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Some Causes of Resonant Frequency Shifts in Atomic Beam Machines. I. Shifts Due to Other Frequencies of Excitation

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The quantum theory of an atomic beam machine is set up in matrix form. A new method is then used to derive the Bloch–Siegert shift in the resonance. The results are extended to the case of Ramsey-type excitation. Finally the Bloch–Siegert shift is computed for the present atomic beam frequency standards and found to be well below the accuracy of measurement.

1. INTRODUCTION

In the analysis of line shapes to be expected from atomic beam resonance experiments it has been customary to assume a special form for the matrix of the interaction which makes the resulting equations easy to solve [reference 1, Eq. V. 2]. These equations lead to expressions for the line shape agreeing with experiment. The effects of including other terms in the actual interaction have been discussed by Bloch and Siegert,[2] Stevenson,[3] and others,[4] and summarized in Ramsey's book.[1] In the case of magnetic resonance the usual approximation corresponds to a rotating (circularly polarized) excitation field, although oscillating (linearly polarized) fields are used in experiments. The effect of the antirotating component of an oscillating field has been found to be a normally negligible shift in the resonance frequency. A similar shift would be expected in observing the resonance in cesium used as a standard of frequency. Because of the high accuracy to which this frequency can be determined, the present calculations were undertaken to determine this shift quantitatively for the cesium beam machine now used as the U. S. standard of frequency.

Although the analytical results here reported have been obtained by others, the methods are somewhat different from those in the literature. In Sec. 2 the approximate equations for the state amplitudes are obtained from the exact ones by a simple, but general, method. A concise, but readily interpretable notation is used. The Rabi line shape is obtained quickly from these equations. Section 3 outlines a general method for treating rapidly oscillating perturbations and applies it to the equations of Sec. 2. This method is shorter than that of Bloch and Siegert,[2] but about comparable to the method of Stevenson[3] in labor required. Another method is indicated in the Appendix. The first-order solution for the amplitude is given in full. Section 4 derives the Ramsey line shape for separated oscillating fields, showing what happens to shifts present in the Rabi line shape. The physical situation treated corresponds more closely to the experimental case than that treated by Ramsey[5] but is no more difficult to work out. Finally, the resonant frequency shifts due to the antirotating field are evaluated numerically for the two cesium beam standards at the National Bureau of Standards.

2. THE RABI LINE SHAPE

Consider a quantum-mechanical system having two states of energies E_1 and E_2. Let a_1, a_2 be the respective probability amplitudes that the system is in one of these states. Let an oscillatory perturbation proportional to $\cos\omega t$ be applied to the system from time t_0 to time $t_0+\tau$. Call the matrix elements of the perturbation connecting the two states $2c$ and the diagonal elements $2d$ and $2f$. Assume that the matrix elements to any other states are zero. The Hamiltonian of the system is then:

$$H=\begin{pmatrix}E_1 & 0\\ 0 & E_2\end{pmatrix}+\begin{pmatrix}2d & 2c\\ 2c & 2f\end{pmatrix}\cos\omega t.$$

Since factors of the form $e^{i\nu t}$ in the amplitudes have no effect on the probabilities it is convenient to factor them out ahead of time so they do not clutter our equations later on. We define new probability amplitudes $\alpha=a_1e^{i\mu t}$, $\beta=a_2e^{i\nu t}$. Using these as a basis the Hamiltonian becomes[6]:

$$\begin{pmatrix}E_1-\mu & 0\\ 0 & E_2-\nu\end{pmatrix}+\begin{pmatrix}d & ce^{i(\mu-\nu)t}\\ ce^{-i(\mu-\nu)t} & f\end{pmatrix}(e^{i\omega t}+e^{-i\omega t}).$$

By choosing $\mu-\nu=\omega$ part of the perturbation matrix becomes time-independent. By choosing $\mu+\nu=E_1+E_2$ the time-independent part of the Hamiltonian H_0 becomes traceless, and hence has eigenvalues $\pm p$. The remaining time-dependent part of the Hamiltonian H_1 is treated as a perturbation. Substituting $\mu=\frac{1}{2}(E_1+E_2+\omega)$ and $\nu=\frac{1}{2}(E_1+E_2-\omega)$ and the ab-

