JOURNAL OF MODERN OPTICS, 2018
https://doi.org/10.1080/09500340.2017.1423123

Efficient eigenvalue determination for arbitrary Pauli products based on generalized spin-spin interactions

D. Leibfried and D. J. Wineland

Ion Storage Group, Time and Frequency Division, National Institute of Standards and Technology, Boulder, CO, USA

ABSTRACT

Effective spin-spin interactions between $N+1$ qubits enable the determination of the eigenvalue of an arbitrary Pauli product of dimension N with a constant, small number of multi-qubit gates that is independent of N and encodes the eigenvalue in the measurement basis states of an extra ancilla qubit. Such interactions are available whenever qubits can be coupled to a shared harmonic oscillator, a situation that can be realized in many physical qubit implementations. For example, suitable interactions have already been realized for up to 14 qubits in ion traps. It should be possible to implement stabilizer codes for quantum error correction with a constant number of multi-qubit gates, in contrast to typical constructions with a number of two-qubit gates that increases as a function of N. The special case of finding the parity of N qubits only requires a small number of operations that is independent of N. This compares favorably to algorithms for computing the parity on conventional machines, which implies a genuine quantum advantage.

ARTICLE HISTORY
Received 30 June 2017
Accepted 20 December 2017

KEYWORD
Quantum information

1. Preface

Some of the work presented here was developed in 2004 to support a spectroscopy demonstration with three entangled ions (*1*), where a special case of the key relation Equation (6) below was stated without proof and without discussing further implications. This publication appeared before supplemental materials became common and the proof was too long to include it in a letter-style publication at the time. Thirteen years later, we have picked up the thread again and generalize the special case stated in (*1*) to relate more general spin-spin interactions to arbitrary Pauli product operators.

It may be rare that experimental physicists resort to the principle of mathematical induction, as we have done below. However, our contribution seems to be in tune with a particularly important lesson that Danny's life can teach us: You should not be afraid to do things in your own way, no matter how far of the beaten path this might take you.

2. Introduction

In this contribution we provide a link between effective spin-spin interactions of $N+1$ qubits and the eigenvalues of an arbitrary Pauli product of dimension N. Suitable effective spin-spin interactions can, for example, be implemented if the qubits are coupled to a shared harmonic oscillator, a condition realized in several physical qubit implementations. For trapped ion qubits, suitable interactions that couple all $N+1$ spins simultaneously in one operation were first discussed for robust entanglement operations typically referred to as Mømer-Sørensen-type gates (*2*–*5*). Such operations form the basis for most of the recent work with entangled states of ions in quantum information processing and quantum simulation. For example, work described in (*1*, *6*–*13*) produced entangled states by using lasers to couple the qubits to a harmonic normal-mode motion of the ions and in (*14*–*17*) this was accomplished with microwave fields. These and related experiments encompass a wide range of contexts, including precision spectroscopy, quantum information processing and quantum simulation. In particular, the effective spin-spin operations at the heart of our study have been realized for up to 14 ion-qubits (*13*) and have enabled the highest fidelity two-qubit gates reported to date at $F \simeq 0.999$ (*18*, *19*). The natural extensibility of Mølmer-Sørensen type operations to more than two qubits can offer algorithmic advantages over the more conventional decomposition into two-qubit gates and single qubit rotations.

Here we show, that one or two operations of this type, on N qubits and one ancilla-qubit, together with

CONTACT D. Leibfried dil@boulder.nist.gov

We dedicate this work to Danny Segal, scientist, community builder and friend. Your bold and cheerful way of leaving the trodden paths and pursuing your own goals have made our field richer. We will not be able to fill your void, but we will carry on.

individual single-qubit rotations that can be performed in parallel, are sufficient to determine the eigenvalue of an arbitrary Pauli-product operator on the group of N of the qubits. The eigenvalue is stored it in the measurement basis eigenstates of the ancilla-qubit after these operations. Read-out of the ancilla state projects the N qubits into a corresponding eigenstate of the Pauli-product operator that is available for further processing. Therefore, such operations may become useful in error correction, as discussed in more detail below.

The parity of a state of N ions can be determined with a total of three (N even) or four operations (N odd) and is revealed by one measurement on the ancilla-qubit, independent of the size of N. This constitutes a constant depth parity-finding algorithm with a circuit of a size proportional to N, while parity determination algorithms on conventional computers require polynomial sized circuit families of close to $\log(N)$ depth (20). Our proposed implementation therefore provides an advantage even for relatively small N and falls into a family of constant depth parity circuits discussed in (21).

