Reprinted from *Printed in U. S. A.*
THE PHYSICS OF FLUIDS VOLUME 11, NUMBER 3 MARCH 1968

Theory for Cyclotron Harmonic Radiation from Plasmas

KARL-BIRGER PERSSON AND E. G. JOHNSON, JR.
National Bureau of Standards, Boulder, Colorado
AND
D. A. UHLENBROCK
Department of Mathematics, University of Wisconsin, Madison, Wisconsin
(Received 5 September 1967; final manuscript received 7 December 1967)

A highly simplified hydrodynamic model for the cause of cyclotron harmonics radiation in a magnetized, abnormal-glow, helium plasma is proposed. The high-velocity electrons are treated as sources for the charge density waves in the plasma. Electromagnetic radiation results from interaction of these wake waves with the statistical fluctuations (the granular structure in the ion density). This radiation contains the cyclotron harmonics. Some numerical curves of the radiation spectra are shown for a number of plasma parameters.

INTRODUCTION

The present paper derives a theory which gives a reason for the incoherent electromagnetic noise that is emitted in the neighborhood of the electron cyclotron frequency and its harmonics under the circumstance that the plasma here considered is an abnormal-glow helium plasma[1] located in a uniform magnetic field. A following paper describes the experimental arrangements necessary to observe the emitted spectrum and compares this and other theories with the observations.[2]

The radiation spectra discussed here have been subjected to rather intense investigations, both theoretical as well as experimental, for the past ten years without leading to positive identification of the mechanism responsible for the radiation. The confusing situation probably exists because there could be several different physical mechanisms involved and because most plasma configurations used will not allow a simple theoretical description.[3] To compound the difficulties the proposed theories are so incomplete that they can not be compared with the experimental observations. A reasonably complete summary and bibliography on this subject is given by Bekefi.[4] As he points out, in order to understand the emission of electromagnetic radiation from magnetized plasmas, it is necessary to determine the coupling between longitudinal and transverse electromagnetic waves and to calculate the absolute radiative intensities. We address ourselves to these points.

Because the abnormal negative glow plasma is sufficiently spatially uniform, we have a situation that permits possible quantitative agreement between theory and experiment. Since we chose to observe only the radiation from the field-free part of the plasma, it was necessary to do some modifications to the discharge tube described in Ref. 1. These modifications are shown in Fig. 1. The discharge tube is divided into parts I and II by the mesh anode. As a consequence, the space I which includes the cathode and the cathode fall range is completely shielded by this fine metal mesh structure. Because the anode is sufficiently transparent. the high-energy electrons generated in the cathode fall range enter the space II and generate a substantial and essentially spatially uniform plasma. The magnetic field is applied parallel to the tube axis. This configuration permits observation of the emitted radiation both parallel and perpendicular to the applied magnetic field. The theory under discussion attempts to explain the cause of radiation emitted in the direction parallel to the magnetic field. This case is chosen because additional information about the radiation process can be gained by distinguishing between left- and right-hand circularly polarized radiation.

The order of magnitude of some of the parameters which characterize the abnormal negative-glow plasma for helium at a pressure of 0.5 Torr follow: the electron density is 10^{12} cm^{-3}, the electron-beam density is 10^{6} cm^{-3}, the temperature of the cold electrons is 0.05 eV, the energy of the beam electrons is 3000 eV, the collision frequency of the cold electrons is 10^{9} sec^{-1}, the collision frequency of the beam electrons is 10^{8} sec^{-1}, and the Debye length is 10^{-4} cm. Because the Debye length is two orders of magnitude smaller than the average distance between electrons in the beam, we can rule out possible collective phenomena involving the beam.

Briefly, this theory assumes that each individual

[1] K. B. Persson, J. Appl. Phys. **36**, 10 (1965).
[2] H. W. Wassink, Phys. Fluids 11, 629 (1968).
[3] V. L. Ginsburg and V. V. Zhelezhiakov, Soviet Astron. **2**, 653 (1958).
[4] G. Bekefi, *Radiation Processes in Plasmas* (John Wiley & Sons, Inc., New York, 1966), Chap. 7.

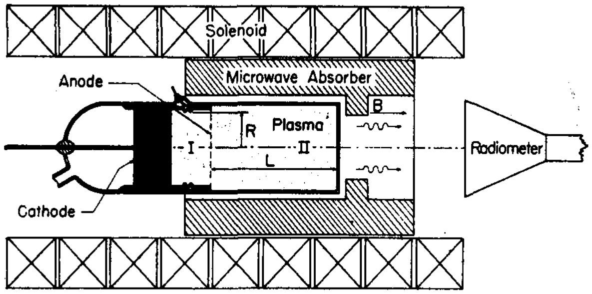

FIG. 1. A block diagram of the plasma tube.

beam electron or any other fast electron in the plasma acts independently of any other electron of the same type during its Coulombic interaction with the cold background plasma. Because the fast electron has velocities at supersonic speeds, there are excited longitudinal space waves in the background plasma which, when the collision frequencies are small, result in a space-charge wake wave that has a time structure harmonically distributed relative to the cyclotron frequency. When the background plasma is perfectly uniform, these longitudinal waves are not coupled to any transverse electromagnetic modes. As a consequence there is no electromagnetic radiation from the induced wave packet. When the exciting electron orbits in a spacially nonuniform plasma, the coupling is operative and harmonic radiation ensues. Because the plasma has a finite number of particles and because it has an amorphous granular structure on the microscopic scale, we find that there can be nonuniformities in the ion density. Crudely, we can vision each ion and the associated free electron in the plasma as an oscillator with a characteristic frequency equal to the plasma frequency. When a fast electron in a cyclotron orbit periodically excites a small number of these randomly situated oscillators, we find cyclotron harmonics. The number of oscillators is controlled by the cyclotron orbit size of the fast electron and by the fact that the only oscillators that are driven are those that are located within about a Debye radius of this orbit. To give some numerical idea of the number of oscillators we would expect on the average, we consider that the fast electrons of energy 1 eV are produced by the ionizing collision of the beam electron with the neutrals. From the typical numbers associated with the plasma parameters in a helium negative-glow plasma, we find the number of ions along the cyclotron orbit is 10^4. This is a sufficiently large number to validate the use of the hydrodynamic equations for the basic electromagnetic processes. At the same time this number is sufficiently small that variations in the actual number of oscillators permit strong coupling between the longitudinal and radiative electromagnetic modes. Please note that the number and location of the ions in the cyclotron orbit are stationary at the time scale of the motion of the fast electron.

