell - \ell_1)\} = \frac{1}{M} \{q(\ell_p) - q(\ell_1)\}$$

$$(\ell_p = \ell_2, \ \ell_3, \ \ldots, \ \ell_L) \tag{22}$$

where

$$c(k) = -k^2 \ell n(k) + \frac{1}{2}(k + 1)^2 \ell n(k + 1) + \frac{1}{2}(k - 1)^2 \ell n(k - 1)$$

and (23)

$$q(\ell) = \frac{1}{2} \{-(M - \ell)^2 \ell n(M - \ell) + (M - \ell + 1)^2 \ell n(M - \ell + 1)$$

$$- (\ell - 1)^2 \ell n(\ell - 1) + \ell^2 \ell n(\ell)\} \ .$$

Now let us consider the case where there is only one calibration in a prediction interval in figure 6. Equation (23) will be reduced to

$$\frac{\langle \epsilon_{0M}^2 \rangle}{\tau^2 \sigma_y^2(\tau)} = \frac{M}{c(1)} \{2q(\ell_1) - M \ \ell n(M)\} \tag{24}$$

where

$$q(\ell_1) = \frac{1}{2} \{-(M - \ell_1)^2 \ell n(M - \ell_1) + (M - \ell_1 + 1)^2 \ell n(M - \ell_1 + 1)$$

$$- (\ell_1 - 1)^2 \ell n(\ell_1 - 1) + \ell_1^2 \ell n(\ell_1)\} \ .$$

In eq (24), for example, let M = 3 and locate the calibration interval in the middle of the prediction interval. Then the result coincides with eq (31) of Barnes' paper published by IEEE in 1966 [8]. And for another example, let $M = 2n + 1$, where n is an integer, and again locate the calibration interval in the middle of the prediction interval--namely, $\ell_1 = n + 1$; the result again coincides with eq (61) of Barnes' paper. Thus eq (24), where ℓ_1 and M are not necessarily integers, seems to be the analytical solution of Barnes' eq (47). Thus approach B, which can include any number of calibrations, can be said to be the generalization of Barnes' approach.

For $f^0 + f^{-1}$ FM noise, the mean square error and matrix equations will be given by (25) and (26).

$$\frac{\langle \epsilon_{0M}^2 \rangle}{\tau^2 \sigma_y^2(\tau)_{f^{-1}}} = rM^2\Big(\sum_{\ell} a_\ell^2 - \frac{1}{M}\Big) + \frac{1}{c(1)}\Big[-2M^2 \sum\sum_{\ell<\ell'} a_\ell a_{\ell'} c(k) + 2M\sum_{\ell} a_\ell q(\ell) - M^2 \ell n(M)\Big] \tag{25}$$

where $r = r(\tau) \equiv \sigma_y^2(\tau)_{f^0} / \sigma_y^2(\tau)_{f^{-1}}$.

$$\{rc(1) + c(\ell_p - \ell_1)\} a_{\ell_1} - \{rc(1) + c(\ell_p - \ell_1)\} a_{\ell_p} + \sum_{\substack{\ell \neq \ell_1 \\ \ell \neq \ell_p}} a_\ell \{c(\ell - \ell_p) - c(\ell - \ell_1)\} = \frac{1}{M}\{q(\ell_p) - q(\ell_1)\}$$

$$(\ell_p = \ell_2, \ell_3, \ldots, \ell_L) \tag{26}$$

Figure 7 shows the mean square prediction error for one calibration in a week. Calibration duration is supposedly one day. The horizontal axis is the position of the calibration interval. With white noise FM, the mean square error does not depend on the position. With flicker noise FM, locating a calibration in the center gives the best result. It can be shown that predictions based on calibrations located symmetrically about the center of the prediction interval give the same error.

Figure 8 shows the mean square error versus number of calibrations in a week. Again calibration duration is supposedly one day. For flicker noise FM, optimum calibration distributions are selected for calibration number 1 to 3. Quick improvement with the number of calibrations is shown. The dotted curve shows the improvement due to a number of clocks without calibration, assuming each has the same noise level, and assuming constant weighting factors for each clock.

Figure 9 shows the influence of calibration duration. Two calibrations per week and optimum--for f^{-1} noise FM--locations for calibrations are selected. The vertical axis is the mean square error normalized by that of the 24-hour calibration duration. As expected, longer calibration duration is preferable.

Figure 10 shows the optimum filter response functions a_ℓ for flicker noise FM. It is interesting to note that two a_ℓ factors for calibrations symmetrically located have the same value, analogous to the white noise FM case.

Figure 11 shows the influence of increasing the size of a single prediction interval, M_0, where a rate of two calibrations per week is used and their positions are fixed. For both white noise FM and flicker noise FM, the error is almost linearly increasing with the size of the prediction interval, instead of increasing with M_0^2 for flicker noise FM. (Similar results were obtained by Barnes [8].) The accumulated mean square error over M/7 prediction periods of seven days each lies essentially on the same curve; we shall discuss this in detail later.

3. ENSEMBLE TIME USING SEVERAL CLOCKS

With approach B, it may be possible to increase the size of a prediction interval and construct a useful ensemble time with only one prediction interval. From the predicted time $\hat{x}_M^i$ of each clock at M, we construct an ensemble time estimate using weights w_M^i for each of the N clocks:

$$\hat{T}_M^{ens} = \sum_{i=1}^{N} w_M^i \hat{x}_M^i, \quad \sum_{i=1}^{N} w_M^i = 1.$$

The correct value for this ensemble time is

$$T_M^{ens} = \sum_{i=1}^{N} w_M^i x_M^i,$$

so the ensemble time error $\hat{T}_M$ is

$$\hat{T}_M = T_M^{ens} - \hat{T}_M^{ens}$$

or

$$\hat{T}_M = \sum_{i=1}^{N} w_M^i \epsilon_M^i . \qquad (27)$$

The mean square of $\hat{T}_M$ in eq (27) is

$$\langle \hat{T}_M^2 \rangle = \sum_i w_M^{i2} \langle \epsilon_{0M}^{i2} \rangle \tag{28}$$

since the clocks are independent.

The optimum value of the weighting factors which makes the mean square of the ensemble time error minimum will be

$$w_{opt}^i = \left(\sum_i \frac{1}{\langle \epsilon_{0M}^{i2} \rangle} \right)^{-1} \frac{1}{\langle \epsilon_{0M}^{i2} \rangle} \tag{29}$$

by the same calculation procedure as described for eq (8). The minimum value of the mean square error is then:

$$\langle \hat{T}_M^2 \rangle_{min} = \left(\sum_i \frac{1}{\langle \epsilon_{0M}^{i2} \rangle} \right)^{-1} \tag{30}$$

For white noise FM or flicker noise FM, the mean square error of one clock involves only the Allan variance to describe the clock. So under the assumption that all the N clocks have the same type of noise--white noise FM or flicker noise FM, the optimum weighting factors and the minimum mean square error will be expressed in simple forms as

$$w_{opt}^i = \left(\sum_i \frac{1}{\sigma_y^{i2}(\tau)} \right)^{-1} \frac{1}{\sigma_y^{i2}(\tau)} \tag{31}$$

and

$$\langle \hat{T}_M^2 \rangle_{min} = K_M \tau^2 \left(\sum_i \frac{1}{\sigma_y^{i2}(\tau)} \right)^{-1} \tag{32}$$

where K_M is constant, calculable from eqs (20), (21), and (30).

In general, for mixed noises, optimum weighting factors can not be expressed only with the Allan variances, but include other parameters related to calibration and prediction methods.

For the general case, with either approach A or B, where many prediction intervals are connected in series (see fig. 19), the change in ensemble time over the (M - 1, M) interval can be written:

$$T_M^{ens} - T_{M-1}^{ens} = \sum_{i=1}^{N} w_M^i (x_M - x_{M-1}), \sum w_M^i = 1,$$

and its estimate

$$\hat{T}_M^{ens} - \hat{T}_{M-1}^{ens} = \sum_{i=1}^{N} w_M^i (\hat{x}_M - \hat{x}_{M-1}) .$$

Then the change in ensemble time error is

$$\hat{T}_M - \hat{T}_{M-1} = \sum_{i=1}^{N} w_M^i \epsilon_M^i ,$$

or

$$\hat{T}_M = \sum_{j=1}^{M} \sum_{i=1}^{N} w_j^i \epsilon_j^i , \quad (\hat{T}_0 \equiv 0) , \tag{33}$$

where the ϵ_j^i is the prediction error for the i^{th} clock on the (j - 1, j) interval, and the initial error is assumed to be zero.

The mean square of eq (33) will be

$$\langle \hat{T}_M^2 \rangle = \sum_i \langle (\sum_j w_j^i \epsilon_j^i)^2 \rangle$$

$$= \sum_i \left[\sum_j w_j^{i2} \langle \epsilon_j^{i2} \rangle + 2 \sum_{j<j'}\sum w_j^i w_{j'}^i \langle \epsilon_j^i \cdot \epsilon_{j'}^i \rangle \right] \tag{34}$$

under independence of each clock.

The mean square ensemble error now contains covariance terms, in addition to the mean square terms we have had before. Equations for the covariance terms can also be derived by calculation procedure similar to those already used.

We can also construct a matrix equation for weighting factors from eq (34)

$$2 w_j^i \langle \epsilon_j^{i2} \rangle + 2 \sum_{j' \neq j} w_{j'}^i \langle \epsilon_j^i \cdot \epsilon_{j'}^i \rangle + \lambda_j = 0 \tag{35}$$
$$(i = 1, 2, \ldots, N \text{ and } j = 1, 2, \ldots, M)$$

and $\sum_i w_j^i = 1$ for all j where λ_j is a Lagrange undetermined multiplier. But the size of this matrix is the square of N times M, which will unlimitedly increase with M, and the weighting factors will be functions of M. This seems impractical.

For white noise FM, however, since the covariance term disappears, the results become simple.

$$w_{j\,\mathrm{opt}}^i = \left(\sum_i \frac{1}{\langle \epsilon_j^{i2} \rangle} \right)^{-1} \frac{1}{\langle \epsilon_j^{i2} \rangle} \longrightarrow \left(\sum_i \frac{1}{\sigma_y^{i2}(\tau)} \right)^{-1} \frac{1}{\sigma_y^{i2}(\tau)} \tag{36}$$

and

$$\langle \hat{T}_M^2 \rangle_{min} = \sum_j \left(\sum_i \frac{1}{\langle \epsilon_j^{i2} \rangle} \right)^{-1} . \tag{37}$$

In general, certain clocks in the ensemble of clocks may come to be rejected with increasing time on account of their prediction errors getting much larger than others. Then it may be preferable to construct the ensemble of clocks with a smaller number of clocks all of which have the same type of noise. Approach B, in this sense, seems better than approach A because it may give a simple form for the weighting factors even for mixed noises (see fig. 11).

Under the assumption of a fixed weighting factor for each interval ($w_j^i = w^i$), again we can get simple forms:

$$\langle \hat{T}_M^2 \rangle = \sum_i w^{i2} \langle (\sum_j \epsilon_j^i)^2 \rangle . \tag{38}$$

So

$$w_{opt}^i = \left(\sum_i \frac{1}{\langle (\sum_j \epsilon_j^i)^2 \rangle} \right)^{-1} \frac{1}{\langle (\sum_j \epsilon_j^i)^2 \rangle} \tag{39}$$

and

$$\langle \hat{T}_M^2 \rangle = \left(\sum_i \frac{1}{\langle (\sum_j \epsilon_j^i)^2 \rangle} \right)^{-1} . \tag{40}$$

We calculated $\langle (\sum_j \epsilon_j^i)^2 \rangle$ in (38) in order to obtain some numerical idea of the mean square error of the ensemble time. For approach B, we already showed an example of this quantity in figure 11, so here some examples for approach A will be explained.