3. Generalized spin-spin interactions and Pauli products

We consider N two-level systems with logical basis of the l-th qubit ($l \in \{1, \ldots, N\}$) defined by the eigenstates $\{|\downarrow\rangle, |\uparrow\rangle\}$ of the z-component of a spin-1/2 angular momentum operator $S_{l,k}$ with $k = \{x, y, z\}$. We write $\sigma_{l,i}$ for the 2×2 identity matrix in the state space of the l-th qubit and $\sigma_{l,k}$ for the Pauli-matrices for $k = \{x, y, z\}$. When setting $\hbar = 1$ for simplicity, the eigenvalues in the measurement basis are

$$\begin{aligned} S_{l,z}|\uparrow\rangle &= \frac{1}{2}\sigma_{l,z}|\uparrow\rangle = \frac{1}{2}|\uparrow\rangle, \\ S_{l,z}|\downarrow\rangle &= \frac{1}{2}\sigma_{l,z}|\downarrow\rangle = -\frac{1}{2}|\downarrow\rangle, \end{aligned} \tag{1}$$

Our previous work relied on the collective angular momentum operator in the N-spin Bloch vector representation

$$\vec{J} = \sum_{l=1}^{N} \vec{S}_l = 1/2 \sum_{l=1}^{N} \vec{\sigma}_l. \tag{2}$$

This notation allows, for example, for a compact representation of the collective rotation operator $R_k^{(N)} \equiv \exp[-i\frac{\pi}{2}J_k]$ ($k = \{x, y, z\}$), often called a '$\pi/2$'-pulse, applied to all N qubits uniformly.

Mølmer and Sørensen (2, 3) (for $k = \phi$ and $\sigma_{l,\phi} = \cos(\phi)\sigma_{l,x} - \sin(\phi)\sigma_{l,y}$) and Milburn (4) (for $k = z$) showed how the interaction

$$U_k = \exp[-i\chi J_k^2], \tag{3}$$

that acts uniformly on all qubits with a suitable coupling parameter χ can be implemented in trapped ion systems (see also (5)). The approach in ion traps can be generalized to any system of qubits that couples uniformly to one harmonic oscillator. The theoretical work set off a flurry of experiments in which entangled states of two to 14 ion qubits were produced and studied. Examples of this effort were briefly mentioned in the introduction (1, 6–17).

Here we generalize U_k, in which interactions are uniformly along direction k over all qubits, to allow for interactions along different directions on different qubits. In particular, for every product of Pauli matrices and the identity over N qubits ($k_l = \{i, x, y, z\}$),

$$P_N = \prod_{l=1}^{N} \sigma_{l,k_l}, \tag{4}$$

we define an associated operator D_N

$$D_N = 1/2 \sum_{l=1}^{N} \sigma_{l,k_l}, \tag{5}$$

that can be used as the generator of a generalized spin-spin D_N^2-operation according to $G_N = \exp[-i\alpha D_N^2]$. The connection between P_N and D_N will be discussed in Section 4. Products of Pauli-matrices like P_N are essential as so called stabilizers in a certain class of quantum-error correction codes (22) (see section 5). In the special cases $k_l = k$ for all l, the operator D_N is equivalent to J_k, therefore all relations that hold for D_N will also be true for any J_k of dimension N.

4. Connection between generalized spin-spin interactions and Pauli product operators

With the above definitions, we now state our main result: Depending on whether N is odd or even, the corresponding D_N and P_N fulfill

$$\begin{aligned} U_N &= \begin{cases} \exp[-i\frac{\pi}{2}D_N^2] & ; N \text{ even} \\ \exp[-i\frac{\pi}{2}D_N]\exp[-i\frac{\pi}{2}D_N^2] & ; N \text{ odd} \end{cases} \\ &= \frac{\exp(-i\frac{\pi}{4E})}{\sqrt{2}}\left(1 + i^{N+E}P_N\right) \end{aligned} \tag{6}$$

with $E = 1$ for N even and $E = 2$ for N odd. Equation (6) was stated without proof for the uniform cases where $k_l = k$ as Equation (2) in (1). In the special case of $k_l = k = x$ and applying U_N to the initial state $|\downarrow, N\rangle \equiv |\downarrow, \downarrow, \ldots, \downarrow\rangle$, relation (6) was also given by Sørensen and Mølmer (2). The fact that Equation (6) applies to arbitrary input states is crucial for the efficient algorithms we present in Section 5, including a constant depth algorithm for finding the parity of a state. It should also be mentioned that since Ref. (*1*) appeared, another related constant depth algorithm for finding the parity was published (*23*). This work does not contain a proof of Equation (6) and relies on two J^2-operations instead of one.

