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MEASUREMENT OF MILLIMETER WAVE POWER BY RADIATION PRESSURE

by

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ABSTRACT

The main part of this report concerns the theory of a device for measuring millimeter power by measuring the radiation pressure force exerted on the reflector of a Fabry-Perot interferometer. It is shown that, because of the phenomenon of multiple reflections, the force is larger than that pertaining to a single reflection by a factor which is approximately the Q of the resonator. It is also shown that such a device theoretically is able to make an absolute measurement of a power level of 0.1 watt with an accuracy of about one-half of a decibel. Appendix A reviews the various derivations of the standard formulas for the force due to radiation pressure and it suggests that a rigorous derivation based upon thermodynamic arguments without direct or indirect use of electromagnetic theory probably can not be given. Appendix B concerns the work performed by radiation pressure. It is shown non-relativistically that the work performed on a reflector which moves through a closed cycle in the presence of a constant field is not zero. For a perfect reflector in simple harmonic oscillation with normal incidence to a plane wave, the power of the reflected wave exceeds the power of the incident wave on the average by an amount equal to the power of the incident wave multiplied by the square of the ratio of the velocity amplitude to the speed of light. In laboratory experiments this effect is negligible.

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INTRODUCTION

The pressure exerted by electromagnetic radiation has been of interest to pure and applied science ever since it was postulated by Kepler in 1619 as the cause of the deflection of tails of comets. However, it was not until the experiments of Lebedew[1] and of Nichols and Hull[2, 3] at the beginning of the present century, that the postulate was completely verified experimentally. Until a few years ago, this phenomenon was associated only with optical and thermal (i.e. black body) radiation. With such radiation the desired effect is very difficult to observe. One reason is that the total force obtainable under laboratory conditions is very small. In the experiments of Nichols and Hull it was less than 10^{-4} dyne. A second reason is that the true effect is often screened by the much larger radiometer effect. This latter effect is the consequence of the recoils of gas molecules, which are unequally distributed over the surface upon which the radiation is incident. This unequal distribution is the result of temperature gradients on the surface.

In more recent years radiation pressure has been observed at microwave frequencies. Here it is feasible to produce larger forces. Cullen[4] observed the effect by the introduction of a small reflecting vane in an X-band waveguide. Using powers of the order of 10 to 50 watts, he obtained excellent agreement between the measured force and that calculated from the power which was measured calorimetrically.

Carra observed radiation pressure[5] and the closely related effect[6] of the angular momentum imparted to an object which is intercepting radiation with circular polarization. Jones[7] and Jones and Richards[8] have observed the force due to radiation pressure on an obstacle placed in a dielectric liquid.

On the other hand, it should be remarked that not all devices with which mechanical forces are produced directly by microwave fields make use of radiation pressure in the strict sense. Forces can also be produced by quasi-static fields. The gold leaf electroscope and the quadrant electrometer with the needle connected to one pair of quadrants can respond to an rf field. In these cases the electric lines of force terminate upon charges residing on the surfaces of the instrument, and the electric lines of force meet the surfaces at right angles ideally. With radiation pressure, the fields are associated with waves emitted as the result of acceleration of distant charges. In the simple case of normal incidence upon the surface upon which the force is exerted, the electric lines of force are parallel to the surface. Some times, both effects may be present, as is probably the case of one commercially available force-operated microwave wattmeter[9].

For a given amount of radiated power, the force exerted by radiation pressure may be magnified many times over that produced by a single plane wave by setting up multiple reflections between the original surface and another parallel surface. These surfaces together form a microwave resonator. It will be shown that the magnification of the force is of the order of the Q of the resonator. This fact seems to have been overlooked in previous work.

Utilization of this magnification seems particularly attractive at millimeter waves, where the diffraction effects can be made small. In this situation the resonator can take the form of a Fabry-Perot interferometer. With power levels produced by receiving type klystrons the force can approach one tenth of a dyne and therefore is at least three orders of magnitude greater than that of the Nichols and Hull experiment.[3] It will be shown that in principle, the power can be determined by measurement of the force and the Q of the resonator. Therefore, this device can be used as a millimeter power meter. Alternatively, the electric intensity of the rf field between the Fabry-Perot plates can be measured in terms of a dc field set up between the plates when the latter has been adjusted to cause the forces set up by the two fields to cancel. In both cases, however, the simple explanation is not the final one as there are a number of corrections due to diffraction, fringing, and impedance mismatches which must be considered.