[1] N. F. Ramsey, *Molecular Beams* (Oxford University Press, London, 1956).
[2] F. Bloch and A. Siegert, Phys. Rev. **57**, 522 (1940).
[3] A. F. Stevenson, Phys. Rev. **58**, 1061 (1940).
[4] S. Autler and C. H. Townes, Phys. Rev. **100**, 703 (1955).
[5] N. F. Ramsey, Phys. Rev. **100**, 1191 (1955). Compare also N. F. Ramsey, "Shapes of Molecular Beam Resonances," in *Recent Research in Molecular Beams*, edited by I. Estermann (Academic Press Inc., New York, 1959), pp. 115–117.
[6] To avoid unnecessary letters in the equations, $\hbar=1$ throughout this report. Thus energy and frequency have the same dimensions.

breviations $\omega_0=E_1-E_2$, $\Delta=(\omega-\omega_0)/2$:

$$H_0=\begin{pmatrix}-\Delta & c\\ c & \Delta\end{pmatrix},$$

$$H_1=\begin{pmatrix}2d\cos\omega t & ce^{i2\omega t}\\ ce^{-i2\omega t} & 2f\cos\omega t\end{pmatrix}.$$

Note that H_0 contains part of the effects of the perturbation, in fact the major part. For $\omega\gg c$ the effect of H_1 is very small. Neglecting H_1 is equivalent to the so-called rotating field approximation.[7] However, the present formulation is independent of whether or not the concept of a rotating excitation field has any physical meaning.

Neglecting H_1 the solution of the Schrödinger equation is

$$\Psi(t)=e^{-iH_0(t-t_0)}\Psi(t_0)$$

where

$$e^{-iH_0(t-t_0)}=\cos p(t-t_0)1-(i/p)\sin p(t-t_0)H_0.$$

If at $t=t_0$ the system is in the state 1, $\alpha(t_0)=1$ and $\beta(t_0)=0$. Then the solution is:

$$\Psi=\begin{pmatrix}\alpha\\ \beta\end{pmatrix}=e^{-iH_0(t-t_0)}\begin{pmatrix}1\\ 0\end{pmatrix}=\begin{pmatrix}\cos p(t-t_0)+i\dfrac{\Delta}{p}\sin p(t-t_0)\\ -i\dfrac{c}{p}\sin p(t-t_0)\end{pmatrix}.$$

Note that α and β oscillate with frequency p. The probability that a transition has occurred by the time $t_0+\tau$ is $|\beta|^2=(c^2/p^2)\sin^2 p\tau$. From det H_0 we find $p^2=c^2+\Delta^2$. Hence,

$$|\beta|^2=\frac{c^2}{c^2+\Delta^2}\sin^2(c^2+\Delta^2)^{\frac{1}{2}}\tau$$

$$=\frac{\sin^2[(2c)^2+(\omega-\omega_0)^2]^{\frac{1}{2}}(\tau/2)}{1+(\omega-\omega_0)^2/(2c)^2},$$

the usual formula for the Rabi line shape [reference 1, Eq. V. 10]. For appreciable transition probability $\Delta=$ order of c, so $\omega\gg p$; i.e., H_1 oscillates rapidly compared to the solution neglecting it. Its effect on α,β should be slight when averaged over a few cycles of ωt.

3. EFFECT OF RAPIDLY VARYING PERTURBATION

The inclusion of H_1 in the Schrödinger equation has two effects. One is the production of a Fourier series in ωt. The rapidly oscillating terms are not observed experimentally and they average to zero. The second effect is a change in the frequency of the unperturbed solution from p to $q\equiv p(1+\lambda)$. This effect is important, as it may result in a shift of the maximum of the resonance line shape. This form of solution is suggested by Floquet's theorem[8] for second-order differential equations with periodic coefficients such as are satisfied by α,β, where iq corresponds to the characteristic exponent.