In addition to the above mechanism for efficient coupling of the longitudinal fields to the radiative fields, there can be generated a harmonics emission spectra if there are spacial time-independent ion-density nonuniformities at a scale of less than the radius of the significant cyclotron orbits.

Ginsburg and Zheleziakov[3] have proposed that fluctuations, primarily electron density, provide efficient scattering centers for the propagating longitudinal waves and efficient coupling to radiative electromagnetic waves. This model probably describes some of the observed microwave emission from hot plasmas; however, the mathematics associated with this model is currently unable to produce numbers that will allow quantitive comparison with available laboratory measurements. We differ from their analysis by considering the electromagnetic radiation induced by a longitudinal wake bounded to the generating electron. In its present form, our model is primarily applicable to the cold plasma which is experimentally well represented by the abnormal negative glow in helium.[1] It does not invoke fluctuations of the electron gas and it does not depend on the presence of propagating longitudinal waves as is the case for the model suggested by Ginzburg and Zheleziakov.

THEORY I. THE ELEMENTARY RADIATOR

This theory is concerned with the emission of microwave radiation by a three-component (ions, electrons, and neutral atoms) nonrelativistic cold plasma with a homogeneous, static magnetic field $\mathbf{B}_0$. The actual geometry of the plasma is a finite-length cylinder of radius R with the cylinder axis parallel to $\mathbf{B}_0$. However, except for the final steps of the calculation the finite size of the plasma will be ignored in the equations and an infinite plasma is assumed.

The basic equations are derived from the Boltzmann–Vlasov equations for the three components by introducing collision terms with empirical effective collision frequencies and taking zeroth- and first-order moments with respect to the velocity. The moment hierarchy is truncated by replacing the square term in the velocities by a diagonal pressure tensor with ideal gas law dependence on the temperature.

Since the ion mass is very large compared to the electron mass, only the limiting equations for infinite ionic (and atomic) mass are considered. They imply that the ionic current and charge densities $\mathbf{J}^{(+)}$ and $q^{(+)}$ are constant in time. Thus $\mathbf{J}^{(+)}$ will vanish if it is assumed to vanish at some initial time, as will be done here. Likewise the equations for the neutral component decouple from those for the electron component and can be ignored.

As a result the effects of the positive and neutral components are represented in the equations for the electron component only in terms of an effective electron–neutral collision frequency ν and the neutralizing, possibly spatially varying ion-charge density $q^{(+)} = q_0^{(+)}(1 + f)$, where f gives the spatial deviation from the average charge density $q_0^{(+)}$. Suggested causes for this f are variations in the production mechanisms for the background plasma or the intrinsic fluctuations in the ion density which are being sampled in such a way that they are significant and hence not negligible. We assume that these fluctuations are essentially direct current compared to the high-frequency processes involved in the production of microwave radiation.

The externally injected fast (keV) beam electrons and the energetic secondaries resulting from initial ionizations are represented in terms of their charge and current-density $q^{(s)}$ and $\mathbf{J}^{(s)}$ as the source in Maxwell's equations and are not included in the Boltzmann distribution functions.

Thus with q, $\mathbf{J}$ as the electron charge and current densities, $\kappa = |e|/m$ the charge to mass ratio of an electron, v^2 the squared electron sound speed, $\omega_c \equiv \kappa\,|B_0|$ and $r_c \equiv v_\perp/\omega_c$ the cyclotron frequency and radius of a source electron with transverse speed v, and the loss rate ν_1, the basic equations including the usual Maxwell's equations (in mks units) are as follows:

Moment equations

$$\frac{\partial q}{\partial t} + \nabla\cdot\mathbf{J} = 0, \tag{1}$$

$$\left(\frac{\partial}{\partial t} + \nu_1 - \kappa\mathbf{B}\times\right)\mathbf{J} + v^2\nabla q + \kappa\mathbf{E}q = 0; \tag{2}$$

Maxwell's equations

$$\nabla\cdot\mathbf{E} = \epsilon_0^{-1}(q + q^{(+)} + q^{(s)}), \tag{3}$$

$$\nabla\cdot\mathbf{B} = 0, \tag{4}$$

$$\nabla\times\mathbf{E} = -\frac{\partial}{\partial t}\mathbf{B}, \tag{5}$$

$$\nabla\times\mathbf{B} = \epsilon_0\mu_0\frac{\partial}{\partial t}\mathbf{E} + \mu_0(\mathbf{J} + \mathbf{J}^{(s)}); \tag{6}$$

Source equations

$$\mathbf{r}^{(s)}(t) = \mathbf{r}_0 + [r_c \cos\omega_c(t - t_0), \cdot r_c \sin\omega_c(t - t_0), z_0 + v_z t], \tag{7}$$

$$T_{[t_-,t_+]}(t) = \Theta(t_+ - t) - \Theta(t - t_-), \tag{8}$$

where Θ is the step function, and t_+ and t_- are the creation and destruction times:

$$q^{(s)}(r, t) = e\,\delta[\mathbf{r} - \mathbf{r}^{(s)}(t)]T_{[t_-,t_+]}(t), \quad e < 0, \tag{9}$$

$$\mathbf{J}^{(s)}(r, t) = q^{(s)}(r, t)\frac{d}{dt}\mathbf{r}^{(s)}. \tag{10}$$

Here $\mathbf{r}^{(s)}$ represents the position of a beam or secondary source electron and $T_{[t_-,t_+]}$ simulates the effect of collisions on them.

Equation (2) is the only nonlinear equation and is linearized by replacing $\mathbf{B}$ with $\mathbf{B}_0$ and q with $-q^{(+)}$, which is found by requiring approximate local neutrality $q + q^{(+)} \cong 0$ of the background plasma. Here $\mathbf{E} = 0$ to this order in the linearization. Also v^2 is taken as a constant.

