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REDISTRIBUTION OF RESONANCE RADIATION.
II. THE EFFECT OF MAGNETIC FIELDS*

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ABSTRACT

Previously obtained results for scattering of radiation in the presence of collisions are restated in a density matrix formalism which employs an irreducible-tensor description of the radiation field. This formalism is particularly useful for problems associated with radiative transfer theory. The redistribution is then extended to include the effect of a weak magnetic field. By averaging over a finite bandwidth which is on the order of the Doppler width, simplified expressions of physical significance for the scattering in the Doppler core and the Lorentz wings are obtained. Expressions are also obtained for the corresponding source function of radiative transfer theory.

Subject headings: magnetic fields — polarization — radiative transfer

I. INTRODUCTION

In a preceding paper (Omont, Smith, and Cooper 1972, hereafter referred to as Paper I), we derived the general expressions for the intensity of scattered light as a function of incident and emergent polarizations and frequencies. The formalism of Fiutak and Van Krandendonk (1962) was closely followed, and the radiation field was described in terms of "pure" polarization states. In the present paper, these results will be restated in a more general formalism which employs an irreducible-tensor description of the radiation field. This formalism will be particularly useful for the multiple-scattering problems which arise in radiative transfer theory. This density matrix formalism is outlined in § II and a detailed discussion is given in Appendix A.

The theory is extended in § III to include the effect of a weak magnetic field. To simplify the mathematics, the incident and scattered radiation are averaged over a finite bandwidth Δ which is on the order of the Doppler width. The physical significance of the results obtained for frequencies in the Doppler core or in the Lorentz wings are discussed in § IV. In § V, the corresponding expressions for the source function of radiative transfer theory are presented; it is shown that these expressions can be reduced to very simple form for the core as well as for the Lorentz wings of the line. In the Doppler core, Hanle effect and depolarizing collisions are important processes; in the wings, only the coherent-scattering term is to be taken into account for polarization computations, as was pointed out by Zanstra (1941).

In this paper (as in Paper I) we stress the details of collisional effects on the frequency redistribution of polarized light. For a broader review of the density matrix formulation applied to emission and absorption of polarized light, the reader is referred to Lamb (1971), Lamb and ter Haar (1971), and the review of optical pumping by Happer (1972).

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II. DENSITY MATRIX FORMALISM

In Paper I we calculated the probability of scattering an incident plane wave with frequency ω_1 , propagation vector \mathbf{k}_1 , and polarization vector $\boldsymbol{\epsilon}_1$ into an outgoing plane wave $\omega_2, \mathbf{k}_2, \boldsymbol{\epsilon}_2$. In equations (41) and (42) of Paper I this probability was expressed in the form

$$F(\omega_1, \omega_2) = \sum_K [M_1^K f_1^K(\omega_1 \omega_2) + M_{23}^K f_{23}^K(\omega_1 \omega_2)], \quad (1)$$

where f_1^K and f_{23}^K are frequency-dependent (line shape) factors. The amplitude factors M_1^K and M_{23}^K were given by equations (47) and (48) in the form

$$M_1^K = 3(2j_e + 1)A_{f_{ie}} \left\{ \begin{matrix} 1 & 1 & K \\ j_f & j_i & j_e \end{matrix} \right\}^2 \sum_Q (-1)^Q P_{-Q}^K(\boldsymbol{\epsilon}_2^* \boldsymbol{\epsilon}_1) P_Q^K(\boldsymbol{\epsilon}_1^* \boldsymbol{\epsilon}_2),$$

$$M_{23}^K = A_{f_{ie}} W(j_f j_i j_e K) \sum_Q (-1)^Q P_{-Q}^K(\boldsymbol{\epsilon}_1 \boldsymbol{\epsilon}_1^*) P_Q^K(\boldsymbol{\epsilon}_2 \boldsymbol{\epsilon}_2^*), \quad (2)$$

where j_i, j_e, j_f refer to the initial, excited, and final atomic states involved in the scattering and P_Q^K is a K th-rank polarization tensor formed from the tensor components of the polarization vectors $\boldsymbol{\epsilon}_1$ and $\boldsymbol{\epsilon}_2$ (see eq. [B4] of Paper I):

$$P_Q^K(\boldsymbol{\epsilon}_2^* \boldsymbol{\epsilon}_1) = - \sum_{qq'} \langle 11qq' | KQ \rangle (\boldsymbol{\epsilon}_2^*)_q (\boldsymbol{\epsilon}_1)_{q'}. \quad (3)$$

In order to express these results in a density matrix formalism, we consider the density matrix (or polarization matrix) for an incident plane wave ω_1, \mathbf{k} . This density matrix is given by equation (A3) of Appendix A in the form

$$\rho_{Rk}(\omega_1) = \sum_{\alpha, \beta} \rho_{k\alpha\beta}(\omega_1) |(\mathbf{k})e_\alpha\rangle \langle (\mathbf{k})e_\beta| \quad (4)$$

(the subscript R is used to distinguish radiation field operators from operators on particle states). Using equations (1)–(4), the density matrix for the outgoing plane wave ω_2, \mathbf{k}' is just

$$\rho_{k'\alpha'\beta'}(\omega_2) = \sum_{\alpha\beta K} A_{f_{ie}} \left[3(2j_e + 1) \left\{ \begin{matrix} 1 & 1 & K \\ j_f & j_i & j_e \end{matrix} \right\}^2 f_1^K(\omega_1 \omega_2) \sum_Q (-1)^Q P_{-Q}^K(e_{\alpha'}^* e_\alpha) P_Q^K(e_\beta^* e_{\beta'}) \right. \\ \left. + W(j_f j_i j_e K) f_{23}^K(\omega_1 \omega_2) \sum_Q (-1)^Q P_{-Q}^K(e_\alpha e_{\beta'}^*) P_Q^K(e_\beta e_{\alpha'}^*) \right] \rho_{k\alpha\beta}(\omega_1), \quad (5)$$

where e_α, e_β and $e_{\alpha'}, e_{\beta'}$ are the various unit polarization vectors orthogonal to \mathbf{k} and \mathbf{k}' respectively.

Notice that this result does not depend on the directions of \mathbf{k} and \mathbf{k}' (except that e_α, e_β and $e_{\alpha'}, e_{\beta'}$ must be orthogonal to \mathbf{k} and \mathbf{k}' , respectively). This is a property of electric dipole processes; that is, the absorption depends on the electric field strength at the atom but not on its spatial distribution. One is therefore led to consider a matrix representation which is independent of \mathbf{k} . Several such representations are discussed in Appendix A, but the one of interest here is the multipolar or irreducible-tensor expansion (see eqs. [A9] *et seq.*)

$$\rho_R = \sum_{KQ} (-1)^Q \rho_{-Q}^K |R; 11KQ\rangle\rangle, \quad (6)$$

$$|R; 11KQ\rangle\rangle = \sum_{qq'} (-1)^{1-q'} \langle 11q - q' | KQ \rangle |v_q\rangle \langle v_{q'}|, \quad (7)$$

where v_q ($q = 0, \pm 1$) is a unit vector in spherical tensor notation (eq. [A7]). The transformation from the $\rho_{\alpha\beta}$ representation to ρ_Q^K is shown to be (see eq. [A14])

$$\rho_Q^K(\omega_1) = \sum_{\alpha\beta} P_Q^K(e_\beta^*, e_\alpha)\rho_{\alpha\beta}(\omega_1), \tag{8}$$

and similarly for $\rho_Q^K(\omega_2)$. We therefore use the identity (verify using Messiah 1966 eq. [C34])

$$\begin{aligned} & \sum_Q (-1)^Q P_{-Q}^K(e_{\alpha'}^* e_\alpha) P_Q^K(e_{\beta'}^* e_{\beta'}) \\ &= \sum_{K'} (2K' + 1) (-1)^{K+K'} \begin{Bmatrix} 1 & 1 & K \\ 1 & 1 & K' \end{Bmatrix} \sum_Q (-1)^Q P_{-Q}^{K'}(e_{\beta'}^* e_\alpha) P_Q^{K'}(e_{\alpha'}^* e_{\beta'}) \end{aligned}$$

to transform the f_1^K term, and the identity

$$P_{-Q}^K(e_\alpha e_{\beta'}^*) P_Q^K(e_{\beta'} e_{\alpha'}^*) = P_{-Q}^K(e_{\beta'}^* e_\alpha) P_Q^K(e_{\alpha'}^* e_{\beta'})$$

to transform the f_{23}^K term. Multiplying equation (5) by $P_{-Q}^K(e_{\beta'}^* e_{\alpha'})$, summing over $\alpha'\beta'$, and using $\sum_{\alpha'} (e_{\alpha'})_{-p} (e_{\alpha'}^*)_q = (-1)^q \delta_{p,q}$ (see eq. [A8] and note that $\sum_{\alpha} \mu_{\alpha p} \mu_{\alpha q}^* = \delta_{pq}$), we finally obtain

$$\begin{aligned} \rho_{-Q}^K(\omega_2) = A_{f_{1e}} & \left[W(j_f j_i j_e K) f_{23}^K(\omega_1, \omega_2) + 3(2j_e + 1) \sum_{K'} (-1)^{K+K'} (2K' + 1) \right. \\ & \left. \times \begin{Bmatrix} 1 & 1 & K' \\ 1 & 1 & K \end{Bmatrix} \begin{Bmatrix} 1 & 1 & K' \\ j_f & j_i & j_e \end{Bmatrix}^2 f_1^{K'}(\omega_1, \omega_2) \right] \rho_{-Q}^K(\omega_1), \tag{9} \end{aligned}$$

which is the desired density matrix formulation of the scattering process.