To determine an approximate solution when H_1 is included, we use an iteration method similar to ordinary perturbation theory, but modified to permit the shift from p to q in the slowly varying part of the solution. We write the Schrödinger equation as

$$i(d/dt)\Psi=H_0(1+\lambda)\Psi+(H_1-\lambda H_0)\Psi$$

and take the second half to be of smaller order. Write $\Psi=\Psi_0+\Psi_1+\Psi_2+\ldots$ and $\lambda=\lambda_2+\lambda_3+\ldots$. Substitute and equate parts of the equation of the same order (order equals the sum of subscripts in any one term)

$$i(d/dt)\Psi_0=H_0(1+\lambda)\Psi_0,$$
$$i(d/dt)\Psi_1=H_0(1+\lambda)\Psi_1+H_1\Psi_0,$$

$$i\frac{d}{dt}\Psi_n=H_0(1+\lambda)\Psi_n+H_1\Psi_{n-1}-\sum_{r=0}^{n-2}\lambda_{n-r}H_0\Psi_r.$$

Since H_0 is independent of t, the formal solution for a first-order linear equation can be used for these matrix equations. Let

$$U(\tau)=e^{-iH_0(1+\lambda)\tau}=\cos q\tau 1-(i/p)\sin q\tau H_0.$$

Then at $t_0+\tau$:

$$\Psi_0=U(\tau)\Psi(t_0),$$

$$\Psi_1=-iU(\tau)\int_0^\tau U^{-1}(t)H_1\Psi_0(t+t_0)dt,$$

$$\Psi_n=-iU(\tau)\int_0^\tau U^{-1}(t)\times[H_1\Psi_{n-1}(t+t_0)-\sum_{r=0}^{n-2}\lambda_{n-r}H_0\Psi_r(t+t_0)]dt.$$

The λ_i are chosen such that no secular terms (terms proportional to t) appear upon integration. Thus q is initially undetermined, but becomes determined to higher orders as the iteration progresses. The integration of $H_1\Psi_0$ where H_1 is purely oscillatory leads to no secular terms, so $\lambda_1=0$. The concept of changing a dominant frequency to eliminate secular terms is essentially the same as that used in Lindstedt's method for finding periodic solutions of nonlinear differential equations.

In our case where H_1 oscillates with frequency $\omega\gg p$ and magnitude c, the integration of $H_1\Psi_0$ will result in a factor c/ω in Ψ_1, thus making it small. However, $H_1\Psi_1$ contains terms in which the oscillatory factors have cancelled and upon integration yield a factor c/q which is not small. Thus to obtain all terms in the solution of

[7] For a spin $\frac{1}{2}$ particle in a rotating magnetic field, H_0 here is identical with $\mathcal{H}'$ in H. Salwen, Phys. Rev. **99**, 1274 (1955).

[8] E. T. Whittaker and G. N. Watson, *A Course of Modern Analysis* (Cambridge University Press, New York, 1927), 4th ed., p. 412.

order c/ω part of Ψ_2 must be computed. This peculiarity of the iteration procedure arises from the fact that we have equated orders in a differential equation and the orders of terms change upon integration. However, this does not invalidate the iteration procedure for no more c/ω terms occur beyond Ψ_2. In general, Ψ_{2n} is of order $(c/\omega)^n$ and higher. The same phenomenon occurs in the iteration procedure used by Bloch and Siegert, disguised by the algebra.

As an example we compute the lowest-order effects of two simple H_1's. To first order the effects of several such perturbations combine linearly, so it is only necessary to consider one at a time. Consider

$$H_0=\begin{pmatrix}-\Delta & c\\ c & \Delta\end{pmatrix}$$

and

$$H_1=\begin{pmatrix}0 & be^{i\Omega t}\\ be^{-i\Omega t} & 0\end{pmatrix}.$$

This is a generalization of the H_1 of Sec. 2, which allows us to consider perturbations at other frequencies.