To make the system of Eqs. (1)–(10) more manageable, Maxwell's equations are re-expressed in terms of the scalar and vector potentials Φ and $\mathbf{A}$:

$$\mathbf{E} = -\nabla\Phi - \frac{\partial}{\partial t}\mathbf{A}, \tag{11}$$

$$\mathbf{B} = \nabla\times\mathbf{A} + \mathbf{B}_0, \tag{12}$$

$$0 = \nabla\cdot\mathbf{A} + \frac{1}{c^2}\frac{\partial}{\partial t}\Phi \text{ (Lorentz condition)}, \tag{13}$$

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2\right)(\epsilon_0\Phi) = q + q^{(+)} + q^{(s)}, \tag{14}$$

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2\right)\left(\frac{1}{\mu_0}\mathbf{A}\right) = \mathbf{J} + \mathbf{J}^{(s)}. \tag{15}$$

When a preferred (right-handed, orthogonal) coordinate system with coordinates (x, y, z) and the z axis parallel along $\mathbf{B}_0$ is introduced, the vectors or operators (like $\mathbf{J}$ or ∇) are usefully resolved into longitudinal and transverse components, $J_\parallel = J_z$ and $J_\perp = (J_+, J_-)$, where in turn $J_\perp$ is conveniently characterized in terms of $J_\pm = J_x \pm iJ_y$. The time dependence is treated by a Fourier integral transformation according to the generic formula

$$f(t) = \int_{-\infty}^{+\infty} d\omega\, e^{-i\omega t}\tilde{f}(\omega) \tag{16}$$

such that a real valued $f(t)$ satisfies the identity $\tilde{f}(\omega)^* = \tilde{f}(-\omega)$ with respect to complex conjugation.

Finally, dimensionless variables and fields labeled with an asterisk are introduced and defined as follows:

$$\mathbf{r} = \frac{c}{\omega_p}\mathbf{r}^*, \qquad t = \frac{1}{\omega_p}t^*, \qquad \Phi = \frac{e\omega_p}{\epsilon_0 c}\Phi^*,$$

$$\mathbf{A} = e\omega_p\mu_0\mathbf{A}^*, \qquad \mathbf{J} = \frac{e\omega_p^3}{c^2}\mathbf{J}^*, \qquad q = \frac{e\omega_p^3}{c^3}q^*,$$

where $\omega_p^2 = \epsilon_0^{-1}\kappa q_0^{(+)}$, the squared (average) plasma frequency.

Using the notation

$$\omega_\nu = \omega + i\nu_1$$

and

$$\Omega = \begin{bmatrix} \Omega_+ & 0 & 0 \\ 0 & \Omega_- & 0 \\ 0 & 0 & \Omega_0 \end{bmatrix} = \begin{bmatrix} \omega(\omega_\nu + \omega_c) & 0 & 0 \\ 0 & \omega(\omega_\nu - \omega_c) & 0 \\ 0 & 0 & \omega\omega_\nu \end{bmatrix} = \begin{bmatrix} (\Omega_\perp) & 0 \\ 0 & \Omega_0 \end{bmatrix} \tag{17}$$

as well as leaving out the asterisk which labels the dimensionless variables and fields, one finds after the Fourier transformation the following set of equations:

$$i\omega\tilde{q} = \nabla_\perp\cdot\tilde{J}_\perp + \nabla_\parallel\cdot\tilde{J}_\parallel, \tag{18}$$

$$i\omega\tilde{\Phi} = \nabla_\perp\cdot\tilde{A}_\perp + \nabla_\parallel\cdot\tilde{A}_\parallel, \tag{19}$$

$$(\nabla_\perp^2 + \nabla_\parallel^2 + \omega^2)\tilde{\Phi} = -(\tilde{q} + \tilde{q}^{(s)} + \tilde{q}^{(+)}), \tag{20}$$

$$(\nabla_\perp^2 + \nabla_\parallel^2 + \omega^2)\tilde{A}_\perp = -(\tilde{J}_\perp + \tilde{J}_\perp^{(s)}), \tag{21}$$

$$(\nabla_\perp^2 + \nabla_\parallel^2 + \omega^2)\tilde{A}_\parallel = -\tilde{J}_\parallel, \tag{22}$$

$$\Omega_\perp\tilde{J}_\perp + v^2\nabla_\perp(i\omega\tilde{q}) + (1+f)\nabla_\perp(i\omega\tilde{\Phi}) + \omega^2\tilde{A}_\perp = 0, \tag{23}$$

$$\Omega_\parallel\tilde{J}_\parallel + v^2\nabla_\parallel(i\omega\tilde{q}) + (1+f)\nabla_\parallel(i\omega\tilde{\Phi}) + \omega^2\tilde{A}_\parallel = 0. \tag{24}$$

Since in Eqs. (7)–(10) the motion of a given source electron is assumed helical, the corresponding variables are best described in terms of a cylindrical coordinate system (r, φ, z) whose z axis coincides with the axis of the cylinder generated by the helix and whose origin is at $\mathbf{r}_0$. Only at the end, when the incoherent contributions of the various source electrons to the radiation are summed subject to the varying initial conditions and orbit parameters, will such a choice of coordinates be impossible. This procedure is adequate because Eqs. (18)–(24) are linear in the variables.

With

$$\begin{aligned}\delta[\mathbf{r} - \mathbf{r}^{(s)}(t)] &= \frac{1}{r}\,\delta(r - r_c)\,\delta_{\mathrm{per}}[\varphi - \omega_c(t - t_0)]\,\delta(z - z_0 - v_z t) \\ &= (2\pi)^{-2}\sum_{\ell=-\infty}^{\infty}\int_{-\infty}^{+\infty} dk_z\,\frac{1}{r}\,\delta(r - r_c) \\ &\quad\cdot\exp\{i\ell[\varphi - \omega_c(t - t_0)] + ik_z(z - z_0 - v_z t)\},\end{aligned} \tag{25}$$

one finds

$$\begin{aligned}\tilde{q}^{(s)}(r, \varphi, z, \omega) &= \sum_{\ell=-\infty}^{\infty} e^{i\ell\varphi}\int_{-\infty}^{\infty} dk_z\, e^{ik_z z} q_\ell^{(s)}(r, k_z, \omega), \\ q_\ell^{(s)}(r, k_z, \omega) &= -(2\pi)^{-3}r_c^{-1}\,\delta(r - r_c)\exp[-i(k_z z_0 - \ell\omega_c t_0)] \\ &\quad\cdot(i\omega_\ell)^{-1}[\exp(i\omega_\ell t_+) - \exp(i\omega_\ell t_-)],\end{aligned} \tag{26}$$

with

$$\omega_\ell = \omega - \ell\omega_c - k_z v_z$$

and

$$\begin{aligned}\tilde{J}_\pm^{(s)}(r, \varphi, z, \omega) &= \pm i r_c\omega_c e^{\pm i\omega_c t_0}\tilde{q}^{(s)}(r, \varphi, z, \omega \pm \omega_c) \\ &= \sum_{\ell=-\infty}^{\infty} e^{i(\ell\pm 1)\varphi}\int_{-\infty}^{\infty} dk_z\, e^{ik_z z} J_{\pm,\ell}^{(s)}(r, k_z, \omega),\end{aligned} \tag{27}$$

where

$$J_{\pm,\ell}^{(s)}(r, k_z, \omega) = \pm i r_c\omega_c q_\ell^{(s)}(r, k_z, \omega)$$

and

$$\tilde{J}_\parallel^{(s)}(r, \varphi, z, \omega) = v_z\tilde{q}^{(s)}(r, \varphi, z, \omega).$$