Figures 12 and 13 show the influence of positions of calibrations. An almost equally spaced distribution will give the best result for the flicker case as shown with (1). Figure 12 is for one calibration used for a prediction and figure 13 is for two calibrations used for a prediction. It should be pointed out that the slope of the mean square errors $\hat{T}_M^2(M)$ may actually be smaller for flicker noise than for white noise, although this is not indicated in the figures, where the normalization influences the slope.

Figure 14 is the accumulated error in one week versus the number of calibrations in a week. Rapid improvement with increasing rate is shown for flicker noise FM.

Figure 15 is the accumulated error versus the number of calibrations used for a prediction. Calibration rate is two calibrations per week. For white noise FM, rapid improvement with L is shown and for flicker noise FM, $L = 1$ gives a local minimum independent of M. For $f^0 + f^{-1}$ FM noise with $r = 10$, it is apparent that there are optimum values L. Improvement with larger L, however, seems quite small, so setting L equal to 1 may be sufficient for flicker noise FM or white and flicker FM noise in approach A.

Figure 18 shows the simulation results of a time scale using one clock for mixed noises shown in figures 16 and 17, using a white noise and flicker noise model [11]. In the diagram, improvement by prediction is clearly shown with curves A and B, which are predicted time errors for curve S, the time error of the mixed frequency noise given in figures 16 and 17. We will discuss figure 18 in more detail later.

4. CONSIDERATION OF "ACCURACY"

All the calculations so far mentioned took into consideration only noises of clocks--white or flicker noise FM. But actually, precision of measurements and "accuracy" of the primary standard must be also considered. Here, effects of random uncorrelated noise as it affects the accuracy of the primary standard, denoted "accuracy," will be discussed; the imprecision of the measurements will be assumed negligible.

We shall assume that the total average fractional frequency offset may be expressed as a sum of two independent random variables,

$$\bar{y}^{i}_{\ell_j} = \bar{y}^{0i}_{\ell_j} + \delta \bar{y}_{\ell_j} \tag{41}$$

where the first term depends on clock noise and the second on noise (inaccuracy) in the primary standard. Then the predicted average fractional frequency offset will be

$$\hat{y}^{i}_{j} = \sum_{\ell_j} a^{i}_{\ell_j} \bar{y}^{i}_{\ell_j} = \sum_{\ell_j} a^{i}_{\ell_j} \bar{y}^{0i}_{\ell_j} + \sum_{\ell_j} a^{i}_{\ell_j} \, \delta \bar{y}_{\ell_j} , \tag{42}$$

and the predicted time error will be

$$\epsilon^{i}_{j} = \tau \hat{y}^{i}_{j} - \Delta x^{i}_{j} \tag{43}$$

$$= \epsilon^{0i}_{j} + \sum_{\ell_j} a^{i}_{\ell_j} (\tau \, \delta \bar{y}_{\ell_j}) \equiv \epsilon^{0i}_{j} + \delta \epsilon^{i}_{j}$$

where

$$\epsilon^{0i}_{j} \equiv \sum_{\ell_j} a^{i}_{\ell_j} (\tau \bar{y}^{0i}_{\ell_j}) - \Delta x^{i}_{j} \quad \text{and} \quad \delta \epsilon^{i}_{j} \equiv \sum_{\ell_j} a^{i}_{\ell_j} (\tau \, \delta \bar{y}_{\ell_j}) . \tag{44}$$

Then the mean square prediction error, finally, must be written as a sum of the noise dependent term and an "accuracy" dependent one as

$$\langle \epsilon_j^{i2} \rangle = \langle \epsilon_j^{0i2} \rangle + \langle \delta\epsilon_j^{i2} \rangle . \tag{45}$$

Assuming $\langle \delta \overline{y}^2_{\ell_j} \rangle = \langle \delta \overline{y}^2 \rangle$ and using equation (44),

$$\langle \delta\epsilon_j^{i2} \rangle = \langle (\sum_{\ell_j} a^i_{\ell_j} \tau \delta \overline{y}_{\ell_j})^2 \rangle = \sum_{\ell_j} a^{i2}_{\ell_j} \cdot (\tau^2 \langle \delta \overline{y}^2 \rangle). \tag{46}$$

where covariance components of inaccuracy are neglected for simplicity. It must be borne in mind that the correlated components of accuracy, which are in general unknown, may be of primary importance, particularly in long term. Then equation (45) will be written as

$$\langle \epsilon_j^{i2} \rangle = \langle \epsilon_j^{0i2} \rangle + \sum_{\ell_j} a^{i2}_{\ell_j} \cdot (\tau^2 \langle \delta \overline{y}^2 \rangle) . \tag{47}$$

Some correction must be made for the matrix equations derived in section 2. But since the second term in equation (47) gives the same effect as white noise FM, no correction of a_ℓ factors for white noise FM is necessary. With the particular cases for flicker noise FM shown in figure 10 where a_ℓ factors are constant, no correction is necessary either.

From equation (33), the ensemble time error and its mean square will be corrected to

$$\hat{T}_M = \sum_i \sum_j w^i_j \epsilon^i_j = \sum_i \sum_j w^i_j (\epsilon_j^{0i} + \delta \epsilon^i_j) \tag{48}$$

and

$$\langle \hat{T}_M^2 \rangle = \langle (\sum_i \sum_j w_j^i \epsilon_j^i)^2 \rangle + \langle (\sum_i \sum_j w_j^i \delta\epsilon_j^i)^2 \rangle \tag{49}$$

respectively.

In order to get a rough numerical estimation of the "accuracy," we assumed constant weighting factors for

$$w_j^i = 1/N = \text{const. and } a_{\ell_j}^i \equiv a_\ell$$

used to obtain equation (50) as the mean square error of the ensemble time. Then:

$$\langle \hat{T}_M^2 \rangle = \frac{1}{N} \langle (\sum_j \epsilon_j^i)^2 \rangle + M \cdot A \cdot \tau^2 \langle \delta \overline{y}^2 \rangle \tag{50}$$

where

$$A \equiv \sum_\ell a_\ell^2 \; (\leq 1) = \text{const.}$$

The first term in equation (50) is the noise dependent one and the second is "accuracy" dependent, assuming constant mean square of the inaccuracy for each interval. The mean square accumulated error for one clock will take the form of a product of the Allan variance and some coefficient, so if

$$\langle (\sum_j \epsilon_j^i)^2 \rangle = \tau^2 \sigma_y^2(\tau) \, k \cdot M ,$$

then eq (50) will be simplified as

$$\langle \hat{T}_M^2 \rangle / \tau^2 \sigma_y^2(\tau) = k \cdot M/N + MA \langle \delta \overline{y}^2 \rangle / \sigma_y^2(\tau) \tag{51}$$

where k is constant.

For the generation of the most accurate and uniform time scale, the first term in eq (51) should be negligibly small compared to the second. The condition for that will be given by

$$k \cdot M/N \ll MA \langle \delta \overline{y}^2 \rangle / \sigma_y^2(\tau) \tag{52}$$

for which

$$\langle \hat{T}_M^2 \rangle \approx MA \tau^2 \langle \delta \overline{y}^2 \rangle . \tag{53}$$

Notice that the ratio of mean square frequency error due to "accuracy" to the Allan variance due to noises is related to this condition. In eq (53) the mean square ensemble time error due to inaccuracy increases linearly with M, as does the prediction error as mentioned before.

Table I gives a numerical comparison between the error dependent on prediction and error dependent on "accuracy," related to condition (52). If one has a clock 10 times as good in stability as a primary standard in accuracy, then in seven days the value of the right hand in eq (52) will be 700. There is also shown in the table numerical examples of the coefficient $k \cdot M$ by approach B. Take (2) for instance, namely two calibrations in a week and 24-hour calibration duration: as may be seen, those values of $k \cdot M$ are sufficiently small compared to 700. So in this case, we can construct a good time scale with only one clock, which is very simple for the generation of a time scale. With the value coming from a condition of two calibrations per week and six-hour calibration duration, one needs several clocks for condition (52), even using such good clocks.

5. COMPARISONS BETWEEN TIME SCALES

We have obtained the result that the mean square of the predicted time error with a primary standard increases with M (or t_1, elapsed time) for white noise FM, flicker noise FM, or primary standard inaccuracy due to random uncorrelated noise. One can see this from the sample shown in figure 18 by comparing curves A and B with curve W which is time error due to pure white noise, which also represents time error due to "inaccuracy": those curves appear to have similar slopes.

It is well-known that the mean square of a simply accumulated time error (without prediction and calibrations) increases proportionally to elapsed time t_1 for white noise FM and to t_1^2 for flicker noise FM. For large value of t_1 ($\geq$ 10 days, for example), however, flicker noise FM will be usually dominant (see fig. 17), and for larger value of t_1, the dominant spectral density $S_y(f)$ may be proportional to f^{-2} for which mean square time error is proportional to t^3.

Curve S in figure 18 which is a time error for one clock can be compared for that with curve A or B or W. (Notice that even in the prediction method, if time error due to a certain noise, f^{-2} for example, is not included in the prediction, then it will become dominant and give the same time error (i.e., t^3) as its simply accumulated time error after long time has elapsed.)

Let us consider the ideal case where the predicted time error is much smaller than that due to inaccuracy; then from eq (53), the mean square time error is given

$$\langle \hat{T}_M^2 \rangle_{pred} \approx t_1 \tau \langle \delta \overline{y}^2 \rangle \tag{54}$$

where $t_1 = M\tau$. The mean square of a simply accumulated time error averaged over N clocks will be [10]

$$\langle \hat{T}_M^2 \rangle_{simple} \approx t_1^2 \sigma_y^2 (t_1)/N \tag{55}$$

where constant weighting factors over clocks are assumed.

Then the ratio of the mean square errors is

$$\langle \hat{T}_M^2 \rangle_{pred} / \langle \hat{T}_M^2 \rangle_{simple} = r_a N/M \tag{56}$$

where $r_a \equiv \langle \delta \overline{y}^2 \rangle / \sigma_y^2 (t_1)$.

Let $r_a = 10^2$ for instance, then the both mean square errors will be the same in 300 days ($\approx$ 1 year) with N = 3 and 1000 days ($\approx$ 3 years) with N = 10. After that a time scale generated by the prediction method with a primary standard will be getting better and better relative to the other. It also has to be considered that the "accuracy" of primary standards will continue to be improved [12].

6. ON AN APPROACH USING TIME DIFFERENCE DATA REFERRED TO THE ENSEMBLE TIME

A time scale generated by the prediction method with a primary standard is restricted in its uniformity by the accuracy of the primary standard used, which will be commonly worse than the frequency stability of clocks. The uniformity of a time scale by the method of a simple weighted sum over clocks, namely without prediction and calibration, may be much improved for a short time range by increasing the number of clocks. (See eq (55).) So, it will be natural to expect to obtain a better uniformity and accuracy in a time scale by generating the time scale with the optimum time prediction and with an initial calibration, but without intermediate calibrations.

In this case, one might use the ensemble time as the reference, instead of a primary standard. So the time difference (or frequency) data of each clock used for the frequency prediction with approach A would be referred to the ensemble time as shown with notation t in figure 19. The following discussion on this approach, however, will show that little improvement can be expected.

Let the predicted frequency or time difference be given as a linear function of time difference data--L in number:

$$\tau\, \hat{y}^{i}_{M} = f^{i}_{M}(\Delta t^{i}_{M-1}, \ldots, \Delta t^{i}_{M-L})$$
$$\equiv \sum_{\ell=1}^{L} a^{i}_{M\ell}\, \Delta t^{i}_{M-\ell} \tag{57}$$

where τ_M is the length of each (prediction) interval, f^{i}_{M} is a linear operator on $\{\Delta t^{i}_{j}\}$ at the M^{th} interval and $\Delta t^{i}_{M-\ell}$ are time difference data for i^{th} clock referred to the ensemble time at $M - \ell^{th}$ interval whose duration is $\tau_{M-\ell}$ (see fig. 19):

$$\Delta t^i_{M-\ell} \equiv t^i_{M-\ell} - t^i_{M-\ell-1} \quad \text{and} \quad t^i_{M-\ell} \equiv x^i_{M-\ell} - \hat{T}_{M-\ell-1} \, ,$$

so

$$\Delta t^i_{M-\ell} = (x^i_{M-\ell} - x^i_{M-\ell-1}) - (\hat{T}_{M-\ell} - \hat{T}_{M-\ell-1})$$

$$\equiv \Delta x^i_{M-\ell} - \Delta \hat{T}_{M-\ell} \quad . \tag{58}$$

In the approaches so far mentioned, the $a^i_{M\ell} = a^i_{\ell}$ in eq (57), so that $f^i_M = f^i$ (see eq (2)).