Let

$$\alpha_0(\tau)=\cos q\tau+i\Delta/p\,\sin q\tau,\ \beta_0(\tau)=-ic/p\,\sin q\tau.$$

Then

$$U(\tau)=e^{-iH_0(1+\lambda)\tau}=\begin{pmatrix}\alpha_0 & -\beta_0\\ \beta_0 & \bar{\alpha}_0\end{pmatrix},$$

$$U(0)=1,$$

where the bar denotes complex conjugate.

$$\Psi_0(\tau+t_0)=\begin{pmatrix}\alpha_0(\tau)\\ \beta_0(\tau)\end{pmatrix}\quad\text{for}\quad\Psi(t_0)=\begin{pmatrix}1\\ 0\end{pmatrix}.$$

The exact integration of the equation for Ψ_1 is tedious and yields terms of all orders. However, we can easily integrate by parts such that the remaining integral is of higher order and is neglected.

$$\Psi_1(\tau+t_0)=-iU(\tau)\left[U^{-1}(t)\int^{t+t_0}H_1(t')dt'\Psi_0(t+t_0)\right]_0^\tau+O\left(\frac{b^2}{\Omega^2}\right),$$

$$=-iU(\tau)\left[U^{-1}(t)\begin{pmatrix}0 & \frac{b}{i\Omega}e^{i\Omega(t+t_0)}\\ \frac{b}{-i\Omega}e^{-i\Omega(t+t_0)} & 0\end{pmatrix}\begin{pmatrix}\alpha_0(t)\\ \beta_0(t)\end{pmatrix}\right]_0^\tau,$$

$$=U(\tau)U^{-1}(\tau)\begin{pmatrix}\frac{-b}{\Omega}e^{i\Omega(\tau+t_0)}\beta_0(\tau)\\ \frac{b}{\Omega}e^{-i\Omega(\tau+t_0)}\alpha_0(\tau)\end{pmatrix}-U(\tau)\begin{pmatrix}0\\ \frac{b}{\Omega}e^{-i\Omega t_0}\end{pmatrix},$$

$$=\begin{pmatrix}-\frac{b}{\Omega}(e^{i\Omega(\tau+t_0)}+e^{-i\Omega t_0})\beta_0(\tau)\\ \frac{b}{\Omega}[e^{-i\Omega(\tau+t_0)}\alpha_0(\tau)-e^{-i\Omega t_0}\bar{\alpha}_0(\tau)]\end{pmatrix}.$$

We now seek the lowest-order parts of Ψ_2.

$$H_1\Psi_1(t+t_0)=\begin{pmatrix}\frac{b^2}{\Omega}[\alpha_0(t)-e^{i\Omega t}\bar{\alpha}_0(t)]\\ -\frac{b^2}{\Omega}(1+e^{-i\Omega(t+2t_0)})\beta_0(t)\end{pmatrix}.$$

We forget the exponential terms as they contribute only to higher orders upon integration

$$U^{-1}H_1\Psi_1=\frac{b^2}{\Omega}\begin{pmatrix}|\alpha_0|^2-|\beta_0|^2\\ \alpha_0\beta_0-\alpha_0\beta_0\end{pmatrix}$$

$$=\frac{b^2}{\Omega}\begin{pmatrix}\frac{\Delta^2}{p^2}+\frac{c^2}{p^2}\cos 2qt\\ -\frac{\Delta c}{p^2}+\frac{\Delta c}{p^2}\cos 2qt+\frac{ic}{p}\sin 2qt\end{pmatrix}.$$

The constant terms yield unwanted secular terms upon integration. Therefore, we choose λ_2 such that

$$\frac{b^2}{\Omega}\begin{pmatrix}\frac{\Delta^2}{p^2} & \\ -\frac{\Delta c}{p^2} & \end{pmatrix}-U^{-1}\lambda_2 H_0\Psi_0=0.$$

U commutes with H_0, hence

$$U^{-1}(t)H_0\Psi_0(t)=H_0\Psi(t_0)=\begin{pmatrix}-\Delta & \\ & c\end{pmatrix}.$$

