UNITED STATES DEPARTMENT OF COMMERCE • Sinclair Weeks, *Secretary*
NATIONAL BUREAU OF STANDARDS • A. V. Astin, *Director*

Phase of the Low Radiofrequency Ground Wave

J. R. Johler, W. J. Kellar, and L. C. Walters

National Bureau of Standards Circular 573

Issued June 26, 1956

For sale by the Superintendent of Documents, U. S. Government Printing Office, Washington 25, D. C.
Price 35 cents

Contents

Phase of the Low Radiofrequency Ground Wave[1]

J. R. Johler, W. J. Kellar, and L. C. Walters

The special theoretical considerations pertinent to the computation of the phase of the ground wave at low radiofrequencies are discussed. The formulas necessary for the numerical evaluation of the amplitude and phase, and the results of the numerical computation are presented. The effects of frequency, conductivity of the earth, altitude above the surface of the earth, and vertical lapse of the permittivity of the atmosphere are evaluated.

1. Introduction

The phase of the low-frequency ground wave is implicit in its theoretical development. Most computations in the past have considered only the amplitude. However, the importance of a precise computation of the phase has increased with the use of various systems of low-frequency radio navigation [21, 22, 23, 26, 27].[2] The phase of the propagated ground wave is also an important consideration in the solution of the more general problem of the propagation of a transient in a radio-communication system. Computations have been performed by the authors during the past 2 years for the purpose of evaluating the effects of propagation on various radio-communication systems and are presented here in a unified form.

A mathematical model for the computation of both the amplitude and the phase of the ground wave near the earth has been presented by Sommerfeld [11], Wise [9, 10], Van der Pol and Niessen [15], Watson [8], Van der Pol and Bremmer [6], Bremmer [7] and Norton [1, 2, 3, 4, 5]. This Circular concerns the computation of the phase of the low-frequency ground wave from this mathematical model. The amplitudes of the ground waves incident to this computation are also presented. The phase variation with frequency is considered. The disturbing influence of the source is considered for the particular source model used in this computation. The effect of the finite conductivity of the earth and the vertical lapse of the permittivity of the atmosphere is also demonstrated by this computation.

2. Theory of the Phase Computation

The prime consideration in this study of the propagation of the ground wave was the nature of the source. The source selected for this computation was the elemental vertical electric dipole of the type originally proposed by Hertz [20] and which is used for the ground wave in various modified forms by Norton [1] and Bremmer [7].

The field developed by the source was described by a primitive quantity, $\overline{\Pi}$, (volts$\times$meters, or joules$\times$meters/coulombs) known as the Hertz vector [20]. The various electric and magnetic components of the field can be computed from this quantity by a differentiation process. As the field can be described by a single quantity, $\overline{\Pi}$, the Maxwell theory for the problem at hand can be represented at a primitive source, $\overline{\Omega}$, as follows:[3]

$$(\nabla^2+k^2)\overline{\Pi}=\frac{-\overline{\Omega}}{\kappa} \qquad (1)$$

---

[1] The work represented by this paper was sponsored by Wright Air Development Center, Wright-Patterson Air Force Base, Dayton, Ohio, in connection with Air Force Contract 33(616)-54-7.

[2] Figures in brackets indicate the literature references on page 10.

[3] The rationalized *mks* system of units is used throughout this Circular.

The current of the source, $\bar{\iota}$, may be derived from the quantity $\bar{\Omega}$ as follows:

$$\bar{\iota}=\frac{\partial\bar{\Omega}}{\partial t} \quad \text{(amperes per square meter)} \tag{2}$$

The Hertz vector for the solution of this equation is as follows [19, 24]:

$$\bar{\Pi}=\frac{1}{4\pi\kappa}\int_v \frac{[\bar{\Omega}]}{d}\,dv, \tag{3a}$$

where

$$\bar{\Omega}=\bar{\Omega}(t) \tag{3b}$$

and

$$[\bar{\Omega}]=\bar{\Omega}\left(t-\frac{\eta d}{c}\right) \tag{3c}$$

The evaluation of the source at a time, $t-(\eta d)/c$, instead of a time, t, accounts for the finite propagation time through space. As the computations in this paper were built upon two separately derived theories (plane- and spherical-earth theories), the source current, $\bar{\iota}$, must be specified closely if consistency is to be maintained at intermediate distances from the source. Hence, the source, $\bar{\iota}$, was assumed (a) to have a uniform distribution along an elemental linear distance,[4] l, of the z axis of the coordinate system [5] (at the origin in cylindrical coordinates; at the surface of the earth, $r=a$, $\theta=0$ in spherical coordinates), and (b) to vary sinusoidally with time at an angular frequency ω. It should be noted that the length, l, would represent the effective length of a practical low-frequency radio-transmitting antenna. The effective length would equal half the physical length if the current on the antenna had a linear distribution along its length.