The predicted time error may be expressed as

$$\epsilon^i_M = x^i_M - \hat{x}^i_M = \Delta x^i_M - \tau \hat{y}^i_M = \Delta x^i_M - f^i_M(\Delta x^i_{M-\ell}) + f^i_M(\Delta \hat{T}_{M-\ell}) \, . \tag{59}$$

The first term in (59) is the real time increase during the M^{th} interval and the second is a prediction term for the first. The third term came from using ensemble time referred data, and is, in a sense, redundant. Using eq (59), the ensemble time error will be expressed from eq (33), assuming fixed weighting factors for each interval as

$$\hat{T}_M = \sum_i w^i \sum_{j=1}^{M} \{\Delta x^i_j - f^i_j(\Delta x^i_{j-\ell}) + f^i_j(\Delta \hat{T}_{j-\ell})\} \tag{60}$$

where $w_j^i = w^i$ for all j is assumed. Again the first term is the real time increase during the 0-M interval and the rest is a prediction for the first.

Now let us reduce the third term. Similarly to eq (60), we will get the following two equations and so an equation for their difference

$$\hat{T}_{j-\ell} = \sum_{i_1} w^{i_1} \sum_{j_1=1}^{j-\ell} \{\Delta x_{j_1}^{i_1} - f_{j_1}^{i_1}(\Delta x_{j_1-\ell}^{i_1}) + f_{j_1}^{i_1}(\Delta \hat{T}_{j_1-\ell})\}$$

and

$$\hat{T}_{j-\ell-1} = \sum_{i_1} w^{i_1} \sum_{j_1=1}^{j-\ell-1} \{\Delta x_{j_1}^{i_1} - f_{j_1}^{i_1}(\Delta x_{j_1-\ell}^{i_1}) + f_{j_1}^{i_1}(\Delta \hat{T}_{j_1-\ell})\} ,$$

so

$$\Delta\hat{T}_{j-\ell} = \sum_{i_1} w^{i_1} \{\Delta x_{j-\ell}^{i_1} - f_{j-\ell}^{i_1}(\Delta x_{j-\ell-\ell_1}^{i_1}) + f_{j-\ell}^{i_1}(\Delta \hat{T}_{j-\ell-\ell_1})\} . \tag{61}$$

Applying the linear function f_j^i and next $\sum_i w^i$, we will obtain

$$\sum_i w^i f_j^i (\Delta \hat{T}_{j-\ell}) = \sum_i w^i \{F_j(\Delta x_{j-\ell}^i) - F_j \cdot f_{j-\ell}^i (\Delta x_{j-\ell-\ell_1}^i)\} + F_j \cdot F_{j-\ell}(\Delta \hat{T}_{j-\ell-\ell_1}) \tag{62}$$

where $F_j \equiv \sum_i w^i \cdot f_j^i$ and may be called an average linear function over clocks.

Substitution of eq (62) into eq (60) gives

$$\hat{T}_M = \sum_i w^i \sum_{j=1}^{M} [\Delta x_j^i - f_j^i(\Delta x_{j-\ell}^i) + F_j(\Delta x_{j-\ell}^i) - F_j \cdot f_{j-\ell}^i(\Delta x_{j-\ell-\ell_1}^i)] + \sum_{j=1}^{M} F_j \cdot F_{j-\ell}(\Delta \hat{T}_{M-\ell-\ell_1}) \tag{63}$$

which again includes the last term--a function of ensemble time difference.

Repeating the above procedure and defining the error function

$$\Delta f_k^i \equiv f_k^i - F_k \text{ for all } k, \tag{64}$$

we will finally obtain eq (65).

$$\begin{aligned}\hat{T}_M = \sum_i w^i \sum_{j=1}^{M} \Big[& \Delta x_j^i - \Delta f_j^i(\Delta x_{j-\ell}^i) - F_j \cdot \Delta f_{j-\ell}^i(\Delta x_{j-\ell-\ell_1}^i) \\ & - F_j \cdot F_{j-\ell} \cdot \Delta f_{j-\ell-\ell_1}^i(\Delta x_{j-\ell-\ell_1-\ell_2}^i) \\ & \cdots \\ & - F_j \cdot F_{j-\ell} \cdot F_{j-\ell-\ell_1} \cdots F_{j-\ell-\ell_1 \cdots -\ell_{M-4}} \cdot \Delta f_{j-\ell-\ell_1 \cdots -\ell_{M-3}}^i(\Delta x_{j-\ell-\ell_1 \cdots -\ell_{M-2}}^i) \\ & - F_j \cdot F_{j-\ell} \cdot F_{j-\ell-\ell_1} \cdots F_{j-\ell-\ell_1 \cdots -\ell_{M-3}} \cdot f_{j-\ell-\ell_1 \cdots -\ell_{M-2}}^i(\Delta x_{j-\ell-\ell_1 \cdots -\ell_{M-1}}^i)\Big] \\ & + \sum_{j=1}^{M} R_{j,\ell} \qquad (j-\ell-\ell_1 \cdots -\ell_{M-1} \le 0)\end{aligned} \tag{65}$$

where

$$R_{j,\ell} = F_j \cdot F_{j-\ell} \cdots F_{j-\ell-\ell_1 \cdots -\ell_{M-2}} (\Delta \hat{T}_{j-\ell-\ell_1 \cdots -\ell_{M-1}}) \rightarrow 0. \qquad (66)$$

The last two terms consist of real time difference data belonging to zero or before zero intervals. The last term R is a function of initial values of the ensemble time error, so it seems reasonable to assume this zero. If the linear functions for all the clocks are all the same, i.e., if the clocks are statistically identical, then all the error functions in eq (64) are equal to zero, then all the medium terms in eq (65) will disappear to give

$$\hat{T}_M = \sum_i w^i \left[(x^i_M - x^i_0) - \sum_{j=1}^{M} F_j \cdot F_{j-\ell} \cdots f^i_{j-\ell-\ell_1 \cdots -\ell_{M-2}} (\Delta x^i_{j-\ell \cdots -\ell_{M-1}}) \right]. \qquad (67)$$

The important thing to notice in this equation is that the ensemble error $\hat{T}_M$ does not depend on the clock values $(x^i_1, \ldots, x^i_{M-1})$ actually occurring in the interval, but only on values predicted for them from values occurring before $j = 1$. Thus, no new information is added in forming the estimate $\hat{T}^{ens}_M$ based on clock frequency estimation using the ensemble values $(\hat{T}^{ens}_1, \ldots, \hat{T}^{ens}_{M-1})$. If, in fact, the clocks started at $j = 0$, $(x^i_m = 0$ for $m \le 0)$, eq (67) is merely the simple weighted average without prediction

$$\hat{T}_M = \sum_{i=1}^{N} w^i (x^i_M - x^i_0) \quad .$$

In either case, the results of using approach A shown in figure 3 show that for flicker noise FM, the error $\hat{T}_M$ will grow rapidly, at least as fast as $\sqrt{M/N}$, as we advance M units from the last calibration.

It is easily shown that this result does not depend on the fact that we have weights $w_m^i = w^i$. When the weights are allowed to vary, the clocks remaining statistically identical, we obtain instead eq (67)

$$\hat{T}_M = \sum_{i=1}^{N} \sum_{j=1}^{M} w_j^i \{(x_j^i - x_{j-1}^i)$$

$$- F_j \cdot F_{j-\ell} \cdots F_{j-\ell \ldots -\ell_{M-3}} f^i_{j-\ell \ldots -\ell_{M-2}} (\Delta x^i_{j-\ell \ldots -\ell_{M-1}})\}$$

where again the second (correction) term does not depend on clock values which occur for $j > 0$, and vanishes for clocks started at $j = 0$.

In the general case where the clocks are required to be statistically identical, we obtain by the same procedure

$$\hat{T}_M = \sum_{i=1}^{N} \sum_{j=1}^{M} \Big[\Delta x_j^i w_j^i - \Delta g_j^i (\Delta x^i_{j-\ell}) - F_j \cdot \Delta g^i_{j-\ell} (\Delta x_j^i - \ell - \ell_1)$$

$$- F_j \cdot F_{j-\ell} \cdot \Delta g^i_{j-\ell-\ell_1} (\Delta x^i_{j-\ell-\ell_1-\ell_2})$$

$$\cdots$$

$$- F_j \cdot F_{j-\ell} \cdots F_{j-\ell \ldots -\ell_{M-4}} \cdot \Delta g^i_{j-\ell \ldots -\ell_{M-3}} (\Delta x^i_{j-\ell \ldots -\ell_{M-2}})$$

$$- F_j \cdot F_{j-\ell} \cdots F_{j-\ell \ldots -\ell_{M-3}} \cdot g^i_{j-\ell \ldots -\ell_{M-2}} (\Delta x^i_{j-\ell \ldots -\ell_{M-1}})\Big]$$

$$+ \sum_{j=1}^{M} R_{j,\ell} \qquad (68)$$

where

$$F_k \equiv \sum_{i=1}^{N} w_k^i f_k^i \,, \qquad g_k^i \equiv w_k^i f_k^i$$

$$\Delta g_k^i \equiv w_k^i \left(f_k^i - \sum_{s=1}^{N} w_k^s f_k^s\right). \tag{69}$$

Equation (68) does contain prediction terms dependent on clock values occurring on the interval $j = (1, M - 1)$, so that new information is being added. But it does not seem likely that the error $\hat{T}_M$ would show significantly reduced growth rate from that obtained in the case with statistically identical clocks. It is important to note that eq (68) for $\hat{T}_M$ is nonlinear (of degree M) in the weights w_M^i, so that the optimum estimation of these weights is very difficult.

The conclusion appears to be that while frequency calibrations against a primary standard can greatly improve the accuracy and stability of the time scale, for the noise processes considered in this paper, no improvement in these quantities can be gained (at least in the case of statistically identical clocks) by using the time scale itself as a primary standard.

7. CONCLUSIONS

We have obtained the result that one can construct with a small number of clocks a time scale whose accuracy is limited by the accuracy of a primary standard. We have also shown that the time dispersion of the time scale thus calculated may be proportional to the square root of the elapsed time t in a statistical sense instead of t^2 for the time scale without intermediate calibrations, so the former may become much better in both accuracy and uniformity than the latter after a long time has elapsed. Furthermore, algorithms using a primary standard have the advantage that it is easier to identify frequency drift, jumps, or aging of the clocks used.

It must be emphasized that these conclusions are based on clocks having only continuous random f^0 and f^{-1} FM noise with constant amplitudes. In practice, these amplitudes can vary in time, and many other anomalous forms of behavior occur, which will strongly influence the choice of both the clock ensemble and the time scale algorithm to be used.

We have shown that (at least for statistically identical clocks), no advantage is gained by using the time scale itself as a "primary standard" for intermediate calibrations.

It is also true, however, that the time scale averaged over a large number of clocks without intermediate calibrations may have a much better uniformity for a short time range than that depending on a primary standard utilized as in the above methods of prediction, where the short term uniformity of the time scale is limited by accuracy of the primary standard. Therefore, it may be possible to generate the two kinds of time scale simultaneously and actually utilize the former time scale periodically calibrated by the latter.