The quantity relevant for the description of radiation propagating parallel to $\mathbf{B}_0$ is

$$\bar{A}_\perp(z, \omega) = \int_0^\infty r\,dr\int_0^{2\pi} d\varphi\,\tilde{A}_\perp(r, \varphi, z, \omega). \tag{28}$$

With a similar notation for the other quantities of Eqs. (18)–(25) it is essential to note that for a spatially homogeneous plasma (with $\nabla_\perp f \equiv 0$) the longitudinal and transverse equations decouple in the following manner:

Longitudinal response:

$$i\omega\bar{q} = \nabla_\parallel\cdot\bar{J}_\parallel, \tag{29}$$

$$i\omega\bar{\Phi} = \nabla_\parallel\cdot\bar{A}_\parallel, \tag{30}$$

$$(\nabla_\parallel^2 + \omega^2)\bar{\Phi} = -(\bar{q} + \bar{q}^{(+)} + \bar{q}^{(s)}), \tag{31}$$

$$(\nabla_\parallel^2 + \omega^2)\tilde{A}_\parallel = -\tilde{J}_\parallel, \tag{32}$$

$$\Omega_\parallel \tilde{J}_\parallel + v^2\nabla_\parallel(i\omega\tilde{q}) + \nabla_\parallel(i\omega\tilde{\Phi}) + \omega^2\tilde{A}_\parallel. \tag{33}$$

Transverse response:

$$(\nabla_\parallel^2 + \omega^2)\tilde{A}_\perp = -(\tilde{J}_\perp + \tilde{J}_\perp^{(s)}), \tag{34}$$

$$\Omega_\perp J_\perp + \omega^2 A_\perp = 0. \tag{35}$$

Thus $A_\perp$ is determined from Eqs. (34) and (35) with the corresponding source $\tilde{J}_\perp^{(s)}$. As in the case of a nonrelativistic electron gyrating in a vacuum there is no radiation in the longitudinal direction at the cyclotron harmonics except at the fundamental frequency.

The situation changes when $\nabla_\perp f$ does not vanish identically. In that case the term $f\cdot\nabla_\perp\Phi$ in Eq. (23) couples the longitudinal and transverse systems, so that harmonic radiation can result. The resulting system of equations leads to unwieldy convolution equations when Fourier integral transforms are applied. Instead we neglect $f \ll 1$ in Eq. (24) and in the $\tilde{A}_\perp$ term of Eq. (23) and keep it only in the critical $\nabla_\perp\Phi$ term of Eq. (23), which produces the coupling.

Hence by combining Eqs. (21) and (23) and forming the Fourier transform in x, y with $k_x = k_y = 0$, it follows that

$$[\omega^2 - \Omega_\perp(\nabla_\parallel^2 + \omega^2)]\tilde{A}_\perp = \Omega_\perp\tilde{J}_\perp^{(s)} + \overline{i\omega\tilde{\Phi}(\nabla_\perp f)}. \tag{36}$$

Here the operator on the left side of Eq. (36) corresponds to the well-known Appleton–Hartree dispersion law for the special case of longitudinal propagation.

A first-order perturbation treatment will determine $\tilde{\Phi}$ to zeroth order in f from Eqs. (18)–(25) with $f \equiv 0$. We substitute the result into Eq. (36) to find $\tilde{A}_\perp$ to first order in f.

Under these $f \equiv 0$ assumptions, the longitudinal source velocity v_z appearing in Eqs. (26) and (27) leads to a Doppler shift by $v_z k_z$ and an inessential $\tilde{J}_\parallel^{(s)}$ current. For simplicity, we ignore the shift by setting $v_z = 0$. As a consequence the helical motion becomes a circular motion and the source electron only "samples" f at $\mathbf{r}_0$ close to the cyclotron orbit. This sampling process is most sensitive to the φ dependence of f, which leads us to adopt a model of the form

$$f(r, \varphi, z) = f(\varphi) = \sum_{\ell=-\infty}^{+\infty} f_\ell e^{i\ell\varphi}, \tag{37}$$

with constants f_ℓ, satisfying $f_\ell^* = f_{-\ell}$, $f_0 = 0$. These constants, reflecting the inhomogenity of the plasma, will, in general, vary for different source electrons. A later statistical treatment will take this into account, when the contributions from different source electrons are compounded. With the notation

$$\tilde{\Phi}(r, \varphi, z, \omega) = \sum_{\ell=-\infty}^{\infty} e^{i\ell\varphi} \int_{-\infty}^{+\infty} dk_z\, e^{ik_z z}\Phi_\ell(r, k_z, \omega), \tag{38}$$

the term of interest in Eq. (36) is given by

$$(\nabla_\perp f)\tilde{\Phi}(z, \omega) = \int_{-\infty}^{\infty} dk_z\, e^{ik_z z} \sum_{\ell=-\infty}^{\infty} \pm 2\pi(\ell \pm 1) f_{\ell\pm1}^* \cdot \int_0^\infty dr\, \Phi_\ell(r, k_z, \omega). \tag{39}$$

In order to determine $\int_0^\infty dr\, \Phi_\ell$ to zeroth-order in f from Eqs. (18)–(25), one can practically neglect the radiation terms. Thus Eqs. (18), (20), (23), and (24) with $f \equiv 0$, when transformed as in Eq. (38), give a determined system from which a fourth-order ordinary differential equation in the r variable for Φ_ℓ is obtained by elimination:

$$(a_2 D_\ell^2 + a_1 D_\ell + a_0)\Phi_\ell = (b_1 D_\ell + b_0) q_\ell^{(s)}, \tag{40}$$

with

$$D_\ell = \left(r^{-1}\frac{\partial}{\partial r} r \frac{\partial}{\partial r} - \ell^2 r^{-2}\right),$$