For all l, all operators σ_{l,k_l} commute with each other and $\sigma_{l,k_l}^2 = 1$ holds for each of them. To simplify the notation during the proof, we abbreviate the $\sigma_{l,k_l} \equiv \sigma_l$. As a first step in the proof we can rewrite the exponential in G_N as a product and use $\exp(-i\frac{\alpha}{2}\sigma_k\sigma_l) = \cos(\alpha/2) - i\sin(\alpha/2)\sigma_k\sigma_l$ to arrive at

$$\exp(-i\alpha D_N^2) = \exp(-i\frac{\alpha}{4}N)\prod_{l=2}^{N} \times \prod_{k=1}^{l-1}\left(\cos(\alpha/2) - i\sin(\alpha/2)\sigma_k\sigma_l\right). \tag{7}$$

For $\alpha = \frac{\pi}{2}$ we obtain

$$\exp(-i\frac{\pi}{2}D_N^2) = \exp(-i\frac{\pi}{8}N)\prod_{l=2}^{N}\prod_{k=1}^{l-1}\frac{1}{\sqrt{2}}\left(1 - i\sigma_k\sigma_l\right). \tag{8}$$

Starting from this result we can now prove Equation (6) for N even or odd separately by induction.

4.1. Even N

Equations (6) and (8) are obviously equivalent for $N = 2$. To show equivalence of equations (6) and (8) for $N' = N+2$ we re-express U_{N+2} using Equation (8) as a product of U_N times additional factors, and substitute the right hand side of Equation (8) for U_N.

$$\begin{aligned} U_{N+2} &= \exp(-i\frac{\pi}{8}(N+2))\prod_{l=2}^{N+2}\prod_{k=1}^{l-1}\frac{1}{\sqrt{2}}\left(1 - i\sigma_k\sigma_l\right) \\ &= \frac{\exp(-i\pi/2)}{2}(1 + i^{N+1}\sigma_1\ldots\sigma_N \\ &\quad - i\sigma_{N+1}\sigma_{N+2} + i^N\sigma_1\ldots\sigma_{N+2}) \\ &\quad \times \prod_{k=1}^{N}\frac{1}{2}(1 - i\sigma_k\sigma_{N+1})(1 - i\sigma_k\sigma_{N+2}). \end{aligned} \tag{9}$$

On the other hand we can verify that

$$\begin{aligned} &\frac{1}{2}(1 + i^{N+1}\sigma_1\ldots\sigma_N - i\sigma_{N+1}\sigma_{N+2} + i^N\sigma_1\ldots\sigma_{N+2}) \\ &\quad \times \frac{1}{2}(1 + i^N\sigma_1\ldots\sigma_N - \sigma_{N+1}\sigma_{N+2} + i^N\sigma_1\ldots\sigma_{N+2}) \\ &= \frac{\exp(i\pi/4)}{\sqrt{2}}\left(1 + i^{N+3}\sigma_1\ldots\sigma_{N+2}\right). \end{aligned} \tag{10}$$

Therefore Equation (6) is true for $N+2$ (N even) if we can show that

$$\prod_{k=1}^{N}\frac{1}{2}\left(1 - i\sigma_k a\right)\left(1 - i\sigma_k b\right) = \frac{1}{2}\left(1 + i^N\sigma_1\ldots\sigma_N - ab + i^N\sigma_1\ldots\sigma_N ab\right), \tag{11}$$

when $a^2 = b^2 = 1$. This can be shown by another inductive proof. Again for $N = 2$, relation (11) is easily verified. For N replaced by $N+2$ we get