ACKNOWLEDGMENTS

The author is sincerely indebted to Dr. S. Jarvis, Jr., Mr. D. W. Allan, and Dr. J. A. Barnes of the National Bureau of Standards for many informative discussions and for significant contribution in making manuscript criticisms. The author also wishes to acknowledge the help of Dr. H. Akima in the Institute of Telecommunication Sciences for his kind aid in computer programming and presentation of his valuable sub-programs, and the assistance kindly given by Mrs. E. Helfrich in manuscript preparation.

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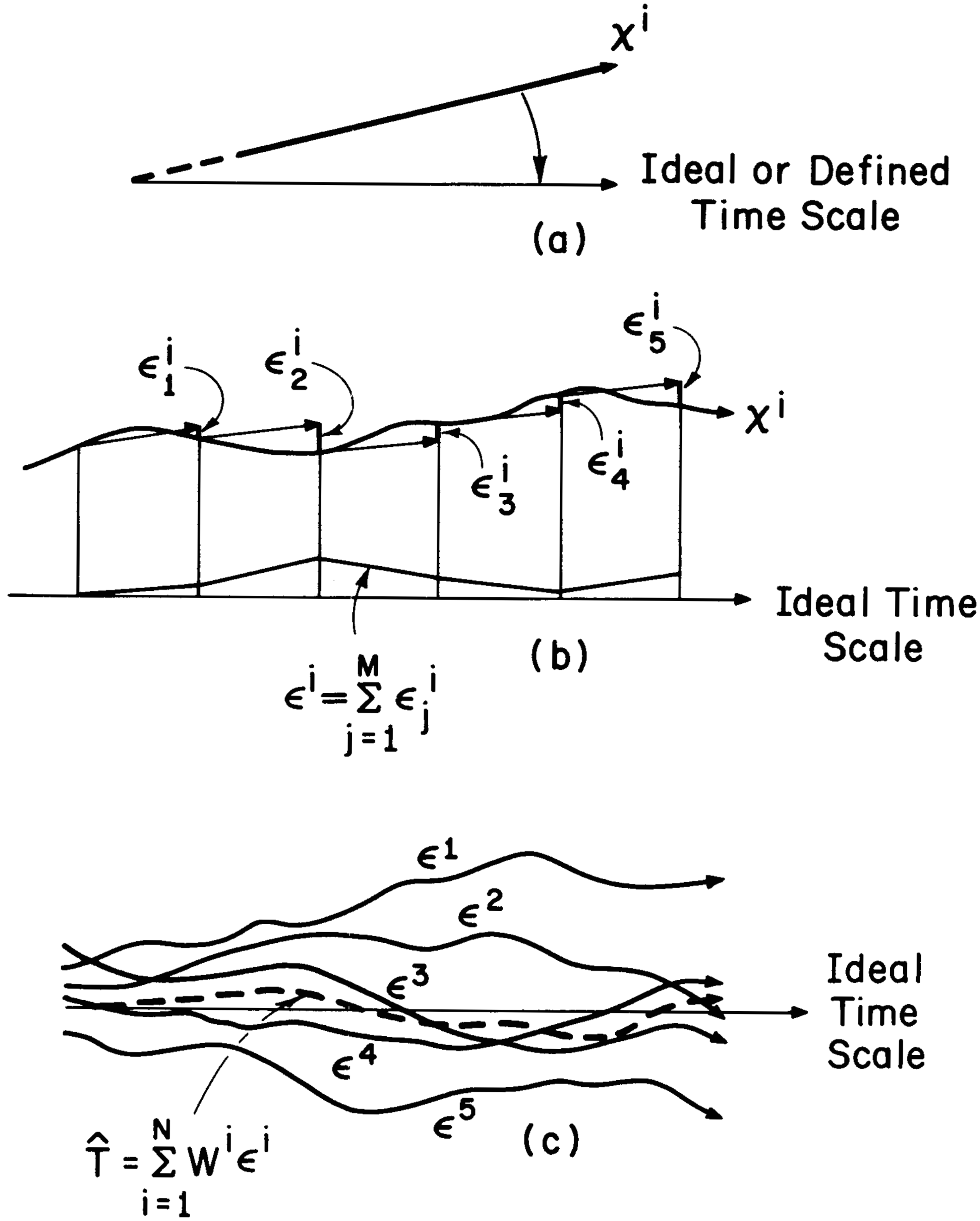

Fig. 1 (a) Linear drift of time. (b) Time error, x^i, due to random noises and its prediction error, ϵ^i. (c) A group of predicted time errors, ϵ^i, and the ensemble time, $\hat{T}$.

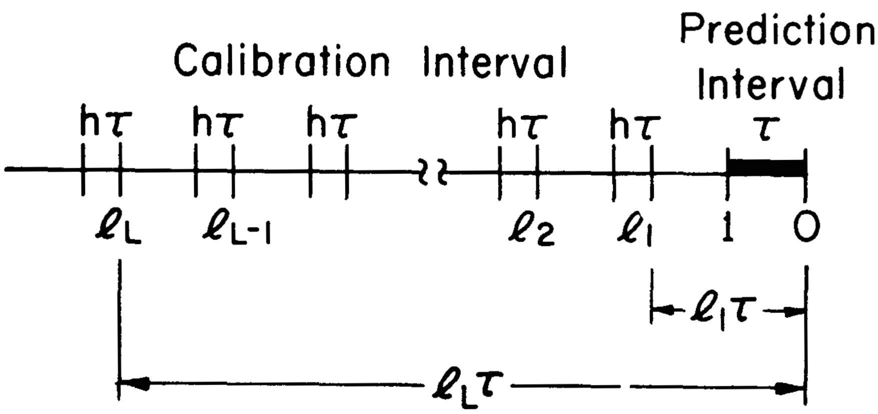

Fig. 2 Approach A.

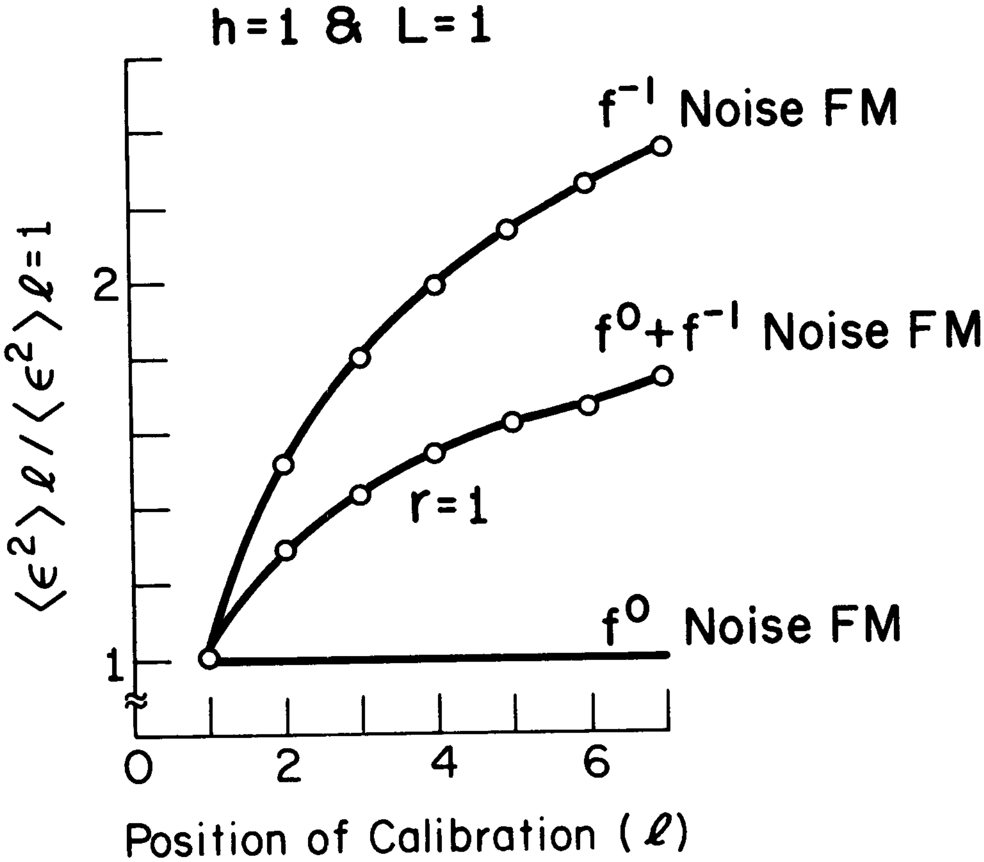

Fig. 3 Mean square prediction error by approach A as a function of position of a calibration. f^0 and f^{-1} noise FM refer to white and flicker noise FM, respectively, and $f^0 + f^{-1}$ noise FM refers to a mixed noise of both with power ratio $r = 1$.

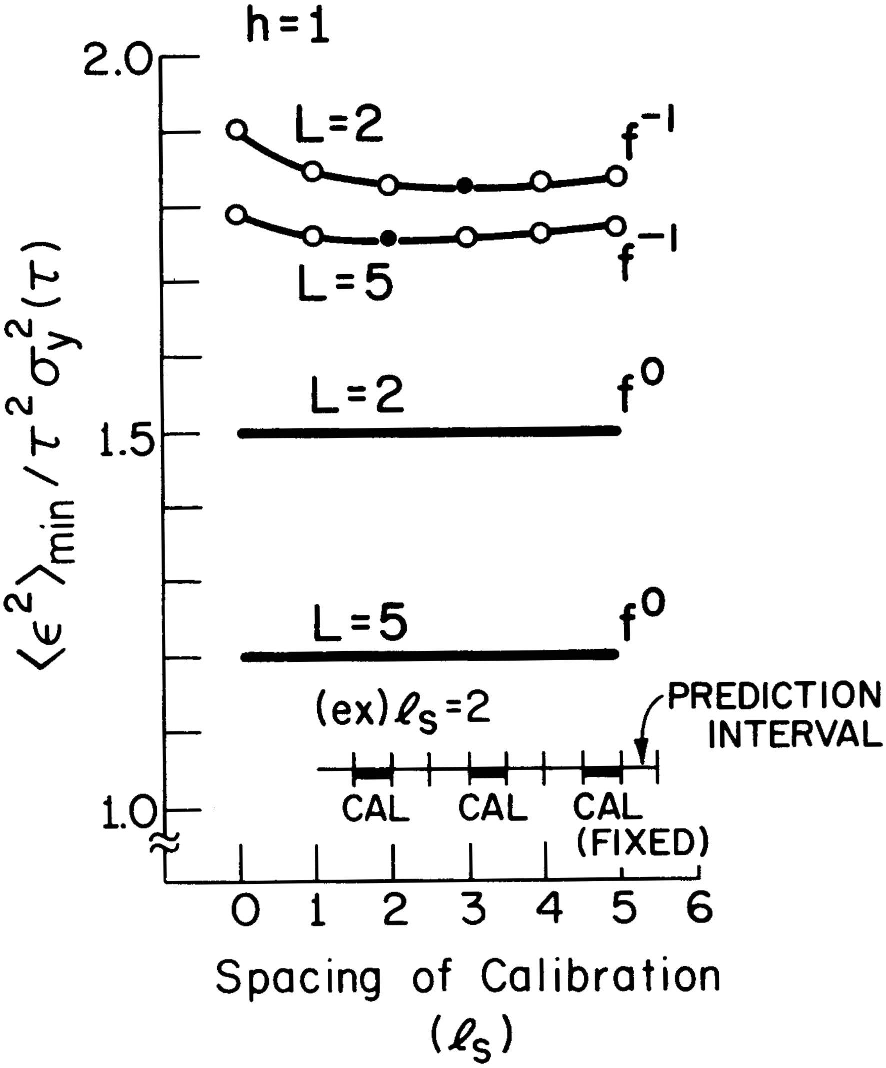

Fig. 4 Mean square prediction error by approach A as a function of spacing of calibration. f^0 and f^{-1} refer to white and flicker noise FM, respectively. An example of the calibration spacing is shown for $\ell_s = 2$.