Then

$$\lambda_2=-b^2\Delta/\Omega p^2.$$

After some more work

$$\Psi_2(\tau+t_0)=(b^2/\Omega p^2)\begin{pmatrix}c\\ \Delta\end{pmatrix}\beta_0(\tau).$$

To first order the complete solution is

$$\Psi=\begin{pmatrix}\alpha_0-\frac{b}{\Omega}(e^{i\Omega\tau}e^{i\Omega t_0}\beta_0-e^{-i\Omega t_0}\beta_0)+\frac{b^2c}{\Omega p^2}\beta_0\\ \beta_0+\frac{b}{\Omega}(e^{-i\Omega\tau}\alpha_0-\bar{\alpha}_0)e^{-i\Omega t_0}+\frac{b^2\Delta}{\Omega p^2}\beta_0\end{pmatrix},$$

and

$$q=p-(b^2\Delta/\Omega p).$$

In the physical situation of an atomic beam machine the observed transition probability includes molecules entering the radiation field region at all initial times t_0, so that an average must be performed over t_0 (or equivalently over the phase of radiation seen by entering atoms). This averaging eliminates the contributions of Ψ_1 to Ψ. The resultant Ψ can be written

$$\langle\Psi\rangle_{t_0\text{average}}=\begin{pmatrix}\alpha_0-\frac{c}{\Delta}\lambda\beta_0\\ \beta_0-\lambda\beta_0\end{pmatrix}+O(\lambda^2).$$

Actually the probabilities rather than amplitudes should be averaged, but the error is of second order.

The transition probability is now

$$|\beta|^2=|-(ic/p)(1-\lambda)\sin q\tau|^2=(c^2/q^2)\sin^2 qt$$

to first order. To this order the effect of H_1 has been to replace p by q. The central maximum of $|\beta_0|^2$ as a function of Δ occurs where $dq/d\Delta=0$. Remembering that p depends on Δ

$$dq/d\Delta=(\Delta/p)-(b^2/\Omega p)+(b^2\Delta^2/\Omega p^3).$$

Neglecting the last term the maximum is at

$$\Delta_{\text{res}}=b^2/\Omega.$$

The Bloch–Siegert shift is obtained by setting $b=c$, $\Omega=2\omega$:

$$\omega_{\text{res}}-\omega_0=c^2/\omega.$$

For the case of an additional perturbation at frequency ω_1 with amplitude b:

$$\Omega=\omega-\omega_1,\quad \omega_{\text{res}}-\omega_0=2b^2/(\omega_0-\omega_1),$$

in agreement with Ramsey [reference 1, Eq. V. 17]. Note that if ω_1 is a sideband caused by frequency modulation of ω, its effect is exactly canceled by the corresponding sideband on the other side of the carrier, provided the power spectrum is symmetric.

Next we look for a shift due to

$$H_1=\begin{pmatrix}d & 0\\ 0 & f\end{pmatrix}2\cos\omega t.$$

Such diagonal elements of the perturbation matrix are usually assumed to be zero. However, this is not the case for $\sigma(\Delta m=0)$ transitions induced by an oscillating field parallel to the constant magnetic field. For then the Hamiltonian is

$$\begin{pmatrix}E_1 & \frac{1}{2}\mu g H_z\\ \frac{1}{2}\mu g H_z & E_2\end{pmatrix},$$

where

$$H_z=H_c+H_v\cos\omega t.$$

But the transitions observed are not between the zero field levels E_1, E_2, but between the levels with H_c diagonal. Diagonalizing the time independent part of the Hamiltonian and then performing the phase factoring as in Sec. 2 gives

$$H=\begin{pmatrix}-\Delta & c\\ c & \Delta\end{pmatrix}+\begin{pmatrix}2b\cos\omega t & ce^{2i\omega t}\\ ce^{-2i\omega t} & -2b\cos\omega t\end{pmatrix}$$

where

$$\Delta=(\omega-\omega_0)/2,$$
$$\omega_0=[(E_1-E_2)^2+\mu^2g^2H_c^2]^{\frac{1}{2}},$$
$$c=(E_2-E_1)\mu_g H_v/4\omega_0,$$
$$b=\mu g H_c\mu g H_v/4\omega_0.$$

In actual practice $b\ll c$.