In the irreducible-tensor density-matrix formulation, the scattering is diagonal in K and independent of Q ; this is a consequence of the spherical symmetry of the problem, and it shows that this formalism is particularly well suited to spherically symmetric problems.

Equation (9) may be regarded as an identity for the 3×3 density matrix which describes the electric dipole part of the radiation field. That is, equation (9) is more general than our derivation (based on the density matrix for a *single* plane wave) would indicate; for example, for a unique plane wave, $\rho(\omega_1)$ could be brought to the form of a 2×2 matrix $\rho_{Rk}(\omega_1)$ by a suitable transformation (see Appendix A), but in the general case this is not possible. We have introduced the density matrix formulation as a simple generalization of the more elementary results of Paper I, although a more formal *a priori* derivation is also possible. One feature which is not obvious from our generalization of Paper I is the fact that, in the presence of a magnetic field, the real functions f_1^K and f_{23}^K become complex functions of both K and Q , as we shall see in the following section. In equation (A12) of Appendix A we show that the Hermitian nature of ρ_R requires that the matrix elements of ρ_Q^K satisfy $(\rho_Q^K)^* = (-1)^Q \rho_{-Q}^K$, which requires that $f_1^{KQ} = (f_1^{K-Q})^*$ and similarly for f_{23}^{KQ} (cf. eq. [9] or [20]).

III. WEAK MAGNETIC FIELD

We now consider the effect of a small magnetic field on the light scattering. The atomic Hamiltonian becomes

$$H_0' = H_0 + H_M = H_0 + \sum_{\alpha} \hbar \omega_{\alpha} J_{\alpha z} \tag{10}$$

where H_0 refers to the unperturbed atom, H_M is the Zeeman Hamiltonian, the z-axis is parallel to the magnetic field \mathcal{H} , and ω_α is the Larmor frequency of the state α . The magnetic field is assumed to be small enough that $\omega_\alpha \tau_c \ll 1$ where τ_c is the mean duration of a binary collision. This means that binary collisions are unaffected by \mathcal{H} , hence the operator Φ which describes collisional relaxation (see eqs. [12] and [14] of Paper I) is unchanged. The effect of the magnetic field is then included in the results of Paper I by simply writing the operators \hat{A} in the form (cf. eq. [12] of Paper I)

$$\hat{A}(\tau) = \exp[-i\tau(L_0 + L_M - i\Phi)], \quad (11)$$

where L_M is the Liouville operator corresponding to the Zeeman Hamiltonian H_M . The operator L_M is defined by its action on any matrix B (where B is a matrix operator on atomic states)

$$L_M B = [H_M, B]. \quad (12)$$

The evaluation of the scattering process could now proceed as in § IV of Paper I using the coupled basis states $|j_a j_b K Q\rangle\rangle$ defined by equation (25) of that paper. The only new feature of the weak-magnetic-field problem is in the operators L_M or H_M which, unlike the unperturbed operators L_0 or H_0 , are not diagonal in $|j_a j_b K Q\rangle\rangle$. This results in a rather complicated coupling of tensors which would tend to obscure the physics of the problem with unnecessary mathematical complications. We will avoid this problem by the artifice of frequency averaging.

We assume that the spectrum of the incident radiation has a spread Δ about the average value $\bar{\omega}_1$. This bandwidth is assumed to be much larger than the natural widths, the collision frequency (pressure width Φ), and all Larmor frequencies. We also assume that the scattered radiation is smeared out over a bandwidth Δ about $\bar{\omega}_2$. This approximates the smearing effect of Doppler broadening when Δ is on the order of the Doppler width Δ_D . This model should adequately describe the depolarization due to collisions or magnetic rotation (Hanle effect). It will not be adequate for the absolute Zeeman shifts which are used in magnetographs (see, e.g., Stenflo 1971); however, these shifts give only small effects, which are hidden in the overall Doppler width.

A comparison of Δ with $|\omega_1 - \omega_{ei}|$, $|\omega_2 - \omega_{ei}|$, and $|\omega_1 - \omega_{ei} - \omega_2 + \omega_{ef}|$ leads to the consideration of five separate cases:

$$\begin{aligned} \text{Case A: } & |\bar{\omega}_1 - \omega_{ei}| \ll \Delta, \quad |\bar{\omega}_2 - \omega_{ef}| \ll \Delta; \\ \text{Case B: } & |\bar{\omega}_1 - \omega_{ei}| \gg \Delta, \quad |\bar{\omega}_2 - \omega_{ef}| \ll \Delta; \\ \text{Case C: } & |\bar{\omega}_1 - \omega_{ei}| \ll \Delta, \quad |\bar{\omega}_2 - \omega_{ef}| \gg \Delta; \\ \text{Case D: } & |\bar{\omega}_1 - \omega_{ei}| \gg \Delta, \quad |\bar{\omega}_2 - \omega_{ef}| \gg \Delta, \quad |\bar{\omega}_1 - \omega_{ei} - \bar{\omega}_2 + \omega_{ef}| \gg \Delta; \\ \text{Case E: } & |\bar{\omega}_1 - \omega_{ei}| \gg \Delta, \quad |\bar{\omega}_2 - \omega_{ef}| \gg \Delta, \quad |\bar{\omega}_1 - \omega_{ei} - \bar{\omega}_2 + \omega_{ef}| \ll \Delta, \end{aligned} \quad (13)$$

where $\bar{\omega}_1$ and $\bar{\omega}_2$ are the mean frequencies of the incident and scattered light.

We now consider the Fourier transform of the \hat{A} operators (see eqs. [9], [10], and [11] of Paper I). The required integrals are all of the general form

$$\int_0^\infty \exp[i(\omega_p - L_0 - L_M + i\Phi)\tau] d\tau = i(\omega_p - L_0 - L_M + i\Phi)^{-1}, \quad (14)$$

where ω_p represents the various frequencies $\pm\omega_1$, $\pm\omega_2$, $\pm(\omega_1 - \omega_2)$, and 0. Since H_0 is degenerate within a multiplet, L_0 may be replaced by its eigenvalues ω_{ab} when it operates on $|ab\rangle\rangle = |j_a m_a j_b m_b\rangle\rangle$, where a or b can equal e, f, i (see eq. [13] of Paper

I). There are three cases of interest (see eqs. [20], [21], and [22] of Paper I) which become, under the frequency averaging discussed above,

$$(i) = (i/\Delta^2) \int_{\bar{\omega}_1 - \Delta/2}^{\bar{\omega}_1 + \Delta/2} d\omega_1 \int_{\bar{\omega}_2 - \Delta/2}^{\bar{\omega}_2 + \Delta/2} d\omega_2 \langle\langle fe | (\omega_2 - \omega_{ef} + L_M - i\Phi)^{-1} | fe \rangle\rangle \times \langle\langle fi | (\omega_1 - \omega_{ei} - \omega_2 + \omega_{ef} - L_M + i\Phi)^{-1} | fi \rangle\rangle \times \langle\langle ei | (\omega_1 - \omega_{ei} - L_M + i\Phi)^{-1} | ei \rangle\rangle, \quad (15)$$

$$(ii) + (iii) = (i/\Delta^2) \int_{\bar{\omega}_1 - \Delta/2}^{\bar{\omega}_1 + \Delta/2} d\omega_1 \int_{\bar{\omega}_2 - \Delta/2}^{\bar{\omega}_2 + \Delta/2} d\omega_2 \langle\langle fe | (\omega_2 - \omega_{ef} + L_M - i\Phi)^{-1} | fe \rangle\rangle \times \langle\langle ee | (L_M - i\Phi)^{-1} | ee \rangle\rangle \times [\langle\langle ie | (\omega_1 - \omega_{ei} + L_M - i\Phi)^{-1} | ie \rangle\rangle - \langle\langle ei | (\omega_1 - \omega_{ei} - L_M + i\Phi)^{-1} | ei \rangle\rangle], \quad (16)$$

where we have used $|fi\rangle\rangle$ as a shorthand notation for $|j_f m_f j_i m_i\rangle\rangle$, etc. The functions $F(i)$ and $F(ii) + F(iii)$ are obtained from equations (15) and (16) after multiplying by the matrix elements of $\epsilon \cdot \mu$ and summing over m as in equations (9), (10), and (11) of Paper I or equations (B1) and (B3) of the present paper.