The distance, d, is the distance from the observer to the volume element of integration in eq (3). The distance, d, is further specified as the distance measured along the surface of the earth.[6]

The source polarization, $\bar{\Omega}$, in the spherical-earth theory was then specified [7] as follows:

$$[\bar{\Omega}]=\bar{r}\,\frac{I_0}{\omega}\exp\{i[kd-\omega t]\} \qquad (d=a\theta) \tag{4a}$$

and in the plane-earth theory:

$$[\bar{\Omega}]=\bar{z}\,\frac{I_0}{\omega}\exp\{i[kd-\omega t]\} \qquad (d=R). \tag{4b}$$

The source was therefore always considered to be polarized perpendicular to the surface of the earth. In either case, however,

$$\bar{\Omega}=\bar{z}\,\frac{I_0}{\omega}\exp\{-i\omega t\} \qquad (d=0). \tag{4c}$$

The nature of the oscillator supplying the source with electric power may be described as follows:

$$\text{Re}\,\bar{\iota}=\bar{z}I_0\sin(\omega t+\pi). \tag{5}$$

It is quite evident that due to the assumptions in eq (4), the phase of the oscillator at time $t=0$ was advanced π radians. This specification simplified graphical presentation of the results of the computation and also avoided negative phase values for the "secondary factor" to be defined later. It is also evident that the amplitude of the oscillator does not vary other than sinusoidally with frequency ω.

[4] The volume element of integration, eq (3), now becomes a linear element of integration.
[5] See figure 1.
[6] See figure 1.
[7] Vectors are defined in figure 1.

The complete field, $\overline{\Pi}$, at a distance, d, from the source can be described in free space for the elemental dipole of Hertz as follows [18, 20]:

$$\overline{\Pi}=\bar{z}\,\frac{I_0 l}{4\pi\kappa\omega d}\exp\{i[kd-\omega t]\} \qquad (\text{volts}\times\text{meters}). \tag{6}$$

The amplitude of the field is inversely proportional to the frequency, ω, and the distance, d, and therefore becomes infinite when either ω or d is zero. With the aid of this representation of the field, the electric and magnetic components of the field can be computed as follows:

$$\overline{E}=\text{curl curl }\overline{\Pi} \quad (\text{volts per meter}) \tag{7}$$

$$\overline{H}=\frac{k^2}{\mu_0 i\omega}\text{ curl }\overline{\Pi} \quad (\text{amperes per meter}). \tag{8}$$

The parameter k is defined for a medium of permittivity, κ, farads per meter; permeability, μ_0, henrys per meter; and conductivity, σ, mhos per meter as follows [18]: [8]

$$k^2=-\mu_0\kappa\frac{\partial^2}{\partial t^2}-\mu_0\sigma\frac{\partial}{\partial t} \quad (\text{radians per meter})^2. \tag{9}$$

The scalar electric and magnetic components of the field can now be computed. The vertical electric field intensity, E_r or E_z is of particular interest.[9]

For the *Spherical-Earth Theory:*

$$E_r=\frac{-1}{r\sin\theta}\frac{\partial}{\partial\theta}\left[\sin\theta\frac{\partial\Pi}{\partial\theta}\right] \quad (\text{volts per meter}) \tag{10}$$

For the *Plane-Earth Theory:*

$$E_z=k^2\Pi+\frac{\partial^2\Pi}{\partial z^2} \quad (\text{volts per meter}) \tag{11}$$

One of the most important and difficult theoretical problems with which early workers in the field of radio-wave propagation were confronted was the evaluation of the Hertz vector $\overline{\Pi}$ near the surface of the earth. The results of the efforts of the many workers in the field are presented in form suitable for computation by Van der Pol and Bremmer [6], Bremmer [7], and Norton [1]. If the Hertz vector, $\overline{\Pi}$, is evaluated for the effect of the proximity of the earth, the ground wave for purposes of this study is now completely specified. However, it has been found convenient to express the total field as the product of two factors: (a) the primary field or free-space field, E_{pr} (volts per meter); and (b) the secondary factor, F,[10] (dimensionless). The primary field is defined as follows:

$$E_{pr}=\frac{I_0 l k_1^2}{4\pi\kappa\omega d}\exp\{i[k_1 d-\omega t]\} \quad (\text{volts per meter}), \tag{12}$$

where

$$k_1=\frac{\omega}{c}\sqrt{\epsilon_1}=\frac{\omega}{c}\,\eta_1 \qquad (\sigma=0,\text{ for air}) \quad (\text{radians per meter}).$$

The wave number, k_1, refers to the air medium at the surface of the earth.