Returning to the d and f case we again approximate the integration finding

$$\Psi_1(t_0+\tau)=-\frac{2i}{\omega}\begin{pmatrix}d[\sin\omega\tau\cos\omega t_0+(\cos\omega\tau-1)\sin\omega t_0]\alpha_0(\tau)\\ f\left[\sin\omega\tau\cos\omega t_0+\left(\cos\omega\tau-\frac{d}{f}\right)\sin\omega t_0\right]\beta_0(\tau)\end{pmatrix}.$$

Averaged over t_0, Ψ_1 vanishes.

$$H_1\Psi_1=-\frac{4i}{\omega}\begin{pmatrix}d^2(\sin\omega t-\sin\omega t_0)\cos\omega t\alpha_0\\ f^2\left(\sin\omega t-\frac{d}{f}\sin\omega t_0\right)\cos\omega t\beta_0\end{pmatrix}.$$

Integration of this gives no secular terms and

$$\Psi_2 = Od^2/\omega^2.$$

So $\lambda_2 = 0$ and thus H_1 is seen to have no effect other than small high frequency oscillations.

Variations of this method are possible, such as transforming to a representation in which H_0 is diagonal. The latter makes the integrations easier, but the transformation must be inverted at the end of the computation.

4. EFFECT ON RAMSEY LINE SHAPE

The preceding derivations apply to the single oscillating field method of atomic beam spectroscopy. The first-order effect is equivalent to replacing Δ by $\Delta' = \Delta - \delta\omega/2$, where $\delta\omega$ is the Rabi resonance frequency minus the Bohr frequency. We can easily use this solution to determine the shift observed with the separated oscillating field method of Ramsey. Let primes denote that Δ has been replaced by Δ' wherever it appears. Within each oscillating field region of length $l = v\tau$, Ψ is transformed by

$$U'(\tau) = \begin{pmatrix} \alpha_0' & -\beta_0' \\ \beta_0' & \bar{\alpha}_0' \end{pmatrix}.$$

In the distance $L = vT$ between the two oscillating field regions, $b = c = \delta\omega = 0$ and U becomes simply

$$V(T) = \begin{pmatrix} e^{i\Delta T} & 0 \\ 0 & e^{-i\Delta T} \end{pmatrix}.$$

Then for an atom traversing the entire apparatus, we have

$$\Psi(\tau + T + \tau + t_0) = U'(\tau)V(T)U'(\tau)\Psi(t_0),$$

$$= \begin{pmatrix} \alpha_0'^2 e^{i\Delta T} + \beta_0'^2 e^{-i\Delta T} \\ \alpha_0'\beta_0' e^{i\Delta T} + \bar{\alpha}_0'\beta_0' e^{-i\Delta T} \end{pmatrix}$$

for

$$\Psi(t_0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

The transition probability is:

$$|\beta|^2 = (4c^2/q^2)\sin^2 q\tau[\cos q\tau \cos\Delta T - (\Delta'/p')\sin q\tau \sin\Delta T]^2.$$

Near the center of the Ramsey pattern $\Delta \ll c$ so $q = p' \approx c$

$$|\beta|^2 \approx \sin^2 2c\tau[\cos\Delta T - (\Delta'/c)\tan c\tau \sin\Delta T]^2.$$

$$d|\beta|^2/d\Delta = (\sin^2 2c\tau)2[\cos\Delta T - (\Delta'/c)\tan c\tau \sin\Delta T] \times [-T\sin\Delta T - (1/c)\tan c\tau \sin\Delta T - (\Delta' T/c)\tan c\tau \cos\Delta T].$$