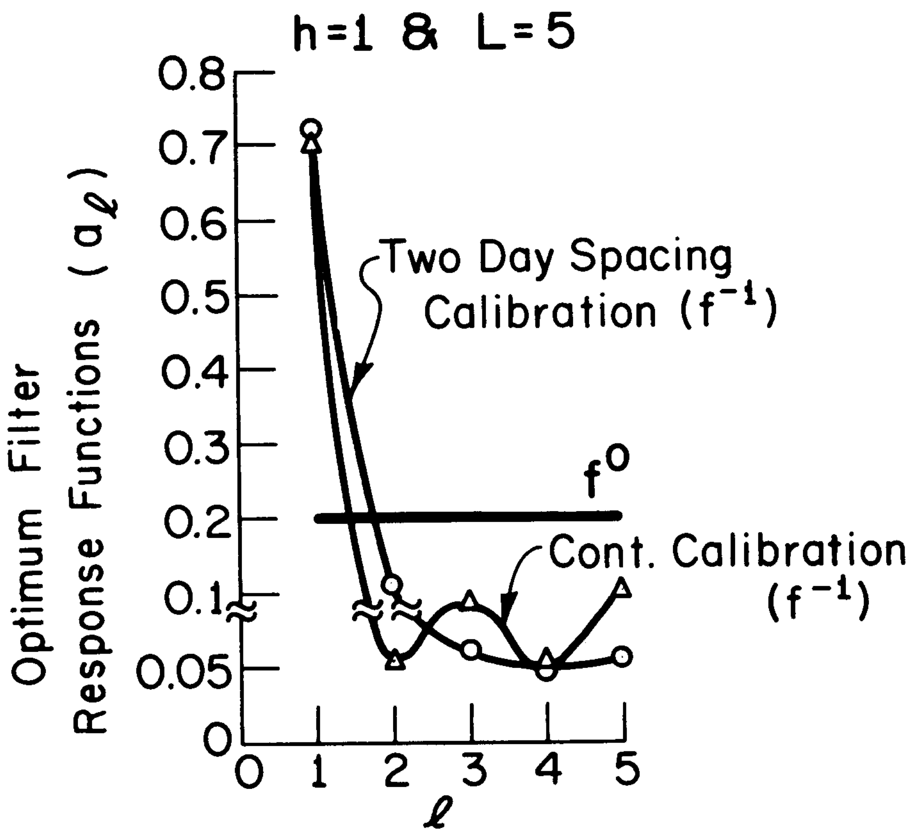

Fig. 5 Optimum filter response functions by approach A. f^0 and f^{-1} refer to white and flicker noise FM, respectively.

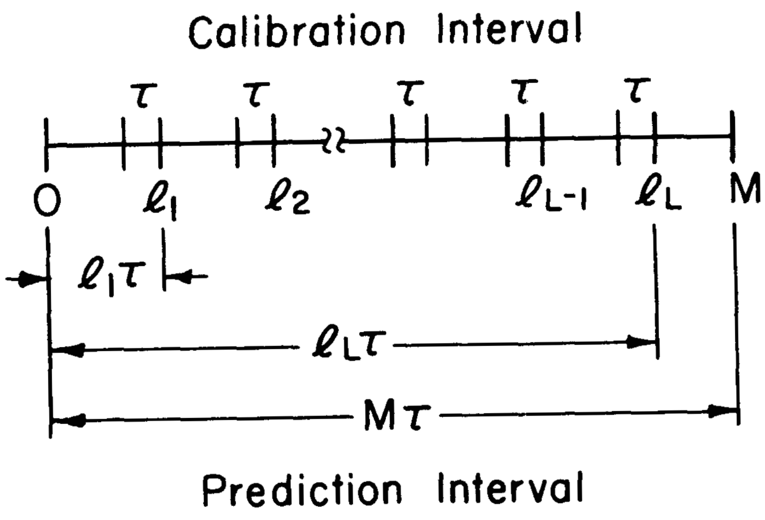

Fig. 6 Approach B.

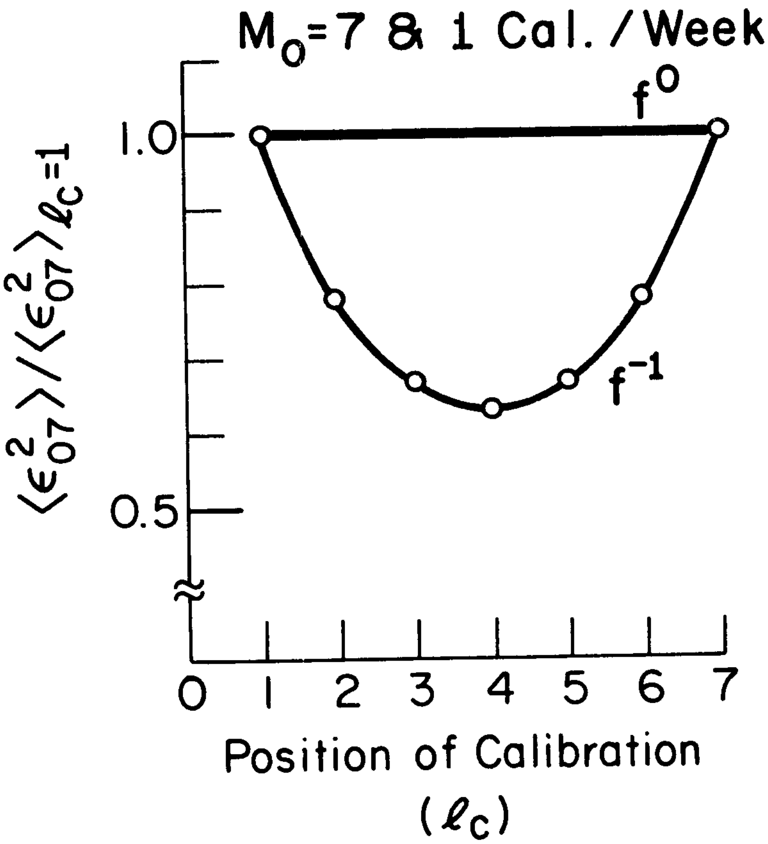

Fig. 7 Mean square prediction error by approach B as a function of position of a calibration. Calibration duration is one day. f^0 and f^{-1} refer to white and flicker noise FM, respectively.

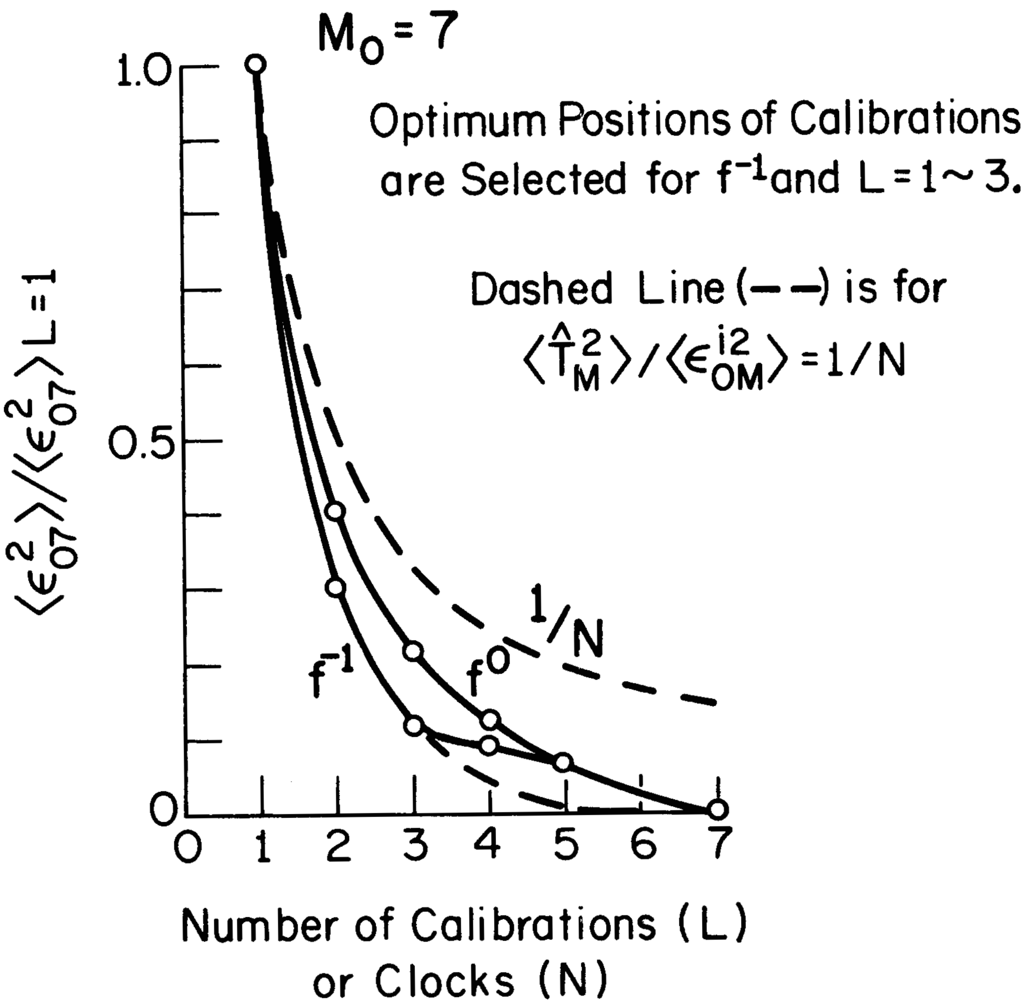

Fig. 8 Mean square prediction error by approach B as a function of number of calibrations in a week. Calibration duration is one day. f^0 and f^{-1} refer to white and flicker noise FM, respectively. The dashed continuation for the f^{-1} curve shows the case where the optimum positions of calibrations are selected for $L = 4 \sim 6$; Saturday and Sunday are used for calibration days.

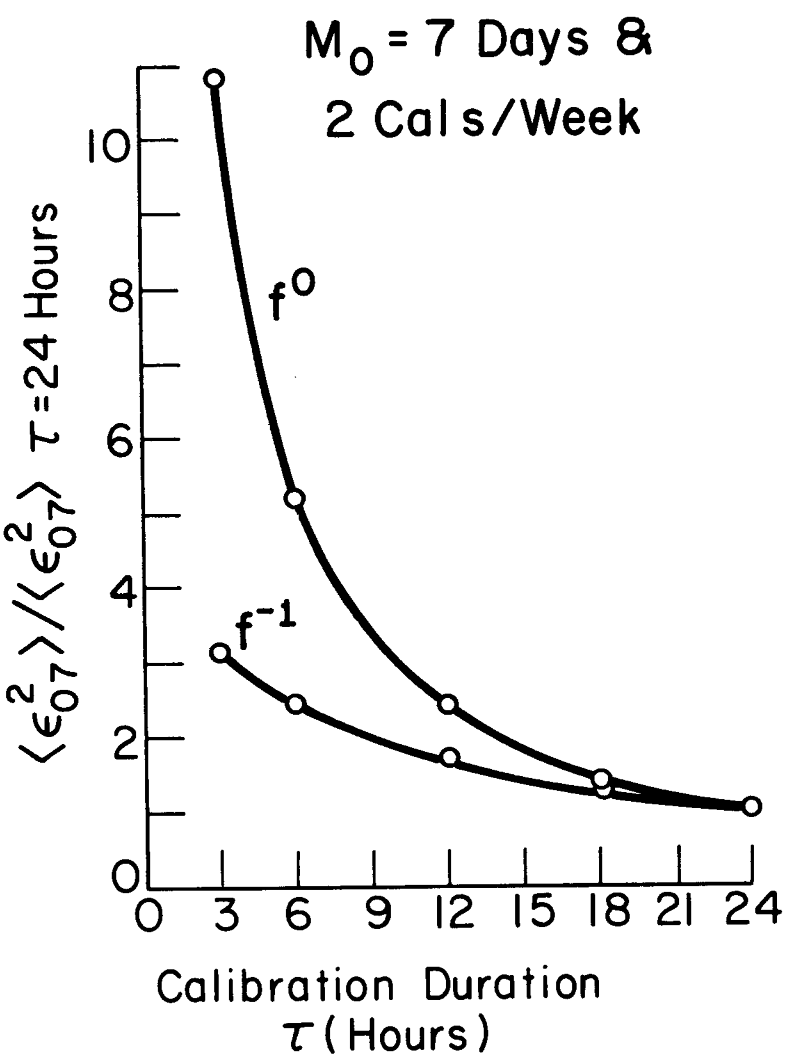

Fig. 9 Mean square prediction error by approach B as a function of calibration duration. f^0 and f^{-1} refer to white and flicker noise FM, respectively.