The secondary factor, F, is defined with E, the electric intensity, as follows:

$$F=\frac{E}{2E_{pr}}.$$

[8] More detailed formulas are given in appendix I.

[9] Other components are given in appendix I.

[10] F is equivalent to the ratio of Hertz vectors $\Pi/2\Pi_{pr}$ in the spherical-earth theory described by Bremmer [7, p. 48].

According to this definition, the secondary factor, F, accounts for the disturbing influence of the source or the earth. At very great distance from the source, in free space, $\sigma=0$, $F=\frac{1}{2}$; or over perfectly conducting earth, $\sigma=\infty$, $F=1$. It is of interest to note, however, that a secondary factor other than unity defined by eq (13) exists over perfectly conducting earth close to the source, or other than $\frac{1}{2}$ in free space close to the source due to the disturbing influence of the source. This consideration at low frequencies is of considerable importance, because "close to the source" can be as far away as several hundred miles. Wait [16] has shown the effect of the horizontal stratification of the ground on the secondary factor, F. This paper is concerned with the homogeneous case only.

At great distances along the surface of the earth, the phase, ϕ_c, of the secondary factor, F, was computed from the spherical-earth theory. Close to the source of the radiation, the computation by this method became unwieldy. The phase was then computed from the plane-earth theory.

The computation of the phase, ϕ', of the primary field as defined by eq (12) was quite simple, and, neglecting the time function, ωt, is

$$\phi'=k_1 d=\frac{\omega}{c}\eta_1 d. \tag{14}$$

In view of the above argument, the total phase, ϕ, is as follows:

$$\phi=\phi'+\phi_c \tag{15}$$

$$\phi=k_1 d+\phi_c, \tag{16}$$

where

$$F=|F|\exp\{i\phi_c\} \tag{17}$$

or ϕ_c is the phase of the secondary factor, F. At great distance from the source, ϕ' is a large number (thousands of radians). On the other hand, ϕ_c is a relatively small number (between 0.1 and 10 radians), and can therefore be regarded as a phase correction that is to be added to the free space or primary field to account for the disturbing influence of the source or the earth.

The basic assumptions and specifications described were so defined as to avoid a negative phase, ϕ_c, for the secondary factor, F, thus simplifying the graphical constructions of the functions. In this regard, the free-space wave field intensity close to the source of the radiation is of interest and may be described as follows:[11]

$$E'_z=\frac{k^2 I_0 l}{4\pi\kappa\omega d}\left\{\left[1-\frac{z^2}{d^2}-\frac{1}{k^2d^2}+\frac{3z^2}{k^2d^4}\right]+i\left[\frac{1}{kd}-\frac{3z^2}{kd^3}\right]\right\}\exp\{i[kd-\omega t]\}\quad\text{(volts per meter).} \tag{18}$$

The secondary factor, F, is as follows:

$$F=F'_z=\left[1-\frac{z^2}{d^2}-\frac{1}{k^2d^2}+\frac{3z^2}{k^2d^4}\right]+i\left[\frac{1}{kd}-\frac{3z^2}{kd^3}\right]. \tag{19}$$

The phase, ϕ_c, of the secondary factor, F, is then as follows:

$$\phi_c=\tan^{-1}\frac{\frac{1}{d^2}-\frac{3z^2}{d^4}}{\frac{k}{d}-\frac{kz^2}{d^3}-\frac{1}{kd^3}+\frac{3z^2}{kd^5}} \tag{20}$$

$$\lim_{d\to 0}\phi_c=\pi\quad\text{(radians).} \tag{21}$$

[11] This may be derived from eq (6) and (11).

For the special case, $z=0$,[12]

$$E'_z=\frac{k^2I_0l}{4\pi\kappa\omega d}\left\{1+\frac{1}{(ikd)^2}-\frac{1}{ikd}\right\}\exp\{i[kd-\omega t]\}\quad\text{(volts per meter)}\tag{22}$$

$$F=F'_z=1+\frac{1}{(ikd)^2}-\frac{1}{ikd}\tag{23}$$

$$\phi_c=\tan^{-1}\frac{\frac{1}{kd}}{1-\frac{1}{k^2d^2}}=\tan^{-1}\frac{1}{kd-\frac{1}{kd}}\tag{24}$$

$$\lim_{d\to 0}\phi_c=\pi\quad\text{(radians)}.\tag{25}$$

The value of the phase, ϕ_c, of the secondary factor, F, is illustrated graphically in figure 2. The phase, ϕ_c, for the free space field secondary factor, F, as shown was computed from eq (24). The value of the phase, ϕ_c, due to the disturbing influence of the earth is also shown at a frequency of 100 kc for conductivities of 5, 0.005, and 0.0001 mho per meter and for plane and spherical earth theory computations of the secondary factor, F.