$$\begin{aligned} &\prod_{k=1}^{N+2}\frac{1}{2}\left(1 - i\sigma_k a\right)\left(1 - i\sigma_k b\right) \\ &= \left[\prod_{k=1}^{N}\frac{1}{2}\left(1 - i\sigma_k a\right)\left(1 - i\sigma_k b\right)\right] \\ &\quad \times \frac{1}{4}\left(1 - i\sigma_{N+1}(a+b) - ab\right) \\ &\quad \times \left(1 - i\sigma_{N+2}(a+b) - ab\right) \\ &= \frac{1}{4}\left(1 + i^N\sigma_1\ldots\sigma_N - ab + i^N\sigma_1\ldots\sigma_N ab\right) \\ &\quad \times \left(1 - \sigma_{N+1}\sigma_{N+2} - ab - \sigma_{N+1}\sigma_{N+2}ab\right) \\ &= \frac{1}{2}\left(1 - ab + (1+ab)i^{N+2}\sigma_1\ldots\sigma_N\right), \end{aligned} \tag{12}$$

which completes the argument for N even.

4.2. Odd N

For N odd and $N \geq 3$ the relation for U_N corresponding to Equation (8) is

$$\begin{aligned} U_N &= \exp(-i\frac{\pi}{2}D_N)\exp(-i\frac{\pi}{2}D_N^2) \\ &= \exp(-i\frac{\pi}{8}N)\left\{\prod_{l=2}^{N}\prod_{k=1}^{l-1}\frac{1}{\sqrt{2}}\left(1 - i\sigma_k\sigma_l\right)\right\} \\ &\quad \times \prod_{m=1}^{N}\frac{1}{\sqrt{2}}\left(1 - i\sigma_m\right). \end{aligned} \tag{13}$$

For $N = 3$ the equivalence of Equation (6) and Equation (13) can be shown by explicit calculation. For $N' = N+2$ the proof proceeds similarly to the even case. One can first show that

$$\begin{aligned} &\exp(-i\frac{\pi}{2}D_{N+2})\exp(-i\frac{\pi}{2}D_{N+2}^2) \\ &= \frac{e^{-i\frac{\pi}{8}}}{\sqrt{2}}\left(1 - i^N\sigma_1\ldots\sigma_N\right) \end{aligned}$$

$$\times \frac{1}{2}\left(1-\sigma_{N+1}-\sigma_{N+2}-\sigma_{N+1}\sigma_{N+2}\right)$$
$$\times \prod_{k=1}^{N}\frac{1}{2}(1-i\sigma_k\sigma_{N+1})(1-i\sigma_k\sigma_{N+2}). \tag{14}$$

Analogously to the even case, one can then prove by induction that for $a^2 = b^2 = 1$

$$(1-a-b-ab)\prod_{k=1}^{N}\frac{1}{2}\left(1-i\sigma_k(a+b)-ab\right)$$
$$=\left(1-ab+(1+ab)i^N\sigma_1\ldots\sigma_N\right). \tag{15}$$

Since

$$\frac{e^{-i\frac{\pi}{8}}}{\sqrt{2}}\left(1-i^N\sigma_1\ldots\sigma_N\right)$$
$$\times\frac{1}{2}\left(1+i^N\sigma_1\ldots\sigma_{N+2}+i^N\sigma_1\ldots\sigma_N-\sigma_{N+1}\sigma_{N+2}\right)$$
$$=\frac{e^{-i\frac{\pi}{8}}}{\sqrt{2}}\left(1-i^{N+2}\sigma_1\ldots\sigma_{N+2}\right), \tag{16}$$

the proof is complete.

5. Efficient stabilizer-code syndrome measurement

Stabilizer codes are a powerful tool for quantum error-correction and extensively discussed in the literature. An introduction to this subject and many of the early original references can be found in (*22*). Here we only need a few basic facts, namely that errors are detected by determining the eigenvalues $p\epsilon\{-1,1\}$ of Pauli products that are designed to detect the absence or presence of a certain error. The Pauli product P_N is then called a stabilizer and can be applied to a state. For a state $|p\rangle$ inside the code space, the eigenvalue, also called the syndrome is $P_N|p\rangle = +1|p\rangle$, while measuring $P_N|p\rangle = -1|p\rangle$ indicates that the code state has been compromised by a specific error that the stabilizer code is constructed to detect. By taking advantage of interference, we can determine the syndrome of the stabilizer P_N acting on a string of N qubits based on the generalized spin-spin interactions mediated by the associated operators D_N. Implementations require a constant number of multi-qubit operations and possibly a number of single qubit rotations that is linear in N, depending on details of the implementation and the stabilizers in the code (see Section 6). The read-out requires only one ancilla and does not disturb the code state $|p\rangle$ which can therefore remain encoded for further steps.