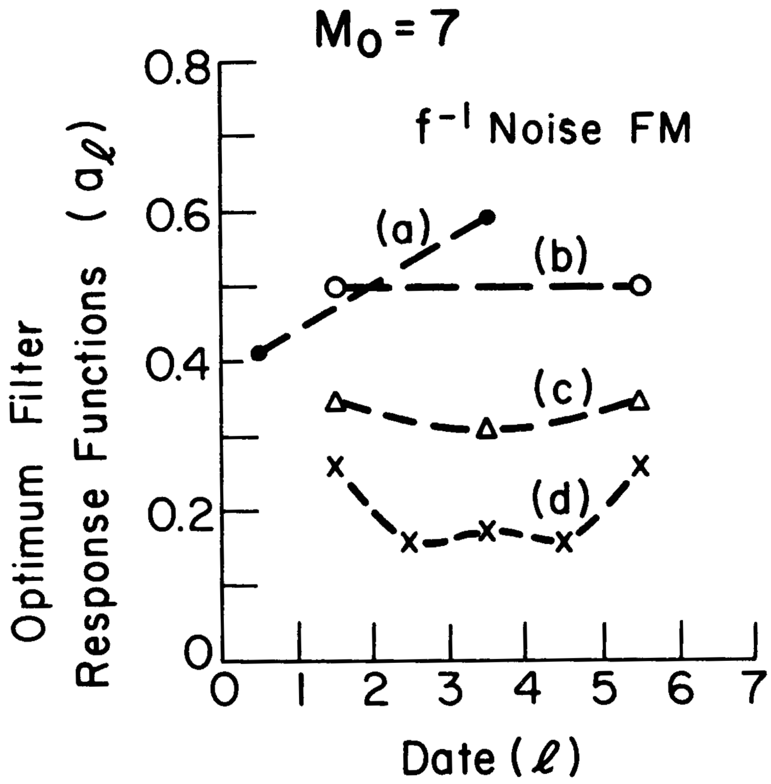

Fig. 10 Optimum filter response functions for flicker noise FM by approach B. (a) and (b): Two calibrations per week. (c): Three calibrations per week. (d): Five calibrations per week. Calibration duration is one day.

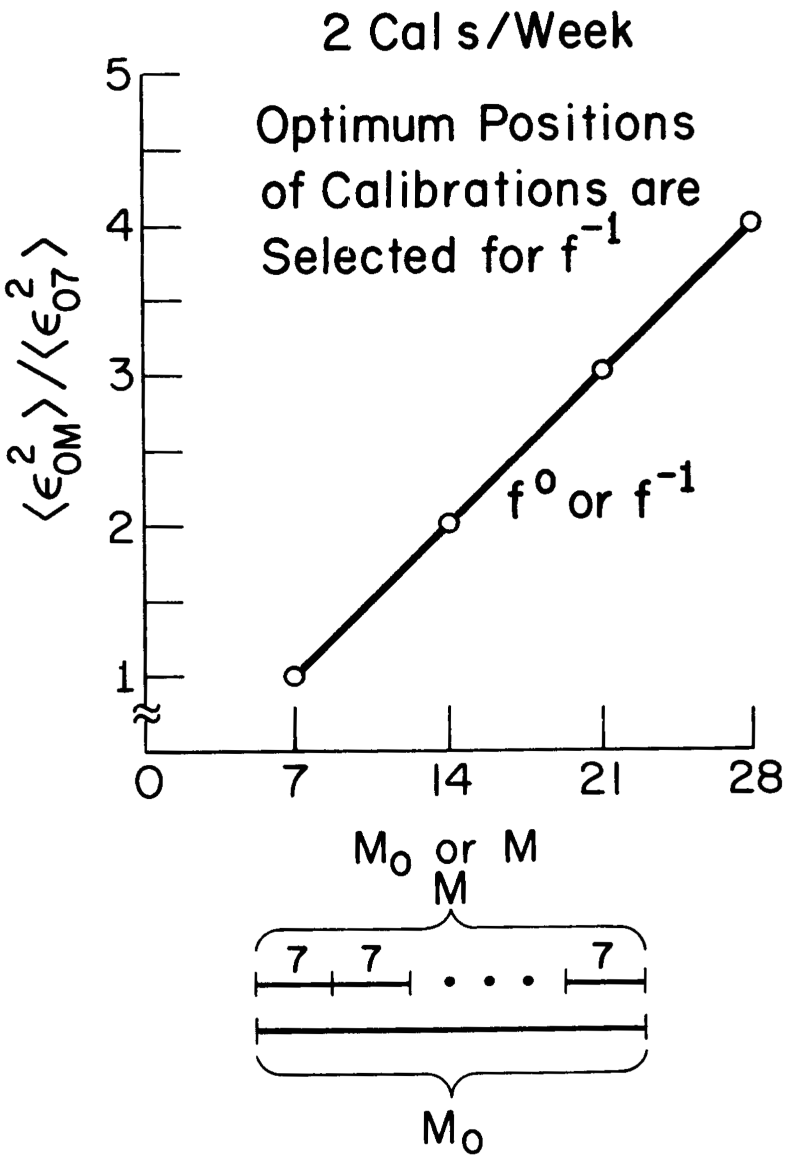

Fig. 11 Mean square prediction error by approach B as a function of the size of one prediction interval, M_0, or M/7 periods of 7 days each, M. Calibration duration is one day. f^0 and f^{-1} refer to white and flicker noise FM, respectively.

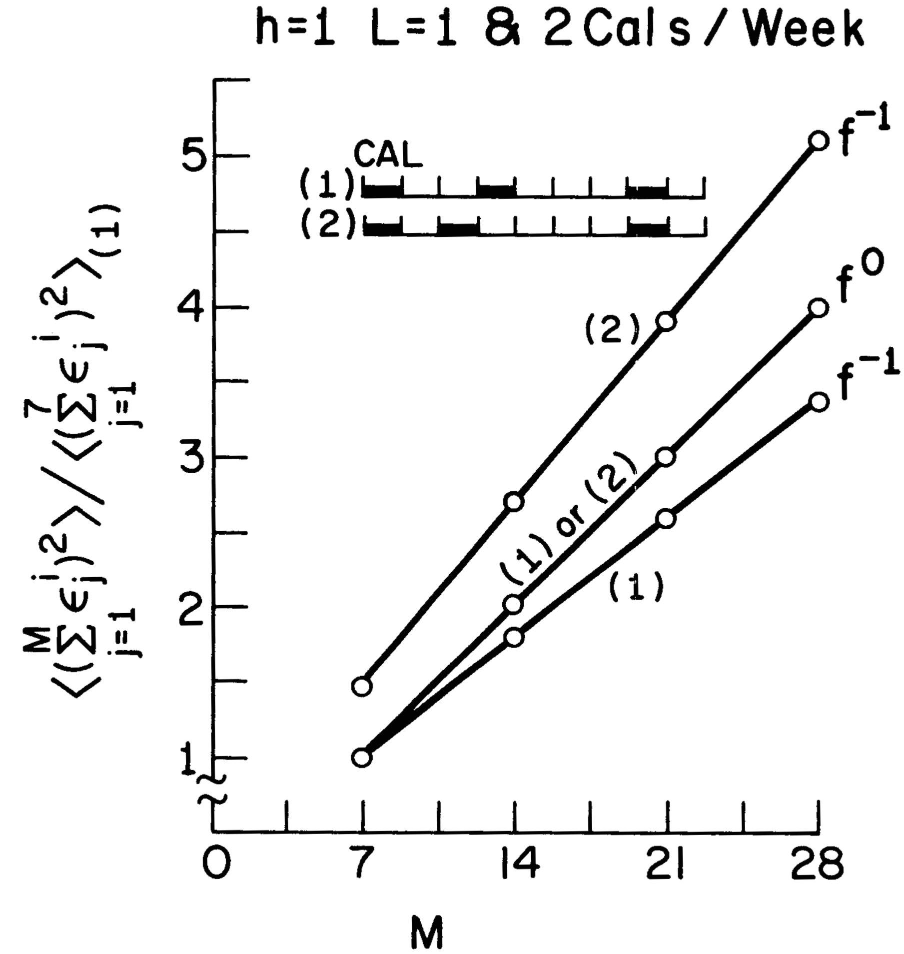

Fig. 12 Mean square of accumulated prediction error by approach A ($L = 1$). Calibration duration is one day. f^0 and f^{-1} refer to white and flicker noise FM, respectively.

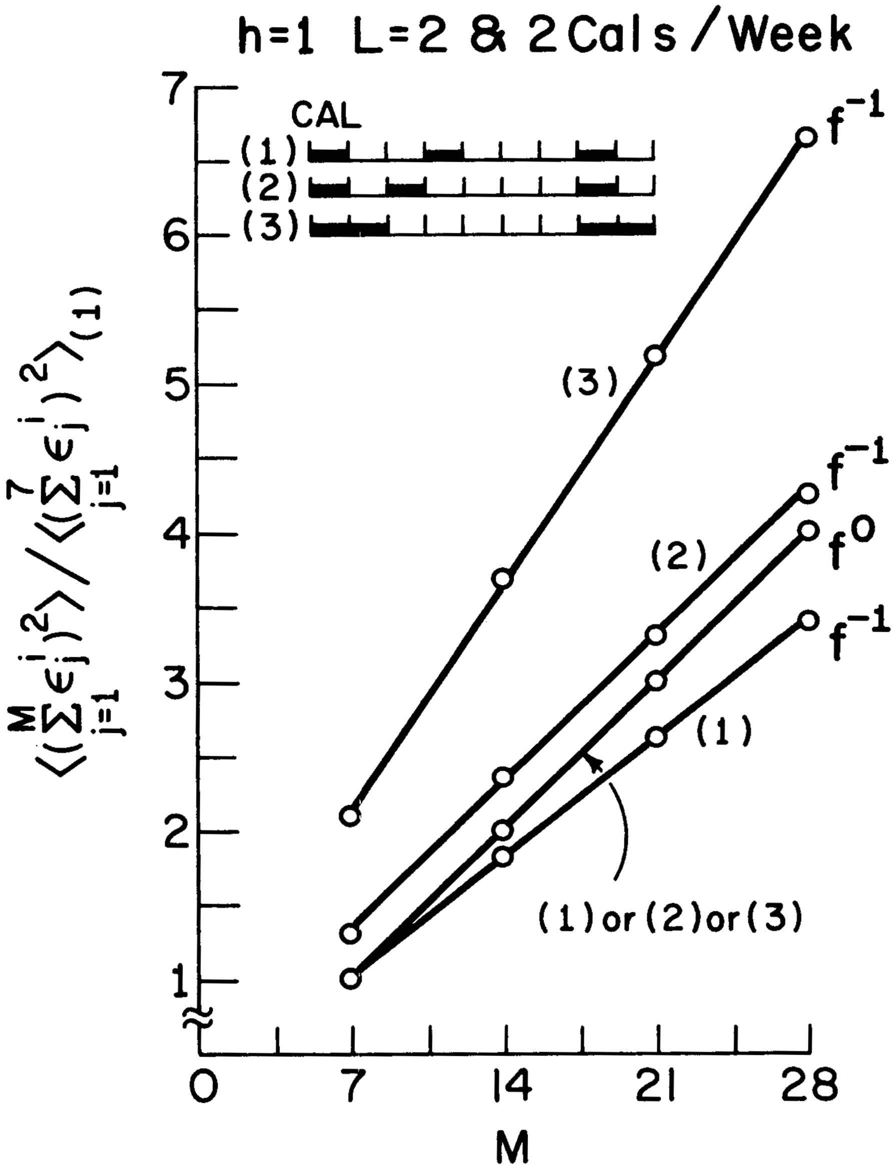

Fig. 13 Mean square of accumulated prediction error by approach A (L = 2). Calibration duration is one day. f^0 and f^{-1} refer to white and flicker noise FM, respectively.