THEORY OF THE APPARATUS

Nearly every textbook of the electromagnetic theory shows that for a plane wave incident on a plane surface in an otherwise unbounded space the force on the surface is given by

$$F = BA\,(u_i + u_r) \quad , \tag{1}$$

where B is a dimensionless quantity, A is the area of the surface, u_i is the electromagnetic energy per unit volume associated with the incident wave, and u_r is the corresponding quantity associated with the reflected wave. It can be further shown that if θ is the angle of incidence,

$$B = \cos^2 \theta \quad . \tag{2}$$

Of course, in the important special case of normal incidence,

$$B = 1 \quad . \tag{3}$$

The phenomenon of radiation pressure is closely related to the transport of momentum by the waves. The force on a surface can be accounted for by the time rate of change of momentum of the waves at the surface. The momentum transported per unit time across a surface at right angles to the direction of propagation is

$$G = p/c \quad , \tag{4}$$

where p is the energy transported per unit time by the wave, and c is the speed of propagation in the medium, which is assumed to be homogeneous, isotropic, and non-dispersive. p is equal, of course, to the magnitude of the Poynting vector multiplied by the area of cross section.

Appendix A, which is at the end of the report, will contain a discussion of some of the methods used to derive these equations. It will also contain some references where complete proofs are given.

In the region between the plates of the interferometer, there are an infinite number of incident and reflected waves. In the application of Eq. (1), $u_i + u_r$ is replaced by u , which represents the total energy density associated with all of these waves. It is assumed that these waves propagate normally to the plates. If the energy density varies with position, the force of one plate becomes

$$F = \int u \, dA \tag{5}$$

by the use of Eqs. (1) and (3) . In Eq. (5) u is evaluated at points just outside the surface of the plate. If the plate is partially transparent, the integral has to be evaluated on both sides, and the difference of the two contributions gives the net force. In this report it is assumed that one of the plates is opaque, and the calculations pertain to this plate.

While u may be supposed to vary over the cross section, it is assumed that at any one point on the cross section it is independent of the coordinate perpendicular to the surface. Then

$$F = U/D \quad , \tag{6}$$

where U is the total stored energy associated with the volume enclosed by the plates and where D is their separation.

The total energy U also appears in the expression for the quality factor Q of the resonator, which by a standard definition is given by

$$Q = \omega \; U/P \quad , \tag{7}$$

where P is the power dissipated within the cavity. If Eqs. (6) and (7) are combined,

$$F = \frac{QP}{\omega D} \quad , \tag{8A}$$

or

$$P = \frac{\omega DF}{Q} \quad . \tag{8B}$$

Equation (7) is a general definition and applies independently of the standing wave condition in the region adjacent to the interferometer: that is, independently of the state of impedance mismatch between the interferometer and its source. Q may be determined experimentally by observation of the frequency response of the interferometer. If this observation is made under the same condition of impedance mismatch as with the measurement of force, the power can be determined from Eq. (8B) from the measurement of the force, since

ω and D can be determined easily. Equation (8B) serves as the basis of the proposedmethod for the measurement of power. From the preceding remarks it can be concluded that with reasonable care in the design of the apparatus, errors due to non-uniformity of the field can be made small.

The spacing D can be determined directly from the resonance condition of the interferometer, which is given approximately as

$$D = m\lambda/2 \quad , \tag{9}$$

where λ is the wavelength and m is an integer. Because of the skin effect D differs slightly from the spacing observed visually. Also because of diffraction effects the wavelength differs slightly from that of a plane wave in unbounded space. However, in practice these differences are small, and knowledge of the visual separation and free space wavelength can be used to ascertain m .