$a_2 \equiv v^2$, the velocity of sound squared.

$$\begin{aligned} a_1 &\equiv \chi\Omega_0 - 1 - v^2k_z^2, \\ a_0 &\equiv \chi k_z^2(1 - \Omega_0 + v^2k_z^2), \\ \chi &\equiv 1 - \left(\frac{\omega_c}{\omega_\nu}\right)^2, \\ b_1 &\equiv -v^2, \\ b_0 &\equiv \chi(v^2k_z^2 - \Omega_0). \end{aligned} \tag{41}$$

In a cold plasma $v^2 \to 0$, and screening distances are correspondingly short. Since $a_2 = v^2$, the limit $v^2 \to 0$ is a singular perturbation limit for Eq. (40).

The limiting second-order equation is

$$(c_0 - D_\ell)\phi_\ell = c_0' q_\ell^{(s)}, \tag{42}$$

where

$$c_0 = \lim_{v^2\to0}\left(-\frac{a_0}{a_1}\right) = \chi k_z^2(1 - \Omega_0)(1 - \chi\Omega_0)^{-1},$$

$$c_0' = \lim_{v^2\to0}\left(-\frac{b_0}{a_1}\right) = -\chi\Omega_0(1 - \chi\Omega_0)^{-1}.$$

The corresponding homogeneous equation to Eq. (42) is Bessel's equation with a complex scale factor for the argument. The regular solution at $r = 0$ behaves as $r^{|\ell|}$ as $r \to 0$.

Short of solving the differential Eqs. (40) or (42) and evaluating the integral $\int_0^\infty dr\, \Phi_\ell$ numerically, the approximation of replacing r by r_c in D_ℓ and integrating the resulting Eq. (42) gives

$$\left(\frac{\partial}{\partial r}\Phi_\ell\right)(0) + \frac{1}{r_c}\Phi_\ell(0) + \left(c_0 + \frac{\ell^2}{r_c^2}\right)\int_0^\infty dr\, \phi_\ell = c_0' \int_0^\infty dr\, q_\ell^{(s)},$$

or since for $\ell^2 \geq 1$ $(\partial_r \Phi_\ell)(0) = 0$ and $\Phi_\ell(0) = 0$, we have

$$\int_0^\infty dr\, \Phi_\ell = \left(c_0 + \frac{\ell^2}{r_c^2}\right)^{-1} c_0' \int_0^\infty dr\, q_\ell^{(s)}, \quad \ell^2 > 1. \tag{43}$$

According to Eq. (26) this implies

$$\int_0^\infty dr\, \Phi_\ell = \frac{ic_0' \exp[-i(k_s z_0 - \ell\omega_c t_0)][\exp(i\omega_\ell t_+) - \exp(i\omega_\ell t_-)]}{8\pi^3 r_c \omega_\ell (c_0 + \ell^2/r_c^2)}.$$

By ignoring the k_s dependence coming from c_0 (i.e., setting $c_0 = 0$) Eq. (39) then gives

$$-\frac{i\omega}{\Omega_\pm}\overline{(\bar{\Phi}\nabla_\perp f)} = \delta(z - z_0)B_\pm',$$

with

$$B_\pm' = \sum_{\ell=-\infty}^{\infty} f_{\ell\pm1}^* \frac{\pm(\ell \pm 1)r_c\omega(\omega_r \mp \omega_c)e^{i\ell\omega_c t_0}(e^{i(\omega-\ell\omega_c)t_+} + e^{i(\omega-\ell\omega_c)t_-})}{2\pi\ell^2(\omega - \ell\omega_c)[\omega(\omega_r^2 - \omega_c^2) - \omega_r]} = \sum_{\ell=-\infty}^{\infty} f_{\ell\pm1}^* H_{\pm\ell} Y_\ell, \tag{44}$$

where

$$Y_\ell \equiv e^{i\ell\omega_c t_0}\frac{e^{i(\omega-\ell\omega_c)t_+} - e^{i(\omega-\ell\omega_c)t_-}}{(\omega - \ell\omega_c)},$$

and

$$H_0 \equiv 0$$

$$H_{\pm\ell} \equiv \frac{\pm(\ell \pm 1)r_c\omega(\omega_r \mp \omega_c)}{\ell^2[\omega(\omega_r^2 - \omega_c^2) - \omega_r]}.$$

By straightforward integration it follows from Eq. (27) that

$$\overline{J_\pm^{(s)}} = \delta(z - z_0)\hat{J}_\pm^{(s)},$$

with

$$\hat{J}_\pm^{(s)} = \mp\frac{r_c\omega_c}{(\omega \pm \omega_c)} e^{\mp i\omega_c t_0}(e^{i(\omega\pm\omega_c)t_+} - e^{i(\omega\pm\omega_c)t_-}) = Y_{\mp1}H_{\pm(a)}, \tag{45}$$

where

$$H_{\pm a} = \mp r_c\omega_c.$$

Within the plasma, the z dependence of $A_\pm$ is governed by Eq. (36), thus

$$(\nabla_\parallel^2 + k_\pm^2)\bar{A}_\pm = B_\pm \delta(z - z_0).$$

where

$$k_\pm^2 = \omega^2(1 - \Omega_\pm^{-1}) \quad \text{and} \quad B_\pm = -\hat{J}_\pm^{(s)} + B_\pm'.$$

If the length of the plasma column in the z direction is denoted by L and it is placed at the interval $[0, L]$ on the z axis, then Eq. (46) applies to the range $0 \leq z \leq L$, while an analogous equation which holds for $z \geq L$ has $k_\pm^2$ replaced by ω^2 and $B_\pm = 0$. The additional conditions which uniquely determine the solution are

$$\bar{A}_\pm(0, \omega) = 0 \quad \text{(total reflection at } z = 0\text{)},$$

$$\bar{A}_\pm(L - 0, \omega) = \bar{A}_\pm(L + 0, \omega), \text{ (continuity at } z = L\text{)}, \tag{47}$$

$$\frac{\partial}{\partial z}\bar{A}_\pm(L - 0, \omega) = \frac{\partial}{\partial z}\bar{A}_\pm(L + 0, \omega), \quad 0 < z_0 < L,$$

with $\bar{A}_\pm(z, \omega)$ outgoing wave for $z > L$.