These expressions will now be discussed for each of the different cases defined in equation (13).

Case A: $|\bar{\omega}_1 - \omega_{ei}| \ll \Delta$, $|\bar{\omega}_2 - \omega_{ef}| \ll \Delta$. The absorption and emission are both resonant, hence we extend the limits of the integrals to $(\infty, -\infty)$. Since the matrix elements of L_M are real and the real part of Φ is positive (cf. eq. [14] of Paper I), the ω_2 integral in equation (15) is of the form

$$\int_{-\infty}^{\infty} \frac{d\omega}{(\omega - \alpha)(\omega - \beta)} = 0, \quad (17)$$

where α and β are complex numbers whose imaginary parts are positive. The integrals over equation (16) are of the form

$$\int_{-\infty}^{\infty} \frac{d\omega}{\omega \pm \alpha} = \ln(-1 \mp i0) = \mp i\pi. \quad (18)$$

The (i) term thus vanishes by equation (17) and

$$(ii) + (iii) = (\pi/\Delta^2) \langle\langle fe | fe \rangle\rangle \times \langle\langle ee | (\Phi + iL_M)^{-1} | ee \rangle\rangle [\langle\langle ei | ei \rangle\rangle + \langle\langle ie | ie \rangle\rangle]. \quad (19)$$

In the weak-magnetic-field case, one still obtains results in the general form of equation (9). For case A which we have been considering, the f_1 term is zero [because the (i) term integrated to zero] and equation (9) becomes

$$\rho_{-Q}^K(\omega_2) = A_{fie} W(j_f j_i j_e K) f_A^{KQ} \rho_{-Q}^K(\omega_1), \quad (20)$$

$$f_A^{KQ} = 2(\pi/\Delta)^2 (\gamma_e^{(K)} + iQ\omega_{He})^{-1}. \quad (21)$$

This result could have been deduced from equation (46) of Paper I by averaging over ω_1 and ω_2 and replacing $\gamma_e^{(K)}$ by $(\gamma_e^{(K)} + iQ\omega_{He})$ corresponding to the diagonal elements of $(\Phi + iL_M)$ (recall that $\gamma_{ab}^{(K)}$ is the real part of $\Phi_{ab}^{(K)}$: see eqs. [30] and [32] of Paper I). Notice that the KQ term of the incident radiation field induces only a KQ component in the excited state.

Case B: $|\bar{\omega}_1 - \omega_{ei}| \gg \Delta$, $|\bar{\omega}_2 - \omega_{ef}| \ll \Delta$. This corresponds to absorption of a photon in the Lorentz wings and reemission in the Doppler core. In this case the integral $\Delta^{-1} \int d\omega_1$ simply replaces ω_1 by $\bar{\omega}_1$. Further, since $|\bar{\omega}_1 - \omega_{ei}| \gg |\bar{\omega}_2 - \omega_{ef}|$, we will expand factors like $(\bar{\omega}_1 - \omega_{ei} - \omega_2 + \omega_{ef} \pm L_M \mp i\Phi)$ to lowest order in powers of $(\bar{\omega}_1 - \omega_{ei})^{-1}$. The ω_2 integral is evaluated by using equation (18), and one obtains

$$(i) = -(\pi/\Delta)(\bar{\omega}_1 - \omega_{ei})^{-2} \langle\langle fe|fe \rangle\rangle \langle\langle fi|fi \rangle\rangle \langle\langle ei|ei \rangle\rangle, \quad (22)$$

$$(ii) + (iii) = (\pi/\Delta)(\bar{\omega}_1 - \omega_{ei})^{-2} \langle\langle fe|fe \rangle\rangle \langle\langle ee|(\Phi - iL_M)^{-1}|ee \rangle\rangle \\ \times [\langle\langle ei|(\Phi + iL_M)|ei \rangle\rangle + \langle\langle ie|(\Phi + iL_M)|ie \rangle\rangle], \quad (23)$$

where it was necessary to expand $(\bar{\omega}_1 - \omega_{ei} - L_M \pm i\Phi)^{-1}$ to second order in $(\bar{\omega}_1 - \omega_{ei})^{-1}$ in equation (23) because the first-order terms cancel.

In Appendix B it is shown that these results can again be put in the form of equation (20) with

$$f_B^{KQ} = (\pi/\Delta)(\bar{\omega}_1 - \omega_{ei})^{-2} (2\gamma_{ei}^{(1)} - \gamma_e^{(K)}) / (\gamma_e^{(K)} + iQ\omega_{He}). \quad (24)$$

Case C: $|\bar{\omega}_1 - \omega_{ei}| \ll \Delta$, $|\bar{\omega}_2 - \omega_{ef}| \gg \Delta$. This case is essentially the same as case B; the (i) term is the same as equation (22) except that $(\bar{\omega}_1 - \omega_{ei})$ is replaced by $(\bar{\omega}_2 - \omega_{ef})$ and, in the (ii) + (iii) term, we again expand $(\omega_2 - \omega_{ef} - L_M - i\Phi)^{-1}$ to second order in $(\bar{\omega}_2 - \omega_{ef})^{-2}$ because the first-order term vanishes due to the requirement that $f^{KQ} = (f^{K-Q})^*$ discussed in § II. In this manner we obtain

$$f_C^{KQ} = (\pi/\Delta)(\bar{\omega}_2 - \omega_{ef})^{-2} (2\gamma_{ef}^{(1)} - \gamma_e^{(K)}) / (\gamma_e^{(K)} + iQ\omega_{He}). \quad (25)$$

Case D: $|\bar{\omega}_1 - \omega_{ei}| \gg \Delta$, $|\bar{\omega}_2 - \omega_{ef}| \gg \Delta$, $|\bar{\omega}_2 - \omega_{ei} - \bar{\omega}_1 + \omega_{ef}| \gg \Delta$. Both ω_1 and ω_2 are replaced by their mean values $\bar{\omega}_1$ and $\bar{\omega}_2$. There is no more integration; hence the expressions are more complicated. Following the methods of Appendix B, one can show that

$$f_D^{KQ} = \frac{1}{2} \frac{(2\gamma_{ef}^{(1)} + iQ\omega_{He})}{(\bar{\omega}_1 - \omega_{ei})^2 (\bar{\omega}_2 - \omega_{ei})^2} \left[\frac{(2\gamma_{ei}^{(1)} + iQ\omega_{He})}{(\gamma_e^{(K)} + iQ\omega_{He})} - 1 \right] + \dots \\ = \frac{1}{2} \frac{(2\gamma_{ei}^{(1)} - \gamma_e^{(K)})}{(\bar{\omega}_1 - \omega_{ei})^2 (\bar{\omega}_2 - \omega_{ei})^2} \left[1 + \frac{(2\gamma_{ef}^{(1)} - \gamma_e^{(K)})}{(\gamma_e^{(K)} + iQ\omega_{He})} \right] + \dots, \quad (26)$$

where the first term in the brackets of the first equation comes from (ii) + (iii), the -1 term comes from (i), and the dots denote additional terms arising from (i) which are independent of the magnetic field and which vanish when $\gamma_{ei}^{(1)} = \gamma_{ef}^{(1)}$ and $\gamma_{if}^{(K)} = 0$ (e.g., when $i = f$ and there is no quenching in the ground state). The calculation of equation (26) is not given in Appendix B, but it is obtained by a straightforward though tedious application of the techniques presented there.