If diffraction and dielectric losses are negligible, it can be shown[10] from Eq. (6) that

$$Q = \frac{2m\pi}{2 - (\Gamma_1|^2 - |\Gamma_2|^2)} \quad , \tag{10}$$

where Γ_1 and Γ_2 are respectively the amplitude (electric field) reflection coefficients of the two plates. The numerator of Eq. (10) contains an approximation which is valid if Γ_1 and Γ_2 are very nearly equal to unity. If Eqs. (9) and (10) are inserted into Eq. (8A),

$$F = \frac{2P}{w_c}\left(\frac{1}{2 - |\Gamma_1|^2 - |\Gamma_2|^2}\right) \tag{11}$$

For a plane wave normally incident upon a nearly perfectly reflecting surface, u_i and u_r in Eq. (11) are both given by $p/(Aw)$. Then the force on this surface is given by

$$F = 2\,p/c\ . \tag{12}$$

The forces produced in the classic experiments on radiation pressure have been given approximately by Eq. (12).

Equation (11) differs from Eq. (12) by the inclusion of the factor in the parentheses, which numerically in a practical case is large compared to unity (about 250 for a typical situation in the millimeter wave region). Therefore, it can be seen that the effect of resonance produced by Fabry-Perot plates can be used to amplify the magnitude of the force obtainable by conventional experiments by a considerable amount. The use of this amplification allows the detection of radiation by its pressure with much smaller power levels than is possible in conventional experiments, or, at power levels greater than the minimum detectable, it allows the measurement of the power level with considerably better precision. The use of this resonance phenomenon is the principal point proposed in this report.

METHOD OF OBSERVATION

In one conceivable arrangement for the absolute measurement of force, the Fabry-Perot plates are horizontal with the opaque one being on top and with it supported from a balance then the radiation force is manifest as an apparent decrease in the weight of the plate. Evidently, this decrease can be measured more easily if the plate is made as small as possible, since the radiation force is independent of the area according to the preceding theory, while the weight increases with the area.

The area of the plates must then be the smallest possible which can intercept the entire beam of radiation. Therefore, it cannot be smaller than the cross section of the horn which supplies the interferometer. To avoid excessive divergence of the beam, the interferometer must lie in the Fresnel diffraction region of the horn. Therefore, it should lie as close as possible to the horn. Assuming that the lower, transparent plate is of negligible thickness, the closest possible distance is with the lowest mode, with the spacing between the plates of one-half wavelength and with about a half wavelength between the horn and the lower transparent plate, or with one wavelength between the horn and the opaque plate.

The limit of the Fresnel region is given by the so-called Rayleigh distance, which, for a square horn of cross section a^2 is given by

$$R = a^2/\lambda \quad , \tag{13}$$

where λ is the free space wavelength. Then, if $R = \lambda$, it follows that the dimensions of the horn $a = \lambda$. To allow a factor of safety for fringing and for misalignment, it follows that the opaque plate should have a cross section of 2λ on a side. The weight of this plate is then

$$W = 4 \rho g \lambda^2 t \quad , \tag{14}$$

where t is the thickness, ρ is the mass density, and g is the acceleration due to gravity. It is apparent, then, that for a given amount of power, the method is more sensitive at shorter wavelengths, where W is smaller.

Consider a practical example in which P is 100 milliwatts (which is typical of small klystrons) and $\lambda = 3$ mm. Reasonable values are $|\Gamma_1|^2 = |\Gamma_2|^2 = 0.998$. According to Eq. (11), $\Gamma = 1.7 \times 10^{-7}N = 1.7 \times 10^{-2}$ dyne.

If the copper plate has a thickness of 2 mm and a density $\rho = 3$, Eq. (4) gives W = 210 dynes or 2.2×10^{-5} kg wt. Thus the apparent change in weight due to radiation pressure is about one part in 1.3×10^{4}. On the other hand, weights of the order of 10^{-5} kg can be determined[11] with an accuracy of about one part in 3×10^{5}. Thus the precision whereby the radiation pressure force can be measured is about one part in 20 for 100 mw. The errors of the other quantities involved in the determination of power are likely to be smaller. One part in 20 corresponds to about 0.2 db, and such accuracies are adequate for most purposes, but this is considerably poorer than what can be accomplished by microcalorimetric methods.