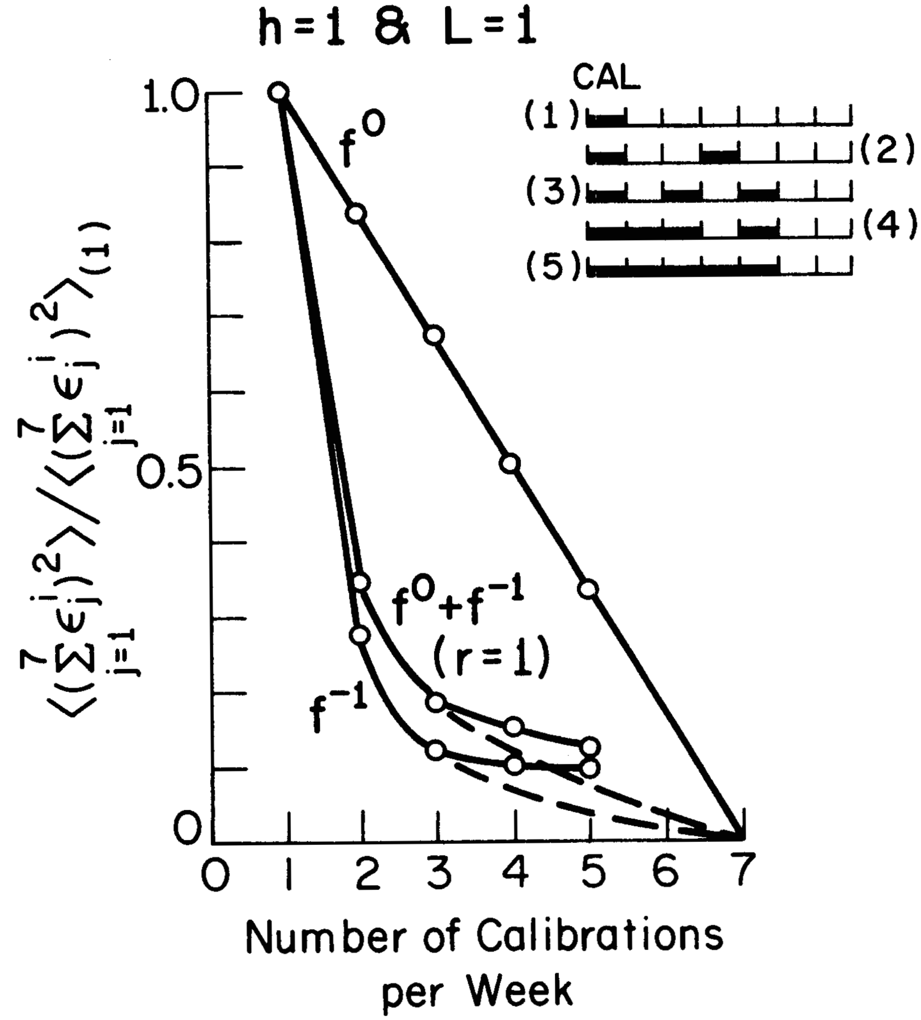

Fig. 14 Mean square of accumulated prediction error by approach A as a function of number of calibrations in a week. Calibration duration is one day. f^0, f^{-1} and $f^0 + f^{-1}$ refer to white noise FM, flicker noise FM, and a mixed noise of both, respectively. The dashed continuations of the curves show the case where the optimum positions of calibrations are selected for the horizontal axis $4 \sim 7$; Saturday and Sunday are used for calibration days.

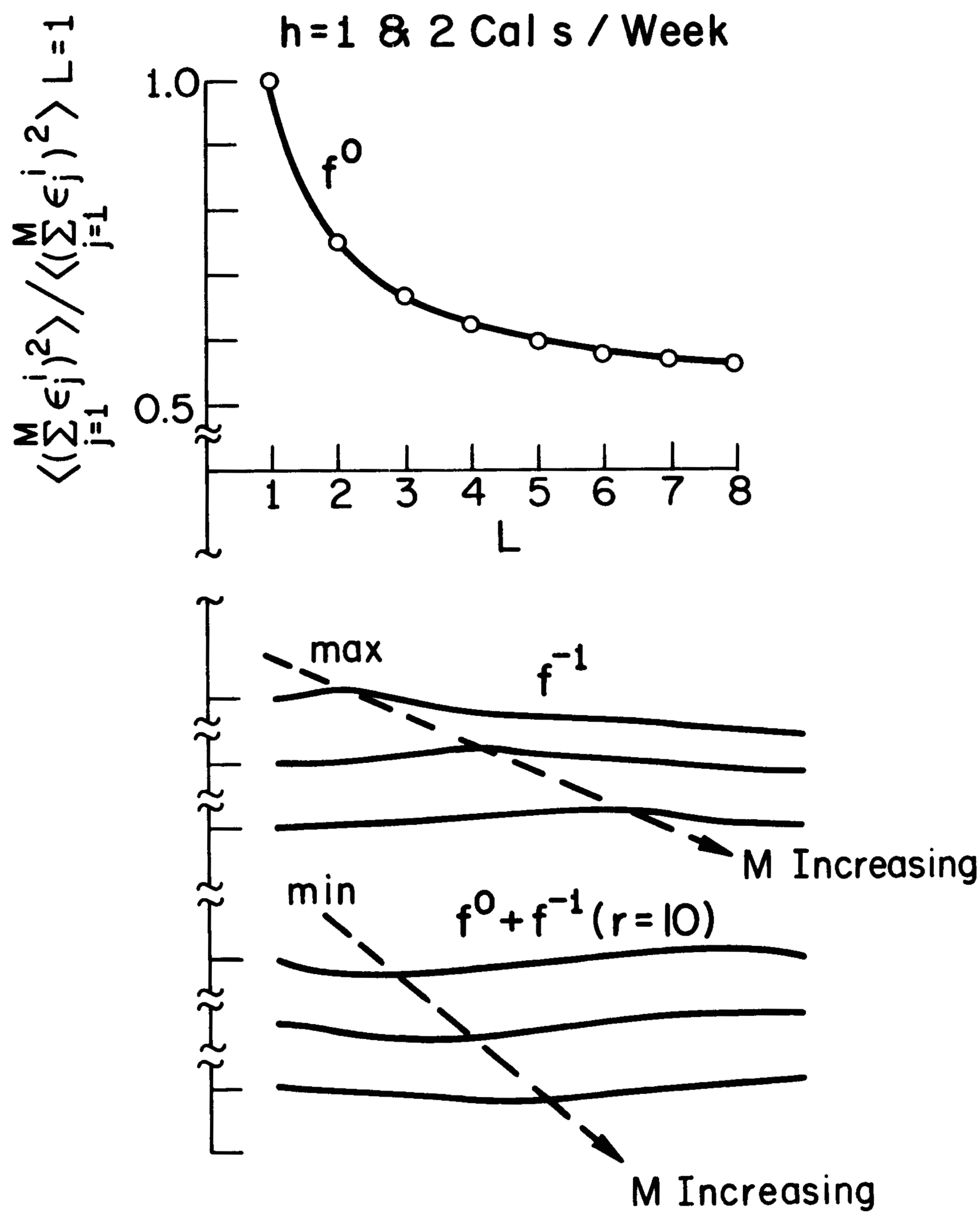

Fig. 15 Mean square of accumulated prediction error by approach A as a function of number of calibrations used for a prediction. Calibration duration is one day. f^0, f^{-1} and f^0+f^{-1} refer to white noise FM, flicker noise FM, and mixed noise of both, respectively.

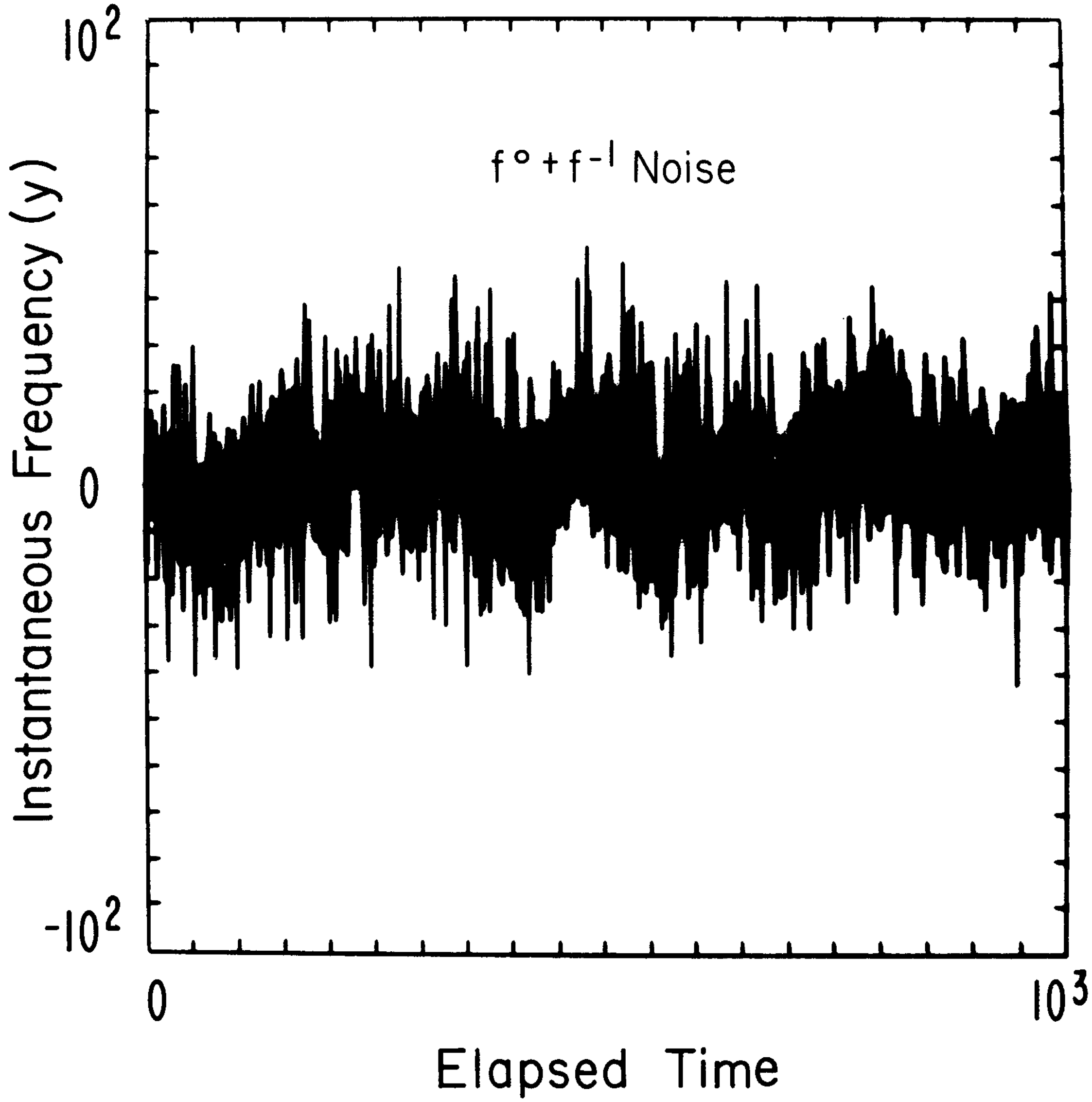

Fig. 16 Noise simulation for $f^0 + f^{-1}$ noise. Unit of time is supposedly one day and the vertical axis is in arbitrary units.