One easily finds outside the plasma that

$$\bar{A}_\pm(z, \omega) = \frac{B_\pm e^{i\omega(z-L)}\sin(k_\pm z_0)}{k_\pm\cos(k_\pm L) - i\omega\sin(k_\pm L)}, \tag{48}$$

and consequently,

$$\bar{A}_\pm(L, \omega) = \frac{B_\pm \sin(k_\pm z_0)}{k_\pm\cos(k_\pm L) - i\omega\sin(k_\pm L)},$$

$$\left(\frac{\partial}{\partial z}A_\pm\right)(L, \omega) = i\omega\bar{A}_\pm(L, \omega). \tag{49}$$

Finally, in order to approximately account for the finite transverse size of the plasma, the far field limit is evaluated in the formula (Green's formula for the Helmholtz equation),

$$A_\pm(r, \omega) = -\frac{1}{4\pi}\int dS' \left\{\frac{e^{i\omega r'}}{r'}\left[\frac{\partial}{\partial z'}A_\pm(r', \omega)\right] - \frac{\partial}{\partial z'}\left(\frac{e^{i\omega r'}}{r'}\right)A_\pm(r', \omega)\right\} \tag{50}$$

with $r' \equiv |\mathbf{r} - \mathbf{r}'|$, $\mathbf{r} = (0, 0, z)$, $\mathbf{r}' = (x', y', L)$, and $dS' = dx'\, dy'$, the element of plasma surfaces at $z = L$.

For $z \gg L$ and $z \ll R_0$, it follows from Eq. (50) that

$$A_{\pm}(r, \omega) = -\frac{1}{2\pi}\frac{e^{i\omega(z-L)}}{(z-L)}\left(\frac{\partial}{\partial z}\bar{A}_{\pm}\right)(L, \omega)$$

$$= \frac{\omega B_{\pm}}{2\pi i(z-L)}\frac{e^{i\omega(z-L)}\sin(k_{\pm}z_0)}{[k_{\pm}\cos(k_{\pm}L) - i\omega\sin(k_{\pm}L)]}. \quad (51)$$

THEORY II. THE POWER SPECTRUM

The results of the previous section will now be used to calculate the power spectrum seen by a radiometer with the receiving area "a" which only receives radiation propagating parallel to the magnetic field. The calculation is done by first forming the Poynting vector at the receiving plane of the radiometer, multiplying it by the receiving area "a", and then integrating it over a time Δt, the integration time of the radiometer. The time Δt, is assumed to be large compared with $T = t_+ - t_-$, the time between two consecutive collisions of the source electron. The resulting radiated energy is a function of the time T. This energy is then averaged over the distribution of times T by using the exponential distribution law with the characteristic time τ. This operation will eventually result in the usual Lorentz line shape around each harmonic. The characteristic time is in dimensionless units written as $\tau = \omega_p \nu_2^{-1}$ where ν_2 is the collision frequency of the source electrons. The energy loss of the source electron is neglected in this process. To remind the reader, ν_1 is the collision frequency of the background electrons. The power radiated by a typical source electron and associated space-charge wave packet is now obtained by multiplying the average energy radiated between two consecutive collisions by the collision frequency τ^{-1} for the source electron. The resulting average radiated power is a function of z_0, the position of a source electron. By assuming that the source electrons are uniformly distributed throughout the plasma and by integrating over the volume of the plasma seen by the radiometer, the dependence on z_0 is removed, and we obtain the total radiated power seen by the radiometer. Multiplying this power with the integration time of the radiometer then gives the response of the radiometer to the observed spectrum. Specifically the following conditions are enforced:

(1) Only plus polarization is accepted by the radiometer; that is, only $\omega = -|\omega|$ is accepted.

(2) The radiometer accepts energy only in a narrow frequency band $\Delta\omega$ which is sufficiently small so that the integration over the frequency can be replaced by the integrand times $\Delta\omega$.

(3) The radiometer integrates the received power over a time Δt which is very large in comparison with the time τ.

(4) The characteristic collision τ is assumed to be sufficiently long so that interference between different "ℓ" values can be neglected.

Executing the operations mentioned above under the stated assumptions, one finds that the power P_e, received by the radiometer from a typical source electron and associated space-charge wave packet located in the plane z_0 can be written as

$$P_e = \frac{\omega^4 \alpha\,\Delta\omega}{2\pi z^2 \tau} FF^* \left\{ \frac{H_{+(a)} H^*_{+(a)}}{(1/\tau)^2 + (\omega + \omega_c)^2} + \sum_{\ell=-\infty}^{\infty} \frac{f_{\ell+1} f^*_{\ell+1} H_{+\ell} H^*_{+\ell}}{(1/\tau)^2 + (\omega - \ell\omega_c)^2} \right\}, \quad (52)$$

where

$$F = \frac{\sin(k_+ z_0)}{k_+ \cos(k_+ L) - i\omega \sin(k_+ L)},$$

and where L, the length of the plasma, has been neglected in comparison with the distance z between the plasma and the radiometer. The asterisk appearing in the formula above labels a conjugate complex quantity.

In order to facilitate numerical calculations and comparison with experimental data it is convenient to introduce the following ratios:

$$x_c = \frac{\omega_c}{|\omega|}, \quad x_p = |\omega|^{-1}, \quad \gamma_1 = \frac{\nu_1}{|\omega|},$$

$$\gamma_2 = \frac{1}{\tau\,|\omega|}, \quad \varphi_0 = |\omega|\, L.$$

Remembering that the radiometer accepts only $\omega < 0$ and changing the notation accordingly, we can write the index of refraction $n = k_+/|\omega|$ explicitly as

$$n = n_r + in_i = \left(1 - \frac{x_p^2}{1 - i\gamma_1 - x_c}\right)^{\frac{1}{2}}. \quad (53)$$

We define further for convenience the expressions

$$Z \equiv \frac{x_p^2(1 - i\gamma_1 + x_c)}{x_c[(1 - i\gamma_1)^2 - x_c^2 - (1 - i\gamma_1)x_p^2]}, \quad (54)$$

$$Z_a \equiv \frac{1}{n\cos(n\varphi_0) + i\sin(n\varphi_0)}, \quad (55)$$

and

$$C_1 \equiv \frac{a v_\perp^2 \omega \gamma_2\,\Delta\omega}{2\pi z^2}. \quad (56)$$