The accuracy of the force measurement increases in proportion to the power. However, ultimately the overall accuracy is limited by other errors. It is doubtful that the reflection coefficients can be determined to an accuracy better than one part in 100, and thus the limiting accuracy of such a device as an absolute instrument is likely to be about one percent. For relative measurements or for uses where it is calibrated in terms of some other instrument such as a microcalorimeter, the precision could be much greater at high powers.

Since the information provided by Mr. Wildhack[11] indicates that for small forces the absolute error in force measurements is independent of the magnitude of the force, there would be little advantage to using thinner reflectors on the Fabry-Perot resonator, and for the same reason the accuracy would not improve at shorter wavelengths, where the reflector would be smaller in cross section.

Therefore, in summary of this section, it can be said that the Fabry-Perot radiation pressure instrument appears to have the advantage of simplicity as a power measuring device at millimeter waves. The accuracy is sufficient to be useful but not as high as other more complicated methods.

As is well known, the sensitivity of such a device can be increased by the use of a mechanical resonance in the instrument in the event that the source is turned on and off at a periodic rate, or if some sort of periodic chopper is placed between the source and the detector. However, for simplicity, this possibility is not considered at present. To adjust the apparatus for electrical resonance with a pulse source appears difficult, but perhaps this matter should be re-examined at some later time.

D. C. COMPENSATION METHOD

One method of operation which is very appealing is to compensate the force due to radiation pressure by an electrostatic force of attraction, which can be created by applying a dc voltage between the plates. Then, in principle, the balance need not be calibrated. However, there is a serious practical difficulty of making an electrical connection to an object supported from a sensitive microbalance without interfering with its operation. Nevertheless, this possibility is being mentioned in the hope that this difficulty can be overcome.

The electrostatic force of attraction is given by Eqs. (5) and (6), where now the energy density is the electrostatic one. It is interesting to note that if the parallel plate capacitor acts as a component of a low frequency resonant L-R-C circuit and if fringing is neglected, the force of attraction is identical with that given by Eq. (8A) except that the right hand side must be multiplied by one half. These observations illustrate the discussion of the Maxwell stress tensor given by Panofsky and Phillips.[12]

ERRORS

The principal indeterminant constant errors are due to diffraction of the waves around the edges of the interferometer and the closely allied phenomenon of field inhomogeneities. If the dc compensation method is used there is the analogous effect of fringing which causes the rf and dc fields to have different distributions. These effects can be held to reasonable limits only by using plates of sufficient size.

If the instrument is used without dc compensation, the principal additional errors are due to measurement of the force, as discussed previously.

In addition, there is the error which results if the device is used under different conditions of impedance mismatch than those under which it is calibrated. Usually the most significant determination of power is measurement of the total incident power, that is, when the device absorbs the total power incident upon it. In such a case, it is necessary to correct for impedance mismatches.

Also, there exists the random error resulting upon failure to adjust the spacing between the reflectors.

Finally, there may be errors due to convection currents and the radiometer effect, which have plagued the optical experiments on radiation pressure.

CONCLUSION

This report shows that the force due to radiation pressure can be magnified by the use of multiple reflections between reflectors. Under practical conditions in the millimeter wave region this amplification can be of the order of 250.

Furthermore it has been shown that it is possible to build a device using this principle which can make an absolute measurement of the power output of a small receiving type klystron with an accuracy of about one-half of a decibel.

Following are two appendices which have no direct bearing upon the design and operation of such a practical device, but which are of considerable academic interest--at least to the author. The first of these discusses the various types of derivations of the basic formula for the force due to radiation pressure. The second discusses the work performed when a reflector moves under the influence of a radiation pressure force. It is shown that, if the reflector is moved through a closed cycle, the total work is not zero and is of such a sign to cause the energy of the electromagnetic field to increase.