Case E: $|\bar{\omega}_1 - \omega_{ei}| \gg \Delta$, $|\bar{\omega}_2 - \omega_{ef}| \gg \Delta$, $|\bar{\omega}_1 - \omega_{ei} - \bar{\omega}_2 + \omega_{ef}| \ll \Delta$. This is the case of "coherent" scattering. The leading term comes from equation (15) where integration over $|\omega_1 - \omega_{ei} - \omega_2 + \omega_{ef}|$ yields

$$f_E = (\pi/\Delta)(\bar{\omega}_1 - \omega_{ei})^{-2}. \quad (27)$$

IV. DISCUSSION OF RESULTS

The physical interpretation of the frequency-dependent (line shape) functions given in § V of Paper I was based on a two-state model atom. This model is not sufficient

for a discussion of magnetic field effects, so we consider instead the case $j_i = j_f = 0$, $j_e = 1$, where the upper level has three magnetic substates; we also neglect perturbation of the lower level. From equation (32) of Paper I we see that $\gamma_e^{(K)}$ is the sum of the collisional damping rate for the 2^K -pole moment of the excited level $\gamma_e^{c(K)}$, and the radiative damping rate Γ_e (the natural life of the excited state is $1/\Gamma_e$). The term $\gamma_e^{c(1)}$ is given by the sum of the mean radiative damping rate $\frac{1}{2}(\Gamma_e + \Gamma_i)$ and the collision broadened line width for $e \leftrightarrow i$ transitions $\gamma_{ei}^{c(1)}$. From equations (30), (61), and (A4) of Paper I we see that $2\gamma_{ei}^{c(1)}$ is just the total collision rate:

$$\gamma_{ei}^{c(1)} = \text{Re} \left\{ 1 - \frac{1}{3} \sum_{m_e} \langle 1m_e | S | 1m_e \rangle \right\}_{\text{Av}}. \quad (28a)$$

Similarly we see that $\gamma_e^{c(0)}$ is the rate at which collisions destroy (quench) the excited states by inducing inelastic collisions out of the $j_e = 1$ level:

$$\gamma_e^{c(0)} = \left\{ 1 - \frac{1}{3} \sum_{m_e m_e'} |\langle 1m_e | S | 1m_e' \rangle|^2 \right\}_{\text{Av}}. \quad (28b)$$

The terms $\gamma_e^{c(1)}$ and $\gamma_e^{c(2)}$ are respectively the collisional destruction rates of the orientation (magnetic dipole) and the alignment (electric quadrupole) of the excited state. These rates are both larger than $\gamma_e^{c(0)}$ because alignment and orientation are destroyed by the $\gamma_e^{c(0)}$ "quenching" collisions and also by "depolarizing" collisions which induce transitions between the magnetic substates m_e of the $j_e = 1$ level.

It is important to make a distinction between $K = 0$ and the cases $K = 1, 2$ when discussing the meaning of the various scattering terms. The $K = 0$ equations describe the scattering isotropically without regard for the polarization whereas the $K = 1, 2$ results are sensitive to both the direction of propagation and the polarization.

Most of the new results due to the magnetic field appear in the magnetic depolarization factor $\gamma_e^{(K)}/(\gamma_e^{(K)} + iQ\omega_{He})$, where $Q\omega_{He}$ is the difference in Larmor precession frequencies for the magnetic substates m_e and m_e' where $Q = m_e - m_e'$. If the Larmor frequencies are much smaller than the relaxation rate, $\omega_{He} \ll \gamma_e^{(K)}$, the relative phases of the various m_e states do not change appreciably during the lifetime of the 2^K -pole moment of the excited state and the magnetic depolarization factor is essentially unity. If the precession frequency is large, the polarization of an emitted photon can be much different from that of the photon which excites such a state, and it is exactly this depolarization which is described by the term $\gamma_e^{(K)}/(\gamma_e^{(K)} + iQ\omega_{He})$. For $K = 0$ we have $Q = 0$, and the magnetic depolarization factor does not appear.

It must also be noted that we have averaged the incident and scattered radiation over a bandwidth Δ which is greater than all Zeeman splittings and collisional line widths. Because of this smearing, our results will not show the detailed structure of the line center which is produced by the splitting and relative intensities of the individual Zeeman components. Our results will describe bulk features such as the relative importance of magnetic as opposed to collisional depolarization and the effects of these mechanisms on the redistribution of radiation from the Lorentz wings to the Doppler core, etc. We also note that, due to the frequency averaging, the probability for resonant emission or absorption (within $\pm\Delta/2$ of the line center) is given by $1/\Delta$ rather than the usual line shape function.

We note finally that a constant $(2\pi^2/\Gamma_e)$ must be factored out of the functions f^{KQ} and combined with A_{fie} to produce the appropriate normalization (see eqs. [49] and [55] of Paper I).

Case A: The scattering proceeds via resonant absorption and emission, each with a probability $1/\Delta$. Equation (21) thus contains the depolarization factor $\gamma_e^{(K)}/(\gamma_e^{(K)} + iQ\omega_{He})$ multiplied by Δ^{-2} , the normalization factor $(2\pi^2/\Gamma_e)$ and the

“branching ratio” $\Gamma_e/\gamma_e^{(K)}$ for radiative decay of the excited state without destruction of the 2^K -dipole moment. This branching ratio was discussed in § V of Paper I (for $K = 0$), where it was noted that our redistribution function is normalized not to unity but rather to $\Gamma_e/\gamma_e^{(K)}$; this represents the fact that inelastic collisions reduce the scattered intensity in the frequency region of interest around $\bar{\omega}_2$.

Case B: With $|\bar{\omega}_1 - \omega_{ei}| \gg \Delta$, the probability for absorption is given by the asymptotic wing of a Lorentz profile $\gamma_{ei}^{(1)}/\pi(\bar{\omega}_1 - \omega_{ei})^2$. This may then be multiplied by the probability $(2\gamma_{ei}^{(1)} - \gamma_e^{(K)})/2\gamma_{ei}^{(1)}$ of having a collision which redistributes radiation over the line profile without destroying the 2^K -pole moment, and by the branching ratio for radiative decay $\Gamma_e/\gamma_e^{(K)}$ (see the discussion following eq. [66] in Paper I). Finally, multiplying by the resonant emission probability $1/\Delta$, the depolarization factor $\gamma_e^{(K)}/(\gamma_e^{(K)} + iQ\omega_{He})$ and the normalization $2\pi^2/\Gamma_e$, we obtain the result stated in equation (24).

At this point it is interesting to elaborate on two of these factors by noting the identity

$$\frac{\Gamma_e}{\gamma_e^{(K)}} \frac{\gamma_e^{(K)}}{(\gamma_e^{(K)} + iQ\omega_{He})} = \frac{\Gamma_e}{2\gamma_{ei}^{(1)}} \frac{2\gamma_{ei}^{(1)}}{(2\gamma_{ei}^{(1)} + iQ\omega_{He})} \times \sum_{n=0}^{\infty} \left[\frac{2\gamma_{ei}^{(1)}}{(2\gamma_{ei}^{(1)} + iQ\omega_{He})} \frac{(2\gamma_{ei}^{(1)} - \gamma_e^{(K)})}{2\gamma_{ei}^{(1)}} \right]^n. \quad (29)$$

The left side of this equation represents the probability of emission before destruction of the 2^K -pole moment multiplied by the amount of magnetic depolarization which occurs before destruction of this moment. The right side of the equation contains the probability $(2\gamma_{ei}^{(1)} - \gamma_e^{(K)})/2\gamma_{ei}^{(1)}$ of having a collision which redistributes the radiation without destroying the 2^K -pole moment multiplied by the amount of magnetic depolarization $2\gamma_{ei}^{(1)}/(2\gamma_{ei}^{(1)} + iQ\omega_{He})$ which takes place between these collisions; this product is raised to the power n , representing a sequence of n such collisions, and summed over all possible sequences from $n = 0$ to $n = \infty$. The remaining multiplying factors on the right-hand side represent the magnetic depolarization which takes place before the first collision occurs multiplied by the probability of radiative decay $\Gamma_e/2\gamma_{ei}^{(1)}$. Thus the right-hand side of this equation is a sum over all possible sequences of events which could occur before destruction of the 2^K -pole moment, and the effect of all these possibilities is represented collectively by the simple product on the left side.

Case C: This case corresponds to resonant absorption in the Doppler core followed by emission in the Lorentz wings. The physical interpretation of the scattering terms is exactly the same as case B.

Case D: The second line of equation (26) can be written as a product of the normalization factor $(2\pi^2/\Gamma_e)$, the probability $\gamma_{ei}^{(1)2}/\pi^2(\bar{\omega}_1 - \omega_{ei})^2(\bar{\omega}_2 - \omega_{ei})^2$ for absorption and emission in the wings, and a factor which is a sum of two terms. The first term contains the probability for a redistributing collision $(2\gamma_{ei}^{(1)} - \gamma_e^{(K)})/2\gamma_{ei}^{(1)}$ multiplied by the probability for radiative decay $\Gamma_e/2\gamma_{ei}^{(1)}$. The second term contains the probability for a redistributing collision multiplied by $(\Gamma_e/\gamma_e^{(K)})[\gamma_e^{(K)}/(\gamma_e^{(K)} + iQ\omega_{He})]$ which was discussed following equation (29). To understand these two terms, we must first recall that the probability for emission or absorption in the wings, where $\omega_1 - \omega_{ei} \gg 2\gamma_{ei}^{(1)} \gg \Gamma_e$, will be negligibly small in the absence of collisions; that is, an atom can “absorb” a photon ω_1 only within $(\omega_1 - \omega_{ei})^{-1}$ seconds of a collision. Bearing this in mind, we interpret the first term in case D as an absorption and emission due to the same collision (cf. eq. [29]). The second term represents an absorption due to one collision followed by a sequence of redistributing collisions and emission due to yet another collision. It is interesting to note that the first term

contains no magnetic depolarization because the time between emission and absorption is the order of $[(\omega_1 - \omega_{ei})^{-1} + (\omega_2 - \omega_{ei})^{-1}]$ which is too short for any appreciable depolarization to occur since $Q\omega_{He}[(\omega_1 - \omega_{ei})^{-1} + (\omega_2 - \omega_{ei})^{-1}] \ll 1$.