The power received by the radiometer as expressed by formula (52) can then be written as

$$P_e = C_1 P_1 \sin\left(n\varphi_0 \frac{z_0}{L}\right) \sin\left(n^* \varphi_0 \frac{z_0}{L}\right), \quad (57)$$

with P_1 defined as

$$P_1 = Z_a Z_a^* \left(\frac{1}{\gamma_2^2 + (1 - x_c)^2} + ZZ^* \sum_{\ell=-\infty}^{\infty} \frac{(\ell+1)^2}{\ell^4} \frac{f_{\ell+1} f_{\ell+1}^*}{\gamma_2^2 + (1 + \ell x_c)^2}\right). \quad (58)$$

The total power P received by the radiometer is obtained by multiplying P_e with the density n_e of source electrons in the plasma and by integrating over the volume of the plasma seen by the radiometer. If this is a tube with the radius R, one finds the total power P received by the radiometer to be

$$P = CP_1 \left(\frac{\sinh(2\varphi_0 n_i)}{n_i} - \frac{\sin(2\varphi_0 n_r)}{n_r}\right), \quad (59)$$

where

$$C = \frac{aR^2 \gamma_2 n_e v_\perp^2 \, \Delta\omega}{8z^2 (2\pi)^2}.$$

The formulas above are written in terms of the dimensionless parameters defined earlier. Translating the formulas back to the mks system, one finds that formula (59) is applicable provided the coefficient C is now written as

$$C = \left(\frac{e^2}{8\epsilon_0 c^2}\right)\left(\frac{aR^2}{(2\pi^2)z^2}\right)(\nu_2 n_e v_\perp^2) \frac{\Delta\omega}{|\omega|} \quad (60)$$

and provided

$$x_c = \frac{\omega_c}{|\omega|}, \quad x_p = \frac{\omega_p}{|\omega|}, \quad \gamma_1 = \frac{\nu_1}{|\omega|}$$

$$\gamma_2 = \frac{\nu_2}{|\omega|}, \quad \varphi_0 = \frac{|\omega|}{c} L.$$

It is obvious from formula (58) that the space charge wave packet generated by the source electron does not radiate at the harmonics of the electron cyclotron frequency unless the coefficients f_ℓ are different from zero. These coefficients are a measure of the nonuniformities of the plasma along the orbit of a typical source electron. It is instructive to consider the effect on the radiated spectrum by two radically different classes of nonuniformities.

The first class is represented by a step in the electron density. A source electron traversing this step will see it at two points along the orbit. Assuming that all locations of the orbit with respect to the step in the electron density are equally probable, it is easily shown by averaging over the position that

$$\langle f_\ell f_\ell^* \rangle = \frac{f_0^2}{2\pi^2(\ell^2 - \frac{1}{4})}, \quad (61)$$

where f_0 is the relative height of the step discontinuity.

The second class of discontinuities is perhaps best illustrated in terms of the crude model of microscopic oscillators described in the introduction. The heavy mass of the ion relative to that of the electron indicates that the statistics of these oscillators is intimately related to the statistics of the location of the ions and less so to the corresponding statistics of the electrons. In the limit of cold and weakly ionized plasmas we can safely make the assumption that the ions are randomly distributed. In this case it is easily shown that the corresponding statistics gives

$$\langle f_\ell f_\ell^* \rangle = \frac{1}{N(2\pi)^2}, \quad (62)$$

where N is the average number of oscillator effectively interacting with the source electron. Note that the coefficients given above are independent of the ℓ value. It is possible that a more detailed analysis of the statistics will show some kind of structure which will make the coefficient functions of ℓ and hence will show up in the emission spectrum. The suggested models for arriving at values for $\langle f_\ell f_\ell^* \rangle$ are strongly idealized and do not exist in the plasma in those forms. However, it can be expected that they give the correct descriptions for low ℓ values. The upper limit is crudely given by $\ell_{max} = 2\pi r_e/d$, where r_e is the radius of cyclotron orbit of the source electron and d is the effective width of a realistic step or pulse in the electron density. A measure for ℓ_{max} is obtained by using the Debye length for d giving

$$\ell_{max} = 2\pi\left(\frac{v_\perp}{v}\right)\left(\frac{\omega_p}{\omega_c}\right), \quad (63)$$

where $v_\perp$ is the velocity of the source electron perpendicular to the magnetic field and v is the thermal velocity of electrons in the cool background plasma. Radiation at high harmonics of the electron cyclotron frequency can, therefore, be seen in plasmas with hot source electrons in a cold background plasma.

The model suggested here for the cyclotron harmonic radiation is based on the assumption that the response of the cold background plasma to the hot source electron is adequately described by the plasma hydrodynamic equations. The introduction of "pulses" of electron density along the orbit of the source electron may seem contradictory to this assumption. However, if one considers the rather small volume of plasma or the corresponding small number of ions that are sampled by a typical source electron, it is no longer surprising that the cyclotron harmonic radiation can be directly attributable to the fine structure of the plasma. The source electrons sample a volume in the form of a toroidal tube $2\pi^2 r_c \Lambda_D^2$, where Λ_D is the Debye length. The average number N of ions contained in this volume can be written as

$$N = \frac{\pi}{2} \frac{\omega_p}{\omega_c} \frac{v_\perp}{v} N_D,$$

where N_D is the number of ions per Debye sphere. This number N is in the range 10 to 10^3 in the helium abnormal negative glow plasma. It is a sufficiently small number so that statistical fluctuations in the number density along the orbit of the source electron must be considered. As the thermal velocity of the ions is sufficiently small relative to the velocity of a typical source electron, the ion configuration along the orbit of the source electron remains fixed during the sampling time of the source electron. With regard to the present calculation the ion configuration is stationary as long as the ratio $(\nu_2/\omega_p)^2$ is much larger than the electron to ion mass ratio. The lack of ℓ dependence of Eq. (62) is most easily understood as being due to the thermal fluctuations of the ion gas.