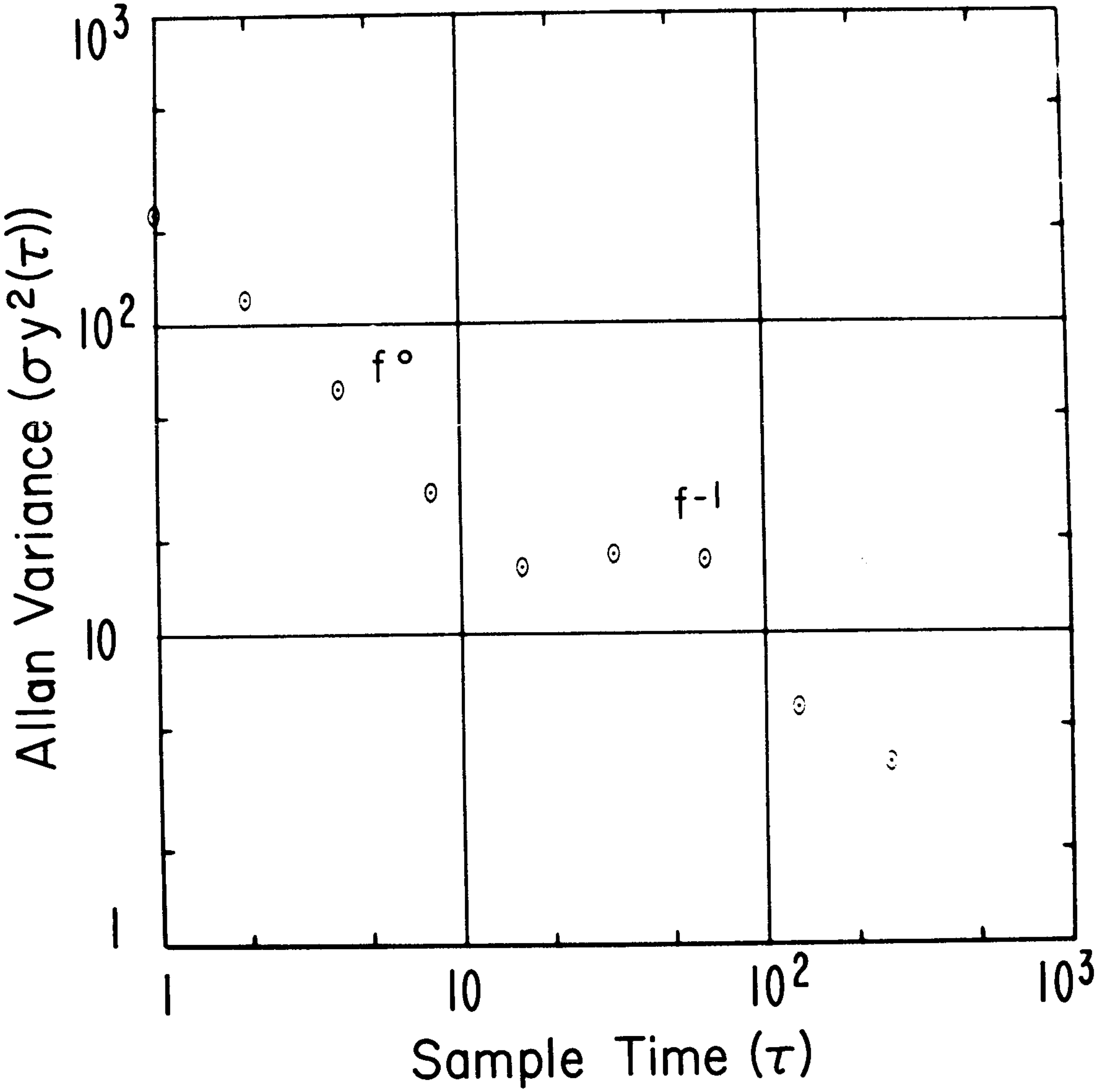

Fig. 17 The Allan variances of the noise given in Fig. 17 as a function of sample time. The unit of Sample Time is supposedly one day and the vertical axis is in arbitrary units.

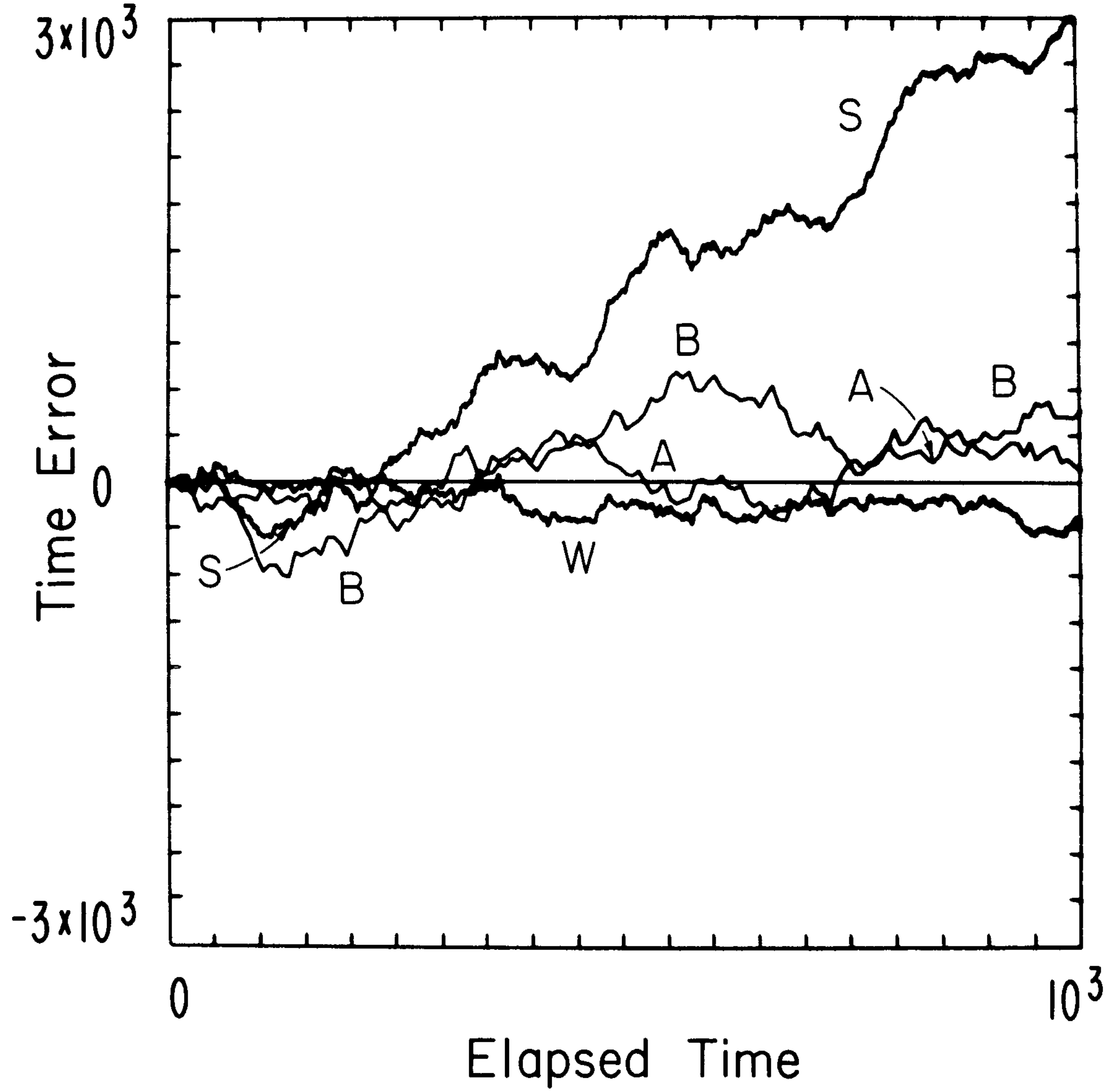

Fig. 18 Time error. The unit of Elapsed Time is supposedly one day and the vertical axis is in arbitrary units. S: simple accumulation of the noise given in figures 17 and 18. A and B: prediction error for S by approach A and B, respectively. W: time error for pure white noise simply accumulated.

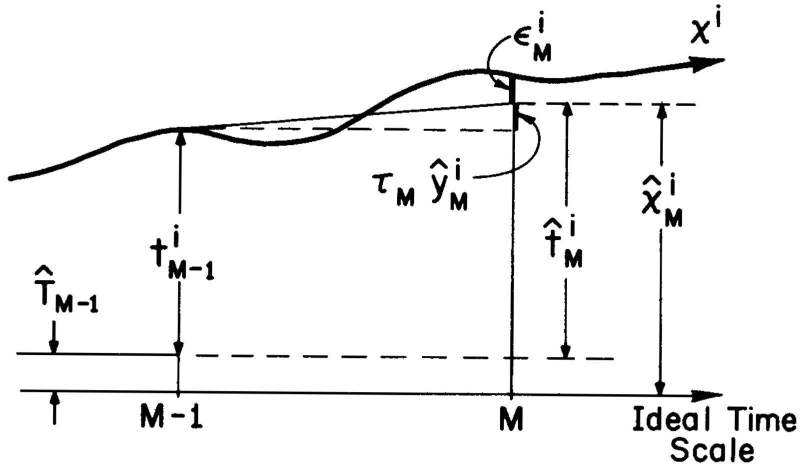

Fig. 19 Prediction interval (M - 1, M). x^i is time error of i^{th} clock and $\hat{x}^i_M$ is prediction for x^i at M. t^i_{M-1} is time error of i^{th} clock at M - 1, referred to the ensemble time error $\hat{T}_{M-1}$ at M - 1 and $\hat{t}^i_M$ is predicted time error of i^{th} clock at M, referred to the ensemble time error $\hat{T}_{M-1}$. τ_M is duration of prediction interval, $\tau_M \hat{y}^i_M$ is predicted time advance of i^{th} clock at M and ϵ^i_M is prediction error for $\tau_M \hat{y}^i_M$.

<table>
<tr><td>$\frac{\langle\delta\bar{y}^2\rangle}{\sigma_y^2(\tau)}$</td><td>$0.1^2$</td><td>$0.2^2$</td><td>$0.5^2$</td><td>$1^2$</td><td>$2^2$</td><td>$5^2$</td><td>$10^2$</td></tr>
<tr><td>$M\times\frac{\langle\delta\bar{y}^2\rangle}{\sigma_y^2(\tau)}$</td><td>0.07</td><td>0.14</td><td>1.75</td><td>7</td><td>28</td><td>175</td><td>700</td></tr>
<tr><td>$K_{f^{-1}}$</td><td colspan="7">$6.84^{(3)}$ $16.8^{(2)}$ $55.4^{(1)}$ $656^{(2)'}$</td></tr>
<tr><td>K_{f^0}</td><td colspan="7">$9.33^{(3)}$ $17.5^{(2)}$ $42^{(1)}$ $364^{(2)'}$</td></tr>
</table>

$$K = \langle\epsilon_{07}^2\rangle / \tau_0^2(=1\ \text{day})\,\sigma_y^2(\tau_0), \qquad M_0 = 7$$

(1): 1 Cal / Week, τ = 1 Day

(2): 2 Cals/Week, τ = 1 Day

(3): 3 Cals/Week, τ = 1 Day

(2)': 2 Cals/Week, τ = 6 Hours

Optimum Positions for Calibrations are Used

Table I

A numerical comparison between predicted time error by approach B shown with $K_{f^{-1}}$ and K_{f^0} and time error due to "accuracy" shown with $M \cdot \langle\delta\bar{y}^2\rangle/\sigma_y^2(\tau)$. $K_{f^{-1}}$ or K_{f^0} is equal to $k \cdot M$ in eq (51) for flicker noise FM or white noise FM, respectively.

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