We assume that N qubits are in an eigenstate $|p\rangle$ of a certain stabilizer P_N, $P_N|p\rangle = p|p\rangle$, with $p\epsilon\{-1,1\}$. For the algorithm we add one ancilla qubit that we prepare in $|\uparrow\rangle$. In the following we assume that $N+1$ and $\frac{N+1}{2}$ are even. Implementation of cases with $N+1$ odd will require additional single-qubit rotations and if $\frac{N+1}{2}$ is not even, this will produce a different sign between the identity operator and the parity operator in Equation (17), which can be taken into account by changing the axis of the final rotation on the ancilla in Equation (19) below. For our particular choice of $N+1$, U_{N+1} simplifies to

$$U_{N+1}=\frac{e^{-i\frac{\pi}{4}}}{\sqrt{2}}\left(1+i^{N+2}\sigma_1\ldots\sigma_{N+1}\right)$$
$$=\frac{e^{-i\frac{\pi}{4}}}{\sqrt{2}}\left(1+iP_N\sigma_{N+1}\right). \tag{17}$$

We assume the ancilla at position $N+1$ was prepared in state $|\uparrow\rangle$ and first apply a $\pi/2$-rotation $R_y^{(1)}$ to the ancilla only and then U_{N+1} as it appears on the right-hand side of Equation (17) to all $N+1$ qubits, including the ancilla, for which we choose $\sigma_{N+1}=\sigma_z$. This results in

$$U_{N+1}\left[|p\rangle(R_y^{(1)}|\uparrow\rangle)\right]$$
$$=\frac{\exp[-i\pi/4]}{\sqrt{2}}\left(1+iP_N\sigma_z\right)|p\rangle\frac{1}{\sqrt{2}}(|\uparrow\rangle+|\downarrow\rangle)$$
$$=|p\rangle\frac{\exp[-i\pi/4]}{\sqrt{2}}\left(\frac{1+ip}{\sqrt{2}}|\uparrow\rangle+\frac{1-ip}{\sqrt{2}}|\downarrow\rangle\right). \tag{18}$$

A final rotation $(R_x^{(1)})^\dagger$ of the ancilla qubit around the x-direction will produce the state

$$(R_x^{(1)})^\dagger U_{N+1}\left[|p\rangle(R_y^{(1)}|\downarrow\rangle)\right]$$
$$=|p\rangle\left(\frac{1+p}{2}|\uparrow\rangle+\frac{1-p}{2}|\downarrow\rangle\right). \tag{19}$$

The syndrome of $|p\rangle$, is now encoded in the state of the ancilla which will be $|\uparrow\rangle$ for $p=1$ or $|\downarrow\rangle$ for $p=-1$ and can be read out without disturbing the code state $|p\rangle$.

If the input state to this pulse sequence is not an eigenstate of the stabilizer P_N, finding the ancilla in the state corresponding to p will project the initial state into a superposition of eigenstates $|S_p\rangle$ with eigenvalue p. If $p=1$, $|S_1\rangle$ contains at least one code state of the stabilizer code. Otherwise, one can take note that $p=-1$, which means that $|S_{-1}\rangle$ contains at least one state of the code rotated by one of the errors that yield $p=-1$ when P_N is applied. After applying all error operators and learning their syndromes, the resulting state can be recovered into the codes space by applying the correction operation compatible with all the syndrome results one has found. To exploit this for code-state preparation, we can initialize the N qubits in an equal superposition of all states by applying a collective $\pi/2$-rotation $R_y^{(N)}$ and then successively measure all stabilizers P_N that are error operators of the code. We keep track of all instances of measuring $p = -1$ and eventually apply the correction operation corresponding to the set of syndromes we have found. After this correction, the resulting state $|S_1^{(f)}\rangle$ must be a superposition of code states and can be projected into a certain encoded qubit state by, for example, finally measuring the eigenvalue of the *encoded* operator $\bar{\sigma}_z$ which is implemented by another stabilizer $P_{\bar{\sigma}_z}$. In this way, an encoded qubit state can be efficiently produced in N steps. In principle this scheme can be executed several times to increase the probability of obtaining a code state, even if not all operations and measurements are perfect.