APPENDIX A

Discussion of the Derivation of the Basic Formulas

For present purposes, a derivation is defined as a quantitative evaluation of the constant B in Eq. (1). For plane waves $B = \cos^2 \theta$, as given in Eq. (2). From this result it may further be shown that, for a superposition of many waves with random angles of incidence $B = 1/3$. On the other hand, thermodynamic proofs have been given to show that $B = 1/3$ for black body radiation, which, of course, is a special case of the situation of many waves with random angles of incidence. The fact that B is shown to be equal to 1/3 by these two supposedly independent methods has an exciting implication: namely, that the second law of thermodynamics is not an independent law but is derivable from Maxwell's equations and the zeroeth and first laws. On the other hand, while the author has been unable to examine all of the available thermodynamic proofs, he has been unable to find one which has been free of objections: either there has been included some illogical step or that the proof has not been truly independent of electromagnetic theory. A bibliography of the early derivations can be found in Reference No. 3.

There exist several electromagnetic derivations, and there appears to be no doubt concerning the validity of any of them. The most physical, and, perhaps, the most elementary proof considers the magnetic force on the current induced by the wave in the reflector.[13] Another proof results when the charge and current densities in the Lorentz force equation are expressed in terms of field quantities by means of Maxwell's equations.[14] Still a third approach is the result of the axiomatic introduction of the Maxwell stress tensor.[15]

The author has been able to examine only two of the early thermodynamic treatments,[16, 17] but these appear to summarize the earlier works. These proofs have not been convincing since they seemed to first express doubt as to the value of the numerical coefficient B and then to adopt the correct value by a process of wishful thinking. A thermodynamic proof which appears to be valid can be found in the textbook of Allis and Herlin.[18] However, their proof is based upon the Planck black body radiation distribution function, whose derivation is based upon the knowledge of the number of normal modes of radiation when contained in an enclosure, and this quantity is derived implicitly from electromagnetic theory.

The possibility of a thermodynamic proof is suggested by the fact that energy density has the same dimensions as pressure, and intuitively it can be supposed, therefore, that radiation pressure is proportional to the energy density. Furthermore, according to the kinetic theory of gases, the pressure of an ideal monatomic gas is proportional to the energy density, and the proportionality constant has the same value as with radiation coming from random directions.[19] However, with sound, the radiation pressure is not the total pressure but an increment superimposed upon the static pressure in the case of a gas. In this case the radiation pressure is not proportional to the sound energy density but to its square root.[20] More completely, the sound radiation pressure is proportional to the geometrical mean of the sound energy density, and the energy density associated with the translational kinetic energy associated with the thermal motions for the sound case is complicated by the fact that the thermal motions are in random directions, while the displacements associated with the sound wave are in one direction. If a general valid thermodynamic proof can be given, it should apply to all types of radiation. The fact that the behavior of sound appears to be different from that of electromagnetic radiation seems difficult to reconcile with the concept of thermodynamic proof. Furthermore, the situation of plane waves also seems difficult to formulate in terms of thermodynamics. For these various reasons, the present author believes that a complete independent thermodynamic proof of the formula for radiation pressure can not be given and, therefore, the second law of thermodynamics is independent of Maxwell's equations.

On the other hand, it is interesting to consider the situation wherein a cylinder and piston enclose either electromagnetic radiation or an ideal monatomic gas. If the piston moves in such a direction to cause the volume to expand, and if the energy density is held constant, some external source must provide an amount of energy equal to twice the work performed, one-half being converted into work and the other half being used to maintain the constant energy density in the increased volume. This result reminds us of the situation in electrostatics when conductors are moved under constraints of constant potential. The similarity, except for a factor of 2 and sign, between radiation pressure and electrostatic forces has been pointed out previously. There is, of course, an analogous situation pertaining to current carrying conductors and magnetic fields.*

APPENDIX B

The Work Done by Radiation Pressure on Moving Reflectors

As reflectors move under the action of radiation pressure, work is performed, obviously. This idea leads directly to the concept of a Carnot engine using radiation as a working substance. The radiation is trapped in a closed cylinder with a piston, whose walls can be made alternately perfectly reflecting or transparent to adjacent temperature reservoirs. Most Carnot engines employ a working substance with constant mass, but a radiation engine is an exception. Such Carnot engines are invoked in the thermodynamic treatments of radiation pressure.[16, 17, 18]