For the central maximum we set the last factor equal to zero:

$$\frac{\tan\Delta T}{T}\left(1 + \frac{\tan c\tau}{cT}\right) = -\frac{\Delta'}{cT}\tan c\tau$$

$$= -\frac{\Delta}{cT}\tan c\tau + \frac{\delta\omega}{2cT}\tan c\tau.$$

For appreciable probability of transition $2c\tau \approx \pi/2$. Also $L \gg l$ or $T \gg \tau$, hence $cT \gg 1$. For small shifts ($\delta\omega \ll c$) $\Delta T \ll 1$ and the left side becomes just Δ. With these simplifying assumptions the shift in the peak of the Ramsey pattern is:

$$\omega_{res} - \omega_0 = \frac{\tan c\tau}{c\tau}\frac{l}{L}\delta\omega.$$

Unlike the shift of the Rabi peak, the shift of the Ramsey peak depends on the velocity of the atom. However, for optimum value of c and the most probable velocities, the velocity dependent factor is about 1.2 and not strongly dependent on velocity, so about this value can be expected if a velocity average were performed. This derivation also holds for any other cause of a shift in the Rabi peak, such as that caused by matrix elements to other far away states, as long as the cause is effective only in the oscillating field regions and does not make the line asymmetric.

5. BLOCH-SIEGERT SHIFT FOR THE CESIUM BEAM FREQUENCY STANDARD

As a numerical example the Bloch-Siegert shift is estimated for the two cesium beam frequency standards currently operated at the National Bureau of Standards. There are actually sixteen hyperfine structure lines in cesium. However, in the frequency standards the oscillating field is parallel to the C field, which provides the Zeeman splitting. Under these conditions only the σ transitions are excited. All matrix elements between states with different m values vanish.[9] For each pair of states with the same m value the two-state analysis of this paper is valid. The observed spectrum is just the superposition of the seven lines for the seven possible σ transitions. In practice these lines are well separated by the order of one hundred times the Ramsey linewidth. As long as they are symmetrical about the center line, which is used as the standard, these other lines should not affect the position of the standard frequency resonance.

The Bloch-Siegert shift in the Rabi line is from Sec. 3, $\delta\omega = c^2/\omega_0$. The fractional shift in the Ramsey peak is then:

$$\text{F.S.} = (\omega_{res} - \omega_0)/\omega_0 = (\tan c\tau/c\tau)(l/L)(c^2/\omega_0^2).$$

Assuming the optimum value of c we have from [reference 1, Eqs. V. 42, 42a]:

$$2cl/\alpha = 0.600\pi$$

and the linewidth

$$\Delta\nu = 0.65\alpha/L.$$

Combining

$$c = 0.300\pi\alpha/l = 1.45(L/l)\Delta\nu$$

[9] H. C. Torrey, Phys. Rev. **59**, 293 (1940), Sec. II.

and

$$c\tau = 0.300\pi\alpha/v = 0.725$$

$$\text{F.S.} = 1.22\frac{l}{L}\left(\frac{1.45L\Delta\nu}{l2\pi\nu_0}\right)^2 = 0.065\frac{L}{l}\left(\frac{\Delta\nu}{\nu_0}\right)^2.$$

In this form we can use the experimental linewidths and need not compute c. For both machines $l=1$ cm, $\nu_0 = 9.2\times10^9$ cps. For NBS I:

$$L = 55 \text{ cm},$$
$$\Delta\nu = 300 \text{ cps},$$
$$\text{F.S.} = 3.8\times10^{-15}.$$

For NBS II:

$$L = 164 \text{ cm},$$
$$\Delta\nu = 120 \text{ cps},$$
$$\text{F.S.} = 1.8\times10^{-15}.$$

These figures are to be compared with the present accuracy of measuring the cesium resonance frequency of 1.7×10^{-11}.[10]