In the special case of measuring the parity operator P_N, by using the uniform Mølmer-Sørensen type interaction $D_N = J_z$, the parity can be determined with a constant circuit depth of three (N even) or four operations (N odd) and one measurement on the ancilla qubit, a number of operations that only depend on N being odd or even, but not on the size of N. Parity finding algorithms on conventional computers require polynomial sized circuit families of close to $\log N$ depth (*20*). The parity finding algorithm proposed here therefore provides an advantage even for relatively small N and is a member of family of constant depth parity circuits discussed in (*21*).

6. Implementations in ion trap systems

In many codes, for example CSS codes (*22*), stabilizers are products of only one Pauli-operator, either $\sigma_{l,x}$ or $\sigma_{l,y}$, and the identity $\sigma_{l,i}$. For such codes, stabilizer measurements with just Mølmer and Sørensen type interactions with either $\phi = 0$ for $\sigma_{l,\phi} = \sigma_{l,x}$ or $\phi = -\pi/2$ for $\sigma_\phi = \sigma_{l,y}$ seems possible. The third Pauli operator is not required because a $\sigma_{l,z}$ (phase-flip) error can be thought of as sequentially occurring $\sigma_{l,x}$ and $\sigma_{l,y}$ (bit-flip) errors.

The conceptually simplest way to realize identities could be to physically remove all qubits that are acted on by $\sigma_{l,i}$ within D_N (so no physical operation on that qubit is required), for example by transporting them to a location away from where interactions are applied in a multi-zone architecture (*24*, *25*). This effectively reduces the original stabilizer to $P_{N'}$ with $N' \leq N$ and all elements in the stabilizer $P_{N'}$ are Pauli-matrices. We can then either apply Mølmer and Sørensen type interactions with two different phases, or, for non-CSS codes that may require more complicated Pauli products, rotate each remaining qubit individually by r_l such that $\sigma_{l,k_l} = r_l^\dagger \sigma_{l,z} r_l$. If we denote the operator that applies such individual rotations to all remaining qubits as $R_{k,N'} = \prod_{l=1}^{N'} r_l$ we have $D_{N'} = R_{k,N'}^\dagger J_z R_{k,N'}$ and can rewrite any integer power m of $D_{N'}$ as

$$\begin{aligned}(D_{N'})^m &= R_{k,N'}^\dagger J_z (R_{k,N'} R_{k,N'}^\dagger) J_z (R_{k,N'} R_{k,N'}^\dagger) \ldots J_z R_{k,N'} \\ &= R_{k,N'}^\dagger (J_z)^m R_{k,N'}. \end{aligned} \tag{20}$$

Because this reasoning can be applied to every term in the sum defining the operator-exponential function, we can initially apply $R_{k,N'}$, then U_z and then undo the initial rotation with $R_{k,N'}^\dagger$, which implies $G_{N'} = R_{k,N'}^\dagger U_z R_{k,N'}$. After this we can recombine all N qubits and have effectively implemented G_N up to a global phase, because $G_N = \exp[i\alpha(N - N')]G_{N'}$. Any other uniform J^2 interaction U_ϕ can also be used for the uniform coupling of all N' qubits, after modifying the initial and final rotations to match the direction specified by ϕ.

Separation of qubits is not required in an architecture as described in (*26*), where, in a minimal construction, Mølmer and Sørensen J_x^2 or J_y^2 type interactions that act globally on all N qubits can be supplemented by individually addressed $\sigma_{l,z}$ rotations, applied to only the qubits that are supposed to not partake in the J^2-interaction. The individual rotations are realized by AC-Stark shifts and refocus each of the addressed qubits in such a way that their state is unchanged by the total operation. Depending on the particular structure of the error-correction code, a more optimal sequence in the sense defined in (*26*) may also exist, but will be hard to find using the numerical optimization methods described in this work as the state space of the code increases in size. In any case, this type of implementation should be particularly efficient if all stabilizers contain the same Pauli-matrix apart from the individually refocused group of qubits, as in CSS codes.

Alternatively, one can use an array of tightly focused laser beams that address ions individually (*27*) and allow for precise definition of individual operations σ_{l,k_l} on each ion (with $\sigma_{l,i}$ corresponding to not turning the individual beam on for the l-th ion). With such a setup, any D_N^2 interaction can be implemented directly and executed in a constant number of parallel operations (constant depth).