As Allis and Herlin point out,[18] the performance of work by radiation pressure is intimately related to the Doppler effect, and a classical quantitative evaluation resembles the treatment of the Doppler effect in introductory textbooks. Such an evaluation may be made by imagining a plane source emitting radiation in a direction normal to its surface towards a parallel plane reflector, which is moving away from the source with a speed v. Because of the motion of the reflector, the volume of the radiation in transit must continually be increasing. Since the rate of emission of radiation by the source is assumed constant, the energy density must decrease from the

*Note added in proof, further discussion on the thermodynamics of radiation, may be found in A. S. Eddington, "The Internal Constitution of the Stars" Dover Publications, New York (1959), Chap. 2.

value it would have if the reflector had remained stationary. A similar statement can be made with reference to reflected radiation, if these statements are expressed mathematically and if Eqs. (1) and (3) and the principle of conservation of energy are employed, the work performed can be evaluated. The author has made such a calculation but will not give it here.

Instead, a simpler and more informative non-relativistic proof based upon the photon hypothesis will be given. This leads to the same result as the purely classical proof, as it should. Suppose first that the reflector is stationary. The number of photons incident upon the reflector is N, and, if it is a perfect one, all are reflected. The momentum per second imparted by the incident beam is Nhf/c, where h is Planck's constant, f is the frequency, and c is the speed of light. The momentum imparted by reaction to the reflected beam is equal. Equating the impulse to the change in momentum the force is given as

$$F_o = 2\ Nhf/c \quad . \tag{15}$$

Now let us suppose that the reflector is moving with the speed v away from the source and that v is small compared to c. However, in a plane of reference in which the reflector is at rest, the source is moving away at a speed v . Hence, because of the Doppler effect, the frequency of the photons now is approximately $f(1 - v/c)$ for the incident beam, and the frequency is unchanged by reflection in this plane of reference. Hence, by repeating the argument which was given above it can be seen that the force in this plane of reference is

$$F = F_o(1 - v/c) \quad . \tag{16}$$

According to the principles of non-relativistic mechanics, this must be the same force which is observed in the original frame of reference in which the reflector moves with the speed v .

If x is the displacement of the reflector, the work performed by the radiation is

$$W = F_o \int dx - (F_o/c) \int v\, dx \quad . \tag{17}$$

Next suppose that the reflector moves parallel to itself in a closed cycle so that it ultimately returns to its original position and velocity as is the case if it undergoes simple harmonic motion. While the first term in Eq. (17) integrates to zero, as one expects intuitively, the remarkable thing is that the second term does not vanish. Except for a factor of mass, the second integral is a phase integral, the same type of integral which is quantized in the Bohr quantum theory. The sign of this integral indicates that work is performed by the outside agency upon the radiation.

This result is easy to understand physically. The incident wave induces instantaneous separations of charge on the reflector, which is being accelerated. According to classical radiation theory, radiation (that is, additional radiation) is produced whenever charges are accelerated. The energy of this radiation depends only upon the square of the magnitudes of the charges and the square of the magnitudes of the accelerations, and, therefore, is independent of the magnitudes of the charges and accelerations. While this physical explanation answers one question as to how this radiation is produced, it raises another question, for which no answer appears to be at hand: how far must a positive and equal negative charge be separated before they produce radiation when they receive equal accelerations?

If the reflector executes simple harmonic motion at an angular frequency and with a velocity amplitude v_o , the second integral of Eq. (17) may be evaluated, and by the help of Eq. (12), it can be shown that the average gain in power of a wave by reflection from such a device is

$$\Delta p = p(v_o/c)^2 \quad . \tag{18}$$

In laboratory experiments v_o/c is small compared to unity, but this author can not help wondering whether in astrophysics there might exist situations where this effect might affect the energy balance of the cosmos. The most favorable laboratory situation of this type pertains to the work in this laboratory by Mockler and his associates on the measurement of the speed of gamma rays by the use of the Mössbauer effect.[21] They believe that 200 cm/sec. can be obtained. With such a speed $\Delta p/p$ is about 5×10^{-17}, which is probably too small to be observed.

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