Case E: This case represents coherent scattering in the line wings (spontaneous emission occurring during a virtual excitation) which is independent of collisions and the magnetic field because the virtual excitation lasts only a very short time. Equation (27) can be interpreted as a normalization factor $(2\pi^2/\Gamma_e)$ multiplied by the absorption probability $\gamma_{ei}^{(1)}/\pi(\bar{\omega}_1 - \omega_{ei})^2$ and the probability $\Gamma_e/2\gamma_{ei}^{(1)}$ that radiation occurs before a collision can take place; the resonance factor $1/\Delta$ comes from the resonance of $(\bar{\omega}_1 - \omega_{ei})$ with $(\bar{\omega}_2 - \omega_{ei})$ (cf. the integral $\Delta^{-1} \int d\omega_1 \delta(\omega_1 - \omega_2)$ in eq. [60] of Paper I).

We next consider the high-field regime where the splitting between the Zeeman components is greater than the collision-broadened line width ($\omega_{He} > 2\gamma_{ei}$). In most problems, one considers incident light produced by a resonance lamp or by a white light source, both of which may be described entirely by cases A and C. For $Q \neq 0$, the scattered intensity is inversely proportional to the magnetic field strength (see eqs. [21] and [25]), hence the $Q = 0$ terms dominate the scattering in the high-field regime. The $Q = 0$ terms would show the normal Zeeman pattern of lines if we had not averaged ω_1 and ω_2 over the large bandwidth Δ (see House 1970 for a discussion of results without this frequency smearing). In order to interpret this result we note that the radiation field need not excite a pure $j_e m_e$ state, it may instead excite a state which is a linear combination of $j_e m_e$ states. The quantum number $Q = m_e - m_e'$ describes the magnetic depolarization of such a linear combination when $Q \neq 0$. For example, during the time the atom is excited, the state m_e will precess about \mathcal{H} at a rate $m_e \omega_{He}$ and, after a time t , the phases of m_e and m_e' will differ by $Q\omega_{He}t$; in the high-field limit ($\Delta > Q\omega_{He} > 2\gamma_{ei}^{(1)}$), this phase difference becomes greater than unity (on the average) before the excited state relaxes, hence the incident polarization is rapidly destroyed for $Q \neq 0$ excitations. Summarizing the above argument, one frequently says (House 1970) that the diagonal elements ($Q = 0$) represent normal Zeeman scattering while off-diagonal elements ($Q \neq 0$) represent modifications of the normal results due to "coherence" or the interference of Zeeman sublevels (the latter being negligible in the high-field limit when the Zeeman levels do not overlap).

In connection with the above, it is interesting to note that coherent scattering in the wings (case E) is an example of a coherence effect which does not vanish for $Q \neq 0$ in the high-field limit. This is due to the fact that the virtual excitation responsible for this process does not last long enough for phase differences $Q\omega_{He}t$ (or collisions, for that matter) to become significant. It should also be mentioned that this process is not important if excitation is produced by a resonance lamp or a white light source (for white light, eqs. [15] and [16] would be integrated over all ω_1 giving the results of cases A and C with Δ replaced by some other normalization constant).

Note also that the result for case D does not go monotonically as $1/Q\omega_{He}$ in the high field regime; it goes instead to a constant value which is $\gamma_e^{(K)}/2\gamma_{ei}^{(1)}$ smaller than its value at $Q\omega_{He} = 0$. The residual contribution in the high-field regime is due to emission closely following the absorption. A similar residual contribution appears in the higher-order terms in cases A, B, and C; to study this effect for these cases it would also be necessary to include corrections due to integrating ω_1 and ω_2 over $(-\Delta/2, +\Delta/2)$ rather than $(-\infty, +\infty)$.

A measurement of incoherent scattering in the line wings would provide an interesting test of the theory as well as a means of measuring both elastic and inelastic cross-sections. In the very low field regime $Q\omega_{He} \ll \gamma_e^{(K)}$ the scattered intensity should be independent of the magnetic field strength. As \mathcal{H} is increased through the region $\gamma_e^{(K)} < Q\omega_{He} < 2\gamma_{ei}^{(1)}$, the scattered intensity will drop off and finally assume a constant value in the high-field regime $Q\omega_{He} \gg 2\gamma_{ei}^{(1)}$. The ratio of intensities in the

very low and high field regions should be $2\gamma_{ei}^{(1)}/\gamma_e^{(K)}$. A measurement of this intensity ratio as well as the collision width or the magnetic field strengths corresponding to transitions into the very low or high field regions should give a fairly accurate measurement of $\gamma_{ei}^{(1)}$ and $\gamma_e^{(K)}$. For such an experiment, one might use a laser tuned to the wings of some atomic resonance line; such an excitation source would be very useful in avoiding signal-to-noise problems associated with the very small cross-section for scattering in the wings (e.g., redistribution from the center to the wings).

V. SOURCE FUNCTION

The expressions derived in the preceding sections give the emissivity of the medium for scattering of spectral radiation. As the absorption ("true" absorption plus scattering) is very well known, it is a simple matter to derive the source function. In this section we write down the expression for the source function in the formalism of the 3×3 density matrix for the radiation field, pointing out its relation to the usual formalism (see, e.g., Chandrasekhar 1950; Jefferies 1968) and discussing the important simplifications arising in typical conditions of stellar atmospheres.

To simplify the discussion, we consider the case $i = f$ and ignore the effect of continuum. We further assume that the absorption profile $\phi(\omega)$ is a Voigt profile with a very small ratio γ_{ei}/Δ_D of the Lorentz and Doppler widths (as noted previously, we assume $\omega_{Ha} \ll \Delta_D$ and we neglect all terms of order ω_{Ha}/Δ_D or smaller). Consequently in the core of the line

$$\phi(\omega) \simeq \phi_c(\omega) = \frac{1}{\Delta_D \sqrt{\pi}} \exp [-(\omega - \omega_{ei})^2/\Delta_D^2], \quad |\omega - \omega_{ei}| \sim \Delta_D; \quad (30)$$

and in the wings

$$\phi(\omega) \simeq \phi_w(\omega) \simeq (\gamma_{ei}/\pi)(\omega - \omega_{ei})^{-2}, \quad |\omega - \omega_{ei}| \gg \Delta_D \gg \gamma_{ei}. \quad (31)$$

For the Voigt profile with $\gamma_{ei} \ll \Delta_D$, the transition between the "core" and the "wings" occurs very sharply at $|\omega - \omega_{ei}| \simeq 3\Delta_D$.

For the radiation field we define quantities $J_Q^K(\omega)$ proportional to the density matrix elements $\rho_Q^K(1, \omega)$, such as

$$J_0^0(\omega) = J(\omega) = \frac{1}{4\pi} \int d\Omega [I_l(\Omega, \omega) + I_r(\Omega, \omega)], \quad (32)$$

where I_l and I_r are the intensities of two orthogonal polarizations of the light traveling in the direction Ω (Chandrasekhar 1950), and $J(\omega)$ is the usual J integral of the radiative transfer theory (Jefferies 1968). The components $S_Q^K(\omega)$ of the source function are defined in the same way; for instance, for *isotropic and unpolarized radiation*

$$S_0^0(\omega) = S_l(\omega) + S_r(\omega). \quad (33)$$

In general the 2×2 matrix source function in the direction (θ, ϕ) is derived from the components S_Q^K of the 3×3 source function through the method of Appendix A (see eq. [A21]).