SOME NUMERICAL RESULTS

The results from the calculations of some spectra based on the model are illustrated in Fig. 2. For the sake of convenience in this calculation we have divided the plasma into groups of electrons; the group of cold electrons that constitute the background plasma and a group of hot electrons that act as source electrons. Furthermore we have assumed that the model is applicable to both groups of electrons and that the collision frequencies, densities, and energies of the groups are such that the coefficient C [Eq. (60)] is the same for the two groups. In the calculation of the radiation from the cold electron group we have also included bremsstrahlung which appropriately is accounted for by adding a term equal to unity inside the bracket of Eq. (58). The normalized collision frequencies for this group of electrons have been set as $\gamma_1 = \gamma_2 = 0.25$. The influence of bremsstrahlung from the hot electron

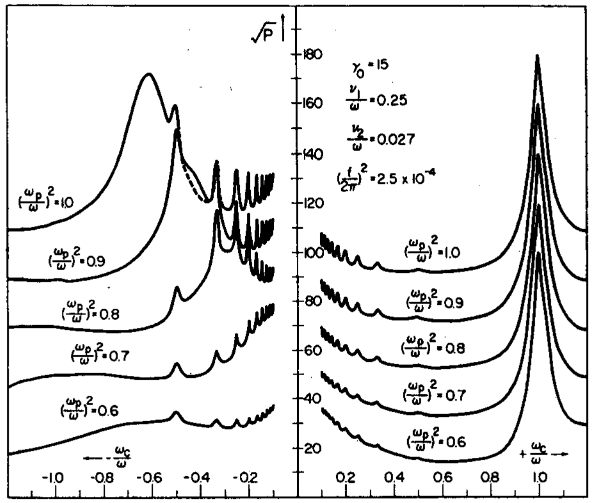

FIG. 2. Typical theoretical emission spectra in the neighborhood of the electron cyclotron frequency.

group was considered insignificant and has been neglected. The normalized collision frequencies for this group of electrons were set as $\gamma_1 = 0.25$ and $\gamma_2 = 0.027$. The parameter φ_0 was set equal to 15 corresponding to plasma length L of approximately 10 cm at an observation frequency of 10 GHz. We used finally the stochastic model for the fluctuations and set $\langle f_\ell f_\ell^* \rangle = 2.5 \times 10^{-4}$. The calculation has been carried out for five different values of the ratio ω_p/ω.

The figure displays both the right -and left-hand circularly polarized radiation as functions of the ratio ω_c/ω with the amplitude proportional to the square root of the power in arbitrary units. The curves are normalized such that the amplitude at the fundamental of the electron cyclotron frequency ω_c/ω is 10^2 and displaced 20 units relative to each other. The spectra are obviously asymmetric with respect to polarization, and this asymmetry is a strong function of the ratio ω_p/ω. The radiation at the fundamental electron cyclotron frequency appears essentially only as plus polarization, as would be the case if the electrons alone were responsible for the radiation. Radiation at the harmonics of the electron cyclotron frequency have both plus and minus polarization. The minus polarization of the harmonic radiation becomes very strong when the ratio ω_p/ω approaches unity. This is partially due to a weak asymmetry in the source function H_ℓ but primarily due to the asymmetric index of refraction coupled with reflections at the boundary as well as due to a resonance in the space-charge wave packet occurring at $\omega^2 = \omega_c^2 + \omega_p^2$ as shown by the factor ZZ^* in expression (58).

SUMMARY AND DISCUSSION

Many attempts have been made in the past to explain the basic mechanism causing the observed harmonic electron cyclotron radiation. Since Bekefi discusses them in his recent book,[4] we will not do so here. It is common to these theories that the radiation losses within the plasma are carried exclusively by the longitudinal waves and that these waves, when they hit the boundaries or some other nonuniformity in the plasma, are converted into electromagnetic waves in a manner presently inaccessible to theory. These theories have neither explained the shape of the spectrum nor accounted for the intensity of the radiation.

Many different mechanisms can give radiation at the harmonics of the electron cyclotron frequency. To distinguish between these mechanisms, it is presently necessary to develop the corresponding theories to a sufficient degree, so that the shape of the spectrum, its intensity, and some other characteristic features can be compared with the experimental data. The model presented in this paper is not necessarily the final explanation for the observed spectrum, but it satisfies the criteria mentioned above and is sufficiently close to the experimental data, in particular the data obtained from the abnormal negative glow plasma, both in the shape of the spectrum and the intensity, to merit further sophistication.

The model proposed in this paper has common elements with theories suggested in the past; nevertheless, it is conceptually different. It requires hot electrons in a relatively cool background plasma. These hot electrons may be supplied by the high-velocity end of the electron-velocity distribution. For the sake of simplicity, this model considers a hot electron which pursue's a cyclotron orbit with a radius much larger than the Debye length. This electron excites plasma oscillations along its orbit which are primarily confined to a Debye length of the orbit. The amplitude of the space charge wave packet resonates when the hybrid frequency $(\omega_c^2 + \omega_p^2)^{\frac{1}{2}}$ is an integer times the cyclotron frequency. We showed that this wave packet has cyclotron harmonics in it; however we also found that this packet does not radiate at the harmonics if the background plasma is perfectly uniform. Because the small volume of plasma excited by the source electron contains so few ions there are significant thermal fluctuations. These nonuniformities allow electromagnetic radiation to occur. Once we had the mechanism for the radiation, we calculated its numerical form by standard methods. We note that the asymmetry between the $\pm$ polarization is due to the differences in the radiative absorption of the plasma.

Although subject to many simplifications, this theory results in data that are compatible with the experimental information. A following paper discusses this point more thoroughly.[2] When a comparison is made with experimental data, the following points should be considered: It is likely that the influence of the boundary condition has been exaggerated in the present calculation. We have assumed a step profile for the plasma at one end and a perfectly reflecting boundary at the other. This is not realized in practice. The influence of the plasma frequency through the index of refraction is therefore stronger than it should be. The plasma frequency appearing in the formula above has been viewed as a fixed parameter common for all points in the plasma. This again is not strictly true in practice due to the presence of macroscopic nonuniformities. The use of two groups of electrons instead of a smooth distribution function is an idealization that will influence somewhat the shape of the radiation spectra. However, the elimination of these theoretical simplifications is expected to change the spectra in minor ways and not its major characteristics.

ACKNOWLEDGMENTS

The authors acknowledge the accurate programming effort of Marlene Pratto.

This work partially supported by Defense Atomic Support Agency.