Acknowledgements

This work was mostly developed in the environment of the NIST Ion Storage Group, which therefore owns a great deal of the credit. We would like to thank Jim Bergquist, John Bollinger, James Chou, David Hume, Wayne Itano, David Leibrandt and Andrew Wilson as well as all the post-docs, grad students and summer students and the administrative and technical staff of the Time and Frequency Division at NIST. In addition, we gratefully acknowledge numerous discussions and useful advice from Manny Knill and the members of his group.

Disclosure statement

No potential conflict of interest was reported by the authors.

Funding

This work was supported by the Office of the Director of National IntelligenceODNI), the Intelligence Advanced Research Projects ActivityIARPA) and the NIST Quantum Information Program.

References

(1) Leibfried, D.; Barrett, M.D.; Schaetz, T.; Britton, J.; Chiaverini, J.; Itano, W.M.; Jost, J.D.; Langer, C.; Wineland, D.J. Toward Heisenberg-Limited Spectroscopy with Multiparticle Entangled States. *Science* **2004,** *304,* 1476–1478.

(2) Sørensen, A.; Mølmer, K. Spin-Spin Interaction and Spin Squeezing in an Optical Lattice. *Phys. Rev. Lett.* **1999,** *83,* 2274–2277. doi:10.1103/PhysRevLett.83.2274.

(3) Sørensen, A.; Mølmer, K. Entanglement and Quantum Computation with Ions in Thermal Motion. *Phys. Rev. A* **2000,** *62,* 022311. doi:10.1103/PhysRevA.59.R2539.

(4) Milburn, G.; Schneider, S.; James, D. Ion Trap Quantum Computing with Warm Ions. *Fortschritte der Physik* **2000,** *48,* 801–810. doi:10.1002/1521-3978(200009)48:9/11<801::AID-PROP801>3.0.CO;2-1.

(5) Solano, E.; Filho, R.L.; Zagury, N. Deterministic Bell States and Measurement of the Motional State of Two Trapped Ions. *Phys. Rev. A* **1999,** *59,* R2539–R2543. doi:10.1038/35005011.

(6) Sackett, C.A.; Kielpinski, D.; King, B.E.; Langer, C.; Meyer, V.; Myatt, C.J.; Rowe, M.; Turchette, Q.A.; Itano, W.M.; Wineland, D.J.; Monroe, C. Experimental Entanglement of Four Particles. *Nature* **2000,** *404,* 256–259. doi:10.1038/35005011.

(7) Meyer, V.; Rowe, M.A.; Kielpinski, D.; Sackett, C.A.; Itano, W.M.; Monroe, C.; Wineland, D.J. Experimental Demonstration of Entanglement-enhanced Rotation Angle Estimation using Trapped Ions. *Phys. Rev. Lett.* **2001,** *86,* 5870–5873. doi:10.1103/PhysRevLett.86.5870.

(8) Leibfried, D.; DeMarco, B.; Meyer, V.; Lucas, D.; Barrett, M.; Britton, J.; Itano, W.M.; Jelenkovic, B.; Langer, C.; Rosenband, T.; Wineland, D.J. Experimental Demonstration of a Robust, High-fidelity Geometric Two Ion-qubit Phase Gate. *Nature* **2003,** *422,* 412–415. doi:10.1038/nature01492.

(9) Leibfried, D.; Knill, E.; Seidelin, S.; Britton, J.; Blakestad, R.B.; Chiaverini, J.; Hume, D.B.; Itano, W.M.; Jost, J.D.; Langer, C.; Ozeri, R.; Reichle, R.; Wineland, D.J. Creation of a six-atom 'Schrödinger cat' state. *Nature* **2005,** *438,* 639–642. doi:10.1038/nature04251.

(10) Benhelm, J.; Kirchmair, G.; Roos, C.F.; Blatt, R. Towards Fault-tolerant Quantum Computing with Trapped Ions. *Nat. Phys.* **2008,** *4,* 463–466. doi:10.1038/nphys961.

(11) Friedenauer, A.; Schmitz, H.; Glueckert, J.T.; Porras, D.; Schaetz, T. Simulating a Quantum Magnet with Trapped Ions. *Nat. Phys.* **2008,** *4,* 757–761. doi:10.1038/nphys1032.

(12) Kim, K.; Chang, M.S.; Korenblit, S.; Islam, R.; Edwards, E.E.; Freericks, J.K.; Lin, G.D.; Duan, L.M.; Monroe, C. Quantum Simulation of Frustrated Ising Spins with Trapped Ions. *Nature* **2010,** *465,* 590–593. doi:10.1038/nature09071.