$S_Q^K(\omega)$ is of course proportional to $\rho_Q^K(\omega_2)$ (§§ II and III) with a summation over all incident frequencies, and a division by the absorption profile $\phi(\omega)$. Consequently, assuming *complete frequency redistribution in the Doppler core*, it is straightforward

to show (using the results of § III), that the explicit expressions of the $S_Q^K(\omega)$ are the following: for the core

$$S_{-Q}^K \text{core}(\omega_2) = W(j_{ij}j_e K) \frac{\Gamma_e}{\gamma_e^{(K)} + iQ\omega_{He}} \left\{ \int_{\text{core}} J_Q^K(\omega_1) \phi_c(\omega_1) d\omega_1 + \frac{2\gamma_{ei} - \gamma_e^{(K)}}{2\gamma_{ei}} \int_{\text{wings}} J_{-Q}^K(\omega_1) \phi_w(\omega_1) d\omega_1 \right\}. \quad (34)$$

This equation is correct only to lowest order in $\gamma_e^{(1)}/\Delta$ because in the calculation for case A we have neglected higher-order terms which are the same order as the first nonvanishing contribution to case B which corresponds to the second term in equation (34). For the wings we have

$$S_{-Q}^K \text{wings}(\omega_2) = W(j_{ij}j_e K) \left\{ \frac{\Gamma_e}{2\gamma_{ei}} J_Q^K(\omega_2) + \frac{2\gamma_{ei} - \gamma_e^{(K)}}{2\gamma_{ei}} \times \frac{\Gamma_e}{\gamma_e^{(K)} + iQ\omega_{He}} \int_{\text{core}} J_{-Q}^K(\omega_1) \phi_c(\omega_1) d\omega + \frac{\Gamma_e(2\gamma_{ei} - \gamma_e^{(K)})}{4\gamma_{ei}^2} \times \left[1 + \frac{2\gamma_{ei} - \gamma_e^{(K)}}{\gamma_e^{(K)} + iQ\omega_{He}} \right] \int_{\text{wings}} J_{-Q}^K(\omega_1) \phi_w(\omega_1) d\omega_1 \right\}, \quad (35)$$

where the definition of the J_Q^K and S_Q^K is referred to the direction of the magnetic field (the last term of eq. [35] is valid only if $\gamma_i^{(K)} = 0$ since it comes from eq. [26]).

For $K = 0$ ($Q = 0$) this reduces to the expression of the two-level problem, particularly well known when $\Gamma_i = \gamma_e^{c(0)} = 0$, where

$$S_0^0 \text{wings}(\omega_2) = \frac{\Gamma_e}{\Gamma_e + 2\gamma_{ei}^c} J_0^0(\omega_2) + \frac{2\gamma_{ei}^c}{\Gamma_e + 2\gamma_{ei}^c} \int J_0^0(\omega_1) \phi(\omega_1) d\omega_1.$$

For $K \neq 0$, there are a number of simplifications due to the following considerations: first

$$\int_w \phi_w(\omega) d\omega / \int_c \phi_c(\omega) d\omega \simeq \int_w \phi_w(\omega) d\omega \simeq 2\gamma_{ei}/3\pi\Delta_D \ll 1;$$

this ratio is typically smaller than 10^{-2} in the solar photosphere. Consequently the last terms of equations (34) and (35) are always very small, and negligible in most of the cases.

Furthermore, at the optical depths where the wings are formed, the radiation in the core is completely depolarized and isotropic, and $J_Q^K(\omega_1) \simeq 0$ in the second term of equation (35).

Consequently, for $K \neq 0$, it seems a very good approximation to retain only the first term in both equations (34) and (35). One sees that in the core, depolarization is caused by the depolarization factor W (eq. [50] of Paper I), depolarizing collisions ($\gamma_e^{c(K)}$), and the magnetic field ($Q\omega_{He}$). In the wings (besides the influence of W) only coherent scattering conserves the polarization, as was pointed out by Zanstra (1941).

The same conclusions remain without the assumption of complete redistribution in the Doppler core; but the first integral of equation (32) should be replaced by the exact Doppler redistribution function (Hummer 1962); the error in the computation of polarization rates with the assumption of complete Doppler redistribution is not always negligible. But of course the calculations in the wings are not directly affected by this problem.

To solve a concrete problem, the simplified equations (34) and (35) have to be completed by the expressions giving the light intensities from the source function. Except in situations of high symmetry, these equations couple J_Q^K not only to S_Q^K but also to other terms with different K 's and Q 's.

APPENDIX A DENSITY MATRIX OF THE RADIATION FIELD

Since the introduction of the Stokes parameters in 1852 (Stokes 1852) the matrix formalism has been extensively used to describe the polarization of light (see, e.g., Born and Wolf 1959; Chandrasekhar 1950; Fano 1949; Schmieder 1969; Whitney 1971); its meaning has become clearer with the development of quantum mechanics and the introduction of the density matrix (see, e.g., Fano 1957; ter Haar 1961). The interest in the density matrix to discuss the properties of the radiation field itself has been underlined in the description of quantum electrodynamics experiments (see, e.g., Fano 1957; Fano and Racah 1959), and the density matrix formalism has reached a new degree of sophistication with the development of quantum optics and coherent light. Nevertheless, the use of the density matrix for light scattering phenomena does not always seem to be well understood. We shall therefore review the basic features of the photon density matrix which are relevant to electric dipole scattering (see also Lamb and ter Haar 1971).

Electromagnetic radiation is usually described by a superposition of plane waves of the form (Messiah 1966, p. 1031 *et seq.*)

$$\psi_R(\mathbf{r}) = \sum_{\mathbf{k}, \alpha} V^{-1/2} \mathbf{e}_\alpha a(\mathbf{k}, \alpha) \exp(i\mathbf{k} \cdot \mathbf{r}), \quad (\text{A1})$$

where \mathbf{k} is the propagation vector, \mathbf{e}_α denotes a polarization vector perpendicular to \mathbf{k} , $a(\mathbf{k}, \alpha)$ is an amplitude factor, and V is an arbitrary volume. The subscript R is henceforth used to distinguish states of the radiation field from particle states; it is not a quantum number or a physical parameter. In a Dirac notation where $\langle \mathbf{r} | (\mathbf{k}) \mathbf{e}_\alpha \rangle$ represents $V^{-1/2} \mathbf{e}_\alpha \exp(i\mathbf{k} \cdot \mathbf{r})$, equation (A1) would be written

$$|\psi_R\rangle = \sum_{\mathbf{k}, \alpha} a(\mathbf{k}, \alpha) |(\mathbf{k}) \mathbf{e}_\alpha\rangle, \quad (\text{A2})$$

which may be regarded as an expansion of a state vector for the radiation field in terms of the plane wave states $|(\mathbf{k}) \mathbf{e}_\alpha\rangle$.

The density matrix for a monochromatic radiation field (i.e., for a given \mathbf{k}) may be defined in the usual way

$$\rho_{R\mathbf{k}} = \sum_{\alpha, \beta} \rho_{\mathbf{k}\alpha\beta} |(\mathbf{k}) \mathbf{e}_\alpha\rangle \langle (\mathbf{k}) \mathbf{e}_\beta|. \quad (\text{A3})$$

This density matrix is proportional to the polarization matrix used in classical optics (Born and Wolf 1959; Cohen-Tannoudji and Lalöe 1967). The diagonal elements $\rho_{\mathbf{k}\alpha\alpha}$ denote the relative intensities of the different polarization components of the plane wave. The off-diagonal elements $\rho_{\mathbf{k}\alpha\beta}$ represent coupling or "coherence" between the polarization states; for a fully polarized wave or "pure" state, $\rho_{\mathbf{k}\alpha\beta} = a(\mathbf{k}, \alpha) a^*(\mathbf{k}, \beta)$.

The density matrix elements in this representation are frequently expressed in terms of Stokes parameters $S_{\mathbf{k}}$ and the 2×2 Pauli σ matrices with the usual convention for these matrices (Messiah 1966):

$$\rho_{R\mathbf{k}} = \frac{1}{2} (\sigma_0 + S_{\mathbf{k}} \cdot \boldsymbol{\sigma}^*), \quad (\text{A4})$$

where σ_0 is the 2×2 unit matrix and the Stokes parameters are

$$\begin{aligned} S_{kz} &= (\rho_{kaa} - \rho_{kbb}) = S_1 = Q/I, \\ S_{kx} &= (\rho_{kab} + \rho_{kba}) = S_2 = U/I, \\ S_{ky} &= -i(\rho_{kab} - \rho_{kba}) = S_3 = V/I. \end{aligned} \quad (A5)$$

The parameters Q , U , and V are unnormalized Stokes parameters, and I is the intensity of the plane wave. (Note that $\mathbf{e}_b \times \mathbf{e}_a = \mathbf{k}/k$.)