APPENDIX

Another, less general method for obtaining the solution to the Bloch–Siegert problem is one which requires deducing the form of the solution, and then solving for some undetermined coefficients. The equations are (cf. Sec. 3)

$$i\dot{\alpha} = -\Delta\alpha + c\beta + be^{i\Omega t}\beta,$$
$$i\dot{\beta} = \Delta\beta + c\alpha + be^{-i\Omega t}\alpha,$$

and the initial conditions $\alpha(t_0)=1$, $\beta(t_0)=0$. Letting the subscript "0" represent the functions in Sec. 3 with argument $t-t_0$, they satisfy:

$$i\dot{\alpha}_0 = (-\Delta\alpha_0 + c\beta_0)(1+\lambda),$$
$$i\dot{\beta}_0 = (c\alpha_0 + \Delta\beta_0)(1+\lambda).$$

We wish to add to α_0, β_0 terms with small coefficients which make the equations satisfied except for small terms. We note that when a function such as $Ae^{i\Omega t}\alpha(t)$ is differentiated, the result contains a term with a factor Ω, that is, a term with a larger order coefficient. Thus, to generate $be^{i\Omega t}\beta$ when α is differentiated, let us add to α_0 a term $Ae^{i\Omega t}\beta_0$. The derivative of this times i gives $A[-\Omega\beta_0 + (c\alpha_0+\Delta\beta_0)(1+\lambda)]e^{i\Omega t}$. Neglecting c and Δ compared to Ω we see that if $A=-b/\Omega$, we obtain $be^{i\Omega t}\beta_0$+smaller terms. Since $\beta=\beta_0$+smaller terms, we have thus satisfied the equation for α to lowest order.

However, we have several small terms left over. Those containing factors $e^{\pm i\Omega t}$ can be canceled by introducing terms of higher order. But the terms without this factor must be canceled by introducing similar terms with small coefficients into the approximate expressions for α and β. Thus we could add $B\beta_0$ to α and perhaps adjust B to cancel the other terms. We cannot add $C\alpha_0$, because this would violate the initial conditions.

After mulling over such considerations we are led to a trial solution in the form:

$$\alpha = \alpha_0 + A\beta_0 e^{i\Omega t} + B\beta_0,$$
$$\beta = \beta_0 + C(\alpha_0 e^{-i\Omega t} - \alpha_0 e^{-i\Omega t_0}) + D\beta_0,$$

where A, B, C, D are undetermined coefficients of order b/Ω and the initial conditions are satisfied. We substitute these expressions into the differential equations and equate the coefficients of terms of each order and functional type. In the α equation equating zero-order coefficients of $\beta_0 e^{i\Omega t}$ gives $-\Omega A = b$ or $A=-b/\Omega$ as noted previously. In the β equation from terms $\alpha_0 e^{-i\Omega t}$ we find similarly $\Omega C = b$ or $C=b/\Omega$. Now equating the first-order coefficients of α_0 and β_0 in the two equations we obtain

$$-\Delta\lambda + Bc = bC - cCe^{-i\Omega t_0},$$
$$c\lambda + B\Delta = -\Delta B + cD,$$
$$c\lambda + \Delta Ce^{-i\Omega t_0} + cD = -\Delta Ce^{-i\Omega t_0},$$
$$\Delta\lambda - cCe^{-i\Omega t_0} + \Delta D = Ab + Bc + \Delta D.$$

All told we have six equations in five unknown. However, these equations are consistent, with solutions:

$$\lambda = -b^2\Delta/\Omega p^2,$$
$$B = (c/\Omega)(b^2/p^2) - (b/\Omega)e^{-i\Omega t_0},$$

and

$$D = (\Delta b^2/\Omega p^2) - (2\Delta b/c\Omega)e^{-i\Omega t_0}.$$

These values give the same solution as in Sec. 3. Note that the first-order coefficients of $\alpha_0 e^{\pm i\Omega t}$, etc., do not cancel. This method, like the others, can be carried to higher order only with greatly increased labor as more undetermined coefficients must be inserted and more cross terms appear in the expansion.

[10] R. C. Mockler, R. E. Beehler, and C. S. Snider, IRE Trans. Instrumentation **I-9**, 120 (1960).