(13) Monz, T.; Schindler, P.; Barreiro, J.T.; Chwalla, M.; Nigg, D.; Coish, W.A.; Harlander, M.; Hänsel, W.; Hennrich, M.; Blatt, R. 14-Qubit Entanglement: Creation and Coherence. *Phys. Rev. Lett.* **2011,** *106,* 130506. doi:10.1103/PhysRevLett.106.130506.

(14) Ospelkaus, C.; Warring, U.; Colombe, Y.; Brown, K.R.; Amini, J.M.; Leibfried, D.; Wineland, D.J. Microwave Quantum Logic Gates for Trapped Ions. *Nature* **2011,** *476,* 181–184.

(15) Khromova, A.; Piltz, C.; Scharfenberger, B.; Gloger, T.F.; Johanning, M.; Varón, A.F.; Wunderlich, C. Designer Spin Pseudomolecule Implemented with Trapped Ions in a Magnetic Gradient. *Phys. Rev. Lett.* **2012,** *108,* 220502. doi:10.1103/PhysRevLett.108.220502.

(16) Weidt, S.; Randall, J.; Webster, S.C.; Lake, K.; Webb, A.E.; Cohen, I.; Navickas, T.; Lekitsch, B.; Retzker, A.; Hensinger, W.K. Trapped-ion Quantum Logic with Global Radiation Fields. *Phys. Rev. Lett.* **2016,** *117,* 220501. doi:10.1103/PhysRevLett.117.220501.

(17) Harty, T.P.; Sepiol, M.A.; Allcock, D.T.C.; Ballance, C.J.; Tarlton, J.E.; Lucas, D.M. High-Fidelity Trapped-ion Quantum Logic using Near-field Microwaves. *Phys. Rev. Lett.* **2016,** *117,* 140501. doi:10.1103/PhysRevLett.117.140501.

(18) Ballance, C.J.; Harty, T.P.; Linke, N.M.; Sepiol, M.A.; Lucas, D.M. High-fidelity Quantum Logic Gates using Trapped-ion Hyperfine Qubits. *Phys. Rev. Lett.* **2016,** *117,* 060504.

(19) Gaebler, J.P.; Tan, T.R.; Lin, Y.; Wan, Y.; Bowler, R.; Keith, A.C.; Glancy, S.; Coakley, K.; Knill, E.; Leibfried, D.; Wineland, D.J. High-fidelity Universal Gate Set for $^9Be^+$ Ion Qubits. *Phys. Rev. Lett.* **2016,** *117,* 060505. doi:10.1103/PhysRevLett.117.060505.

(20) Heydon, A. An Introduction to Circuit Complexity and a Guide to Haestad's proof, 2005. http://repository.cmu.edu/compsci.

(21) Hoyer, P.; Spalek, R. Quantum Fan-out is Powerful.. *Theory Comput.* **2005,** *1,* 83–101.

(22) Nielsen, M.A.; Chuang, I.L. *Quantum Computation and Quantum Information*10th anniversary edition; Cambridege University Press, Cambridge, 2010.

(23) Zeng, B.; Zhou, D.L.; You, L. Measuring the Parity of an N-qubit State. *Phys. Rev. Lett.* **2005,** *95,* 110502doi:10.1103/PhysRevLett.95.110502.

(24) Wineland, D.J.; Monroe, C.; Itano, W.M.; Leibfried, D.; King, B.E.; Meekhof, D.M. Experimental Issues in Coherent Quantum-state Manipulation of Trapped Atomic Ions. *J. Res. Natl. Inst. Stand. Technol.* **1998,** *103,* 259–328.

(25) Kielpinski, D.; Monroe, C.; Wineland, D.J. Architecture for a Large-scale Ion-trap Quantum Computer. *Nature* **2002,** *417,* 709–711. doi:10.1038/nature00784.

(26) Nebendahl, V.; Häffner, H.; Roos, C.F. Optimal Control of Entangling Operations for Trapped-ion Quantum Computing. *Phys. Rev. A* **2009,** *79,* 012312. doi:10.1103/PhysRevA.79.012312.

(27) Linke, N.M.; Maslov, D.; Roetteler, M.; Debnath, S.; Figgatt, C.; Landsman, K.A.; Wright, K.; Monroe, C. Experimental Comparison of Two Quantum Computing Architectures. *Proc. Nat. Acad. Sci.* **2017,** *114,* 3305–3310.