Since there are two orthogonal polarization vectors orthogonal to \mathbf{k} , the density matrix ρ_{Rk} defined by equation (A3) is a 2×2 matrix which depends on \mathbf{k} . We could obtain a 3×3 representation of ρ_R by expressing the polarization vectors \mathbf{e}_α in terms of the three Cartesian coordinate vectors \mathbf{u}_x , \mathbf{u}_y , and \mathbf{u}_z :

$$\rho_R = \sum_{ij=x,y,z} \rho_{ij} |\mathbf{u}_i\rangle \langle \mathbf{u}_j|, \quad \rho_{ij} = \sum_{\alpha\beta} \mu_{\alpha i} \mu_{\beta j}^* \rho_{k\alpha\beta}, \quad \mu_{\alpha i} = \mathbf{e}_\alpha \cdot \mathbf{u}_i, \quad (A6)$$

where $\mu_{\alpha i}$ denote the components of the vector \mathbf{e}_α in the Cartesian coordinate system. Another 3×3 representation, which is convenient from the point of view of tensor analysis, is based on the coordinate vectors

$$\mathbf{v}_0 = \mathbf{u}_z, \quad \mathbf{v}_{\pm 1} = \mp (\mathbf{u}_x \pm i\mathbf{u}_y)/\sqrt{2}. \quad (A7)$$

In this representation

$$\begin{aligned} \rho_R &= \sum_{qq'} \rho_{qq'} |\mathbf{v}_q\rangle \langle \mathbf{v}_{q'}|, \\ \rho_{qq'} &= \sum_{ij} \mu_{iq} \mu_{j q'}^* \rho_{ij} = \sum_{\alpha\beta} \mu_{\alpha q} \mu_{\beta q'}^* \rho_{k\alpha\beta}, \\ \mathbf{e}_\alpha &= \sum_q \mu_{\alpha q} \mathbf{v}_q = \sum_q (-1)^q (\mathbf{e}_\alpha)_{-q} \mathbf{v}_q, \end{aligned} \quad (A8)$$

where $(\mathbf{e}_\alpha)_q$ denote the spherical tensor components of the vector \mathbf{e}_α (see eq. [33] of Paper I). Yet another representation may be formed by considering the linear combination (cf. eq. [25] of Paper I).

$$|R; 11KQ\rangle\rangle = \sum_{qq'} (-1)^{1-q'} \langle 11q - q' | KQ \rangle |\mathbf{v}_q\rangle \langle \mathbf{v}_{q'}|, \quad (A9)$$

where $\langle 11q - q' | KQ \rangle$ is a Clebsch-Gordan coefficient in the notation of Messiah. In this representation ρ_R is given by

$$\rho_R = \sum_{KQ} (-1)^Q \rho_{-Q}^K |R; 11KQ\rangle\rangle, \quad (A10)$$

$$(-1)^Q \rho_{-Q}^K = \sum_{qq'} (-1)^{1-q'} \langle 11q - q' | KQ \rangle \rho_{qq'}. \quad (A11)$$

This representation is nothing more than a multipole or irreducible-tensor expansion of ρ_R . Such a representation can be quite useful for analysis of the angular distribution and polarization of radiation emitted by systems which are simultaneously influenced by other radiation or fields (Fano and Racah 1959). It should be noted that our 3×3 density matrix ρ_R was constructed from the three components of the polarization vector $\hat{\mathbf{e}}$ for a single plane wave. Faroux (1970) has shown (see chap. 5) that these three components correspond to the three components of the state vector for the electric

dipole part of the radiation field. Equation (A10) therefore represents a multipolar expansion of the density matrix for the *electric dipole part* of the radiation field; this should not be confused with the more general result discussed by Fano and Racah (1959).

We obtain a useful identity for the complex conjugate of ρ_Q^K by using the fact that the density matrix ρ_R is a Hermitian operator. Using $\rho_{qq'} = \rho_{q'a}^*$ in equation (A11), it readily follows that

$$(\rho_Q^K)^* = (-1)^Q \rho_{-Q}^K. \quad (\text{A12})$$

This identity should not be confused with the definition of Hermitian conjugation of a tensor operator $T(k, q)$ as given, for example, by Edmonds (1960, p. 77). Our ρ_Q^K is defined in terms of the *matrix elements* of an operator ρ_R ; hence ρ_Q^K is not itself an operator in the sense used by Edmonds (p. 77).

We are now in a position to work out the change in equation (5) when we transform from the 2×2 representation to the multipolar representation. Substituting $\rho_{qq'}$ from equation (A8) into equation (A11) and using (see eq. [B6] of I)

$$(e_\beta^*)_q = (-1)^Q [(e_\beta)_{-q}]^*, \quad (\text{A13})$$

we find

$$\rho_Q^K = \sum_{\alpha\beta} \rho_{\kappa\alpha\beta} P_Q^K(e_\beta^*, e_\alpha), \quad (\text{A14})$$

where P_Q^K was defined in equation (3).

In practice it is convenient to have explicit formulae relating the density matrix elements between the various representations. Such relations between ρ_Q^K and the Stokes parameters for an arbitrary direction of k are given in Appendix A of Lamb and ter Haar (1971) in a notation slightly different from ours. In table 1 we give the unitary matrix relating the Cartesian components ρ_{ij} of equation (A6) to the ρ_Q^K :

$$\rho_{ij} = \sum_{KQ} A_{ij,KQ} \rho_Q^K, \quad \rho_Q^K = \sum_{ij} A_{ij,KQ}^* \rho_{ij}. \quad (\text{A15})$$

It is also useful to have explicit relations between the ρ_Q^K referred to a coordinate frame xyz and the Stokes parameters for a plane wave propagating parallel to an axis z' . We define a coordinate frame $x'y'z'$ related to xyz by a rotation through the

TABLE 1
COEFFICIENTS $A_{ij, KQ}$ (eq. [A15]) RELATING THE DENSITY MATRIX ELEMENTS
IN CARTESIAN COORDINATES ρ_{ij} (eq. [A6]) AND IN THE IRREDUCIBLE
REPRESENTATION ρ_Q^K (eq. [A10])

ρ_{ij}	ρ_Q^K								
	00	10	11	1-1	20	21	2-1	22	2-2
xx	$1/\sqrt{3}$	0	0	0	$+1/\sqrt{6}$	0	0	$-1/2$	$-1/2$
yy	$1/\sqrt{3}$	0	0	0	$+1/\sqrt{6}$	0	0	$+1/2$	$+1/2$
zz	$1/\sqrt{3}$	0	0	0	$-2/\sqrt{6}$	0	0	0	0
xy	0	$-i/\sqrt{2}$	0	0	0	0	0	$+i/2$	$-i/2$
yx	0	$+i/\sqrt{2}$	0	0	0	0	0	$+i/2$	$-i/2$
yz	0	0	$+i/2$	$-i/2$	0	$-i/2$	$-i/2$	0	0
zy	0	0	$-i/2$	$+i/2$	0	$-i/2$	$-i/2$	0	0
zx	0	0	$+1/2$	$+1/2$	0	$+1/2$	$-1/2$	0	0
xz	0	0	$-1/2$	$-1/2$	0	$+1/2$	$-1/2$	0	0

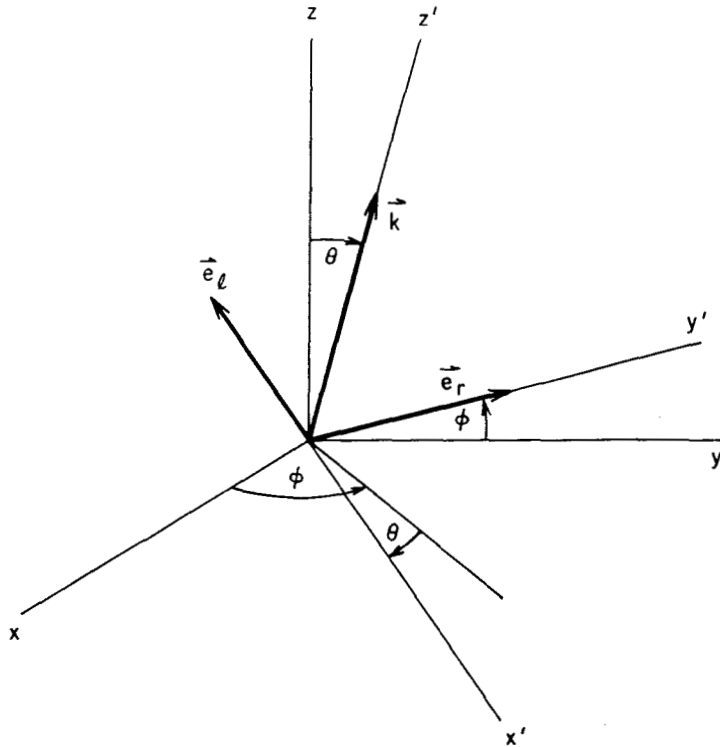


FIG. 1.—Polarization directions for Stokes parameters. The coordinate frames xyz and $x'y'z'$ are related through Euler angles $(\phi, \theta, 0)$. The polarization vectors are $e_l = -e_x$, $e_r = e_{y'}$, and the propagation vector k points in the z' direction.

Euler angles $(\phi, \theta, 0)$ as shown in figure 1. We define Stokes parameters relative to the polarization vectors $e_l = -e_x$ and $e_r = e_{y'}$:

$$I = I_l + I_r, \quad Q = I_l - I_r, \tag{A16}$$

etc. The relation between ρ_Q^K and S_n is given in terms of the matrix $B_{KQ,n}(\theta, \phi)$ of table 2 in the form

$$\begin{aligned} \rho_Q^K &= \sum_n B_{KQ,n}(\theta, \phi) S_n / \sqrt{2}, \\ S_n / \sqrt{2} &= \sum_{KQ} B_{KQ,n}^*(\theta, \phi) \rho_Q^K, \\ \sum_{KQ} B_{KQ,n} B_{KQ,n'}^* &= \delta_{nn'}. \end{aligned} \tag{A17}$$

In radiative transfer theory, one generally uses the unnormalized Stokes parameters $I, Q, U,$ and V (see eq. [A5]). The unnormalized tensor components proportional to ρ_Q^K are thus given by

$$I_Q^K(\theta, \phi) = 2^{-1/2} [B_{KQ,0}(\theta, \phi)I + B_{KQ,1}(\theta, \phi)Q + B_{KQ,2}(\theta, \phi)U + B_{KQ,3}(\theta, \phi)V], \tag{A18}$$

TABLE 2
 COEFFICIENTS $B_{KQ,n}$ (eq. [A17]) RELATING THE IRREDUCIBLE COMPONENT, ρ_Q^K , TO
 THE STOKES PARAMETERS S_n IN THE DIRECTION θ, ϕ

ρ_Q^K	$S_n/\sqrt{2}$		
	0	1	2
00.....	$(2/3)^{1/2}$	0	0
10.....	0	0	$-\cos \theta$
1 ± 1.....	0	$\pm 3^{1/2} \sin^2 \theta$	$\pm 2^{-1/2} \sin \theta e^{\pm i\phi}$
20.....	$(3 \cos^2 \theta - 1)/2\sqrt{3}$	$\mp 2^{-1/2} \sin \theta \cos \theta e^{\pm i\phi}$	0
2 ± 1.....	$\pm 2^{-1/2} \sin^2 \theta \cos \theta e^{\pm i\phi}$	$\mp 2^{-1/2} \sin \theta \cos^2 \theta/2\sqrt{2}$	$i2^{-1/2} \sin \theta e^{\pm i\phi}$
2 ± 2.....		$-e^{\pm 2i\phi}(1 + \cos^2 \theta)/2\sqrt{2}$	$\pm i2^{-1/2} \cos \theta e^{\pm 2i\phi}$

3

and one defines the corresponding integrals (eq. [32])

$$J_Q^K = \sqrt{3} \frac{1}{4\pi} \int I_Q^K(\theta, \phi) \sin \theta d\theta d\phi \quad (\text{A19})$$

where the constant $\sqrt{3}$ is introduced to ensure that (eq. [32])

$$J_0^0 = J = \frac{1}{4\pi} \int (I_l + I_r) \sin \theta d\theta d\phi. \quad (\text{A20})$$

The source functions for Stokes parameters ($S_l = S_l + S_r$, $S_Q = S_l - S_r$, etc.), are deduced from the S_Q^K 's by equation (A15):

$$\begin{aligned} S_l(\theta, \phi) &= \sqrt{2} \sum_{KQ} B_{KQ,0}^*(\theta, \phi) S_Q^K, \\ S_Q(\theta, \phi) &= \sqrt{2} \sum_{KQ} B_{KQ,1}^*(\theta, \phi) S_Q^K, \end{aligned} \quad (\text{A21})$$

etc.

APPENDIX B

The purpose of this Appendix is to derive equation (24) from equations (22) and (23). We give here a derivation for pure polarization states only, but the generalization is obvious. The explicit expression of $F(i)$ obtained by substituting equation (22) of this paper into equation (9) of Paper I is

$$\begin{aligned} F(i) &= -X \operatorname{Re} \sum_{m_e m_e' m_l m_l'} \langle j_l m_l | \epsilon_1^* \cdot \mu | j_e m_e \rangle \\ &\quad \times \langle j_e m_e | \epsilon_2 \cdot \mu | j_l m_l \rangle \langle j_l m_l | \epsilon_2^* \cdot \mu | j_e m_e' \rangle \langle j_e m_e' | \epsilon_1 \cdot \mu | j_l m_l \rangle \end{aligned} \quad (\text{B1})$$

where

$$X = (2\pi/\Delta)(2j_l + 1)^{-1}(\bar{\omega}_1 - \omega_{el})^{-2}. \quad (\text{B2})$$

In a similar manner we use equation (23) of this paper and equations (10) and (11) of Paper I to obtain

$$\begin{aligned} F(\text{ii}) + F(\text{iii}) &= X \operatorname{Re} \sum_{m_l m_l' m_f m_e m_e' r_e r_e''} \\ &\quad \times \langle j_f m_f | \epsilon_2 \cdot \mu | j_e m_e \rangle \langle j_e r_e | \epsilon_2 \cdot \mu | j_f m_f \rangle \langle \langle j_e m_e j_e r_e | (\Phi + iL_M)^{-1} | j_e m_e' j_e r_e' \rangle \rangle \\ &\quad \times [\langle j_l m_l | \epsilon_1^* \cdot \mu | j_e r_e' \rangle \langle \langle j_e m_e' j_l m_l | (\Phi + iL_M) | j_e r_e'' j_l m_l' \rangle \rangle \langle j_e r_e'' | \epsilon_1 \cdot \mu | j_l m_l' \rangle \\ &\quad + \langle j_e m_e' | \epsilon_1 \cdot \mu | j_l m_l \rangle \langle \langle j_l m_l j_e r_e' | (\Phi + iL_M) | j_l m_l' j_e r_e'' \rangle \rangle \langle j_l m_l' | \epsilon_1^* \cdot \mu | j_e r_e'' \rangle]. \end{aligned} \quad (\text{B3})$$

The matrix elements of L_M are (cf. eqs. [10] and [12]) diagonal in $j_a m_a$ and are given by

$$\langle \langle j_a m_a j_b m_b | L_M | j_a m_a j_b m_b \rangle \rangle = (m_a \omega_{H_a} - m_b \omega_{H_b}). \quad (\text{B4})$$

The Φ operator is diagonal in $|j_a j_b K Q\rangle$ states (eq. [28] of Paper I); hence we evaluate the Φ terms in the brackets of equation (B3) exactly as in equation (36) of Paper I (except that we are now neglecting the pressure shift produced by the imaginary part of Φ), and this bracket becomes

$$[2\gamma_{ie}^{(1)} + i(m_e' - r_e')\omega_{He}] \langle j_e m_e' | \epsilon_1 \cdot \mu | j_l m_l \rangle \langle j_l m_l | \epsilon_1^* \cdot \mu | j_e r_e' \rangle. \quad (\text{B5})$$

We again use the tensors D_{-Q}^K defined in equation (38) of Paper I to obtain

$$F(\text{ii}) + F(\text{iii}) = X \operatorname{Re} \sum_{KQ} D_{-Q}^{(K)*} (j_e j_e j_l \epsilon_2 \epsilon_2^*) D_{-Q}^{(K)} (j_e j_e j_l \epsilon_1 \epsilon_1^*) \\ \times (2\gamma_{ei}^{(1)} + iQ\omega_{He}) \langle\langle j_e j_e KQ | (\Phi + iL_M)^{-1} | j_e j_e KQ \rangle\rangle, \quad (\text{B6})$$

and we see that the f_{23} function (cf. eqs. [42] and [44] of Paper I) in case B is just

$$f_{23B}^{KQ} = (\pi/\Delta)(\bar{\omega}_1 - \omega_{ei})^{-2} (2\gamma_{ei}^{(1)} + iQ\omega_{He}) (\gamma_e^{(K)} + iQ\omega_{He})^{-1}. \quad (\text{B7})$$

The f_1 function for case B follows in the same manner from equation (B1):

$$f_{1B}^{KQ} = -(\pi/\Delta)(\bar{\omega}_1 - \omega_{ei})^{-2}. \quad (\text{B8})$$

Notice that the f_1 function does not depend on KQ ; such a situation was discussed in Paper I, where it was shown (eq. [52]) that f_1 simply adds to the f_{23} term to give equation (24):

$$f_B^{KQ} \equiv f_{1B}^{KQ} + f_{23B}^{KQ} \\ = (\pi/\Delta)(\bar{\omega}_1 - \omega_{ei})^{-2} (2\gamma_{ei}^{(1)} - \gamma_e^{(K)}) / (\gamma_e^{(K)} + iQ\omega_{He}). \quad (\text{B9})$$

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