The details of the computation of the secondary factor, F, or the Hertz vector, $\overline{\Pi}$, near the earth have been elegantly worked out by the aforementioned authors. The vertical electric field intensity at great distance close to the earth in the spherical-earth theory is, after the various approximations [6, 7], as follows:[13]

$$E_r=2E_{pr}F_r\quad\text{(volts per meter)}.\tag{26}$$

The secondary factor, F, is as follows:

$$F=F_r=\left[2\pi\alpha^{2/3}(k_1a)^{1/3}\left(\frac{d}{a}\right)\right]^{1/2}\sum_{s=0}^{\infty}\frac{f_s(h_1)f_s(h_2)}{[2\tau_s-1/\delta_e^2]}\exp\left\{i\left[(k_1a)^{1/3}\tau_s\alpha^{2/3}\frac{d}{a}+\frac{\alpha d}{2a}+\frac{\pi}{4}\right]\right\}\tag{27}$$

The phase, ϕ_c, can be computed from the following quantity:

$$\phi_c=\arg F_r\quad\text{(radians)}.\tag{28}$$

Close to the source (wavelengths), the effect of the source on the secondary factor, F, must be considered. In this case, at the surface of the earth, i. e., $f_s(h_2)=f_s(h_1)=1$:

$$F=F_r+F'_z-1\qquad(z=0).\tag{29}$$

Because the computation by means of the residue series described in eq (28) becomes unwieldy at short distances, it can be assumed that the earth is a plane. In the plane-earth theory, the vertical electric field intensity at the surface of the earth, E_z, was found to be given by the following approximate formula [2, 3]:

$$E_z=2E_{pr}F_z\quad\text{(volts per meter)}.\tag{30}$$

The secondary factor, F, is as follows:

$$F=F_z=y(\rho_1)f(\sigma,\epsilon)-\frac{1}{ik_1d}+\frac{1}{(ik_1d)^2},\tag{31}$$

where

$$y(\rho_1)=1+i\sqrt{\pi\rho_1}e^{-\rho_1}erfc[-i\sqrt{\rho_1}]\tag{32}$$

$$f(\sigma,\epsilon)=1-\frac{k_1^2}{k_2^2}+\frac{k_1^4}{k_2^4}.\tag{33}$$ [14]

[12] See figure 1.

[13] Symbols are defined in glossary, page 37.

[14] See table 42 and eq (94) in appendix I.

and ρ_1 is the numerical distance of Sommerfeld [11].[15]

It has been found convenient to express the phase, ϕ, as a time, t, as follows:

$$t=\frac{\phi}{\omega}\cdot 10^6 \quad \text{(microseconds)}. \tag{34}$$

For the phase, ϕ_c, of the secondary factor, F, a time t_c can be ascribed as follows:

$$t_c=\frac{\phi_c}{\omega}\cdot 10^6 \quad \text{(microseconds)}. \tag{35}$$

The vertical lapse of the permittivity of the earth's atmosphere was reckoned from a mathematical model of the atmosphere which assumed a linear decrease with altitude from some surface value η_1, eq (12), and some value, η_2, h kilometers aloft. This is expressed as ΔN units per kilometer as follows:[16]

$$\Delta N=\frac{\eta_1-\eta_2}{h}\cdot 10^6 \tag{36}$$

$$N=(\eta_n-1)10^6. \tag{37}$$

The parameter, α, is introduced by the following:

$$\alpha=\frac{a}{a_e}=\frac{1}{k'}, \tag{38}$$

where a is the radius of the earth, a_e is the "effective radius" of the earth, and k' is the "effective radius factor" of the earth. ΔN is related to the parameter, α, in eq (26), (27), and (28) as follows:

$$\alpha=1+\Delta N\cdot 10^3. \tag{39}$$

A physical interpretation of the α parameter results in an adjustment of the amplitude and phase of the ground wave which would be expected from the assumed mathematical model of the atmosphere, i. e., the assumed vertical lapse of the permittivity of the atmosphere. The derivation of this parameter by Bremmer [7] is based on a model of the earth's atmosphere which has a linear decrease with altitude. The derivation is completely independent of *ad hoc* geometrical-optical considerations with which it is often associated. It is independent of the thickness of the earth's atmosphere (wavelengths) as long as a wavelength is small compared with the radius of the earth. Satisfactory experimental vertification of this model has not as yet been achieved. Other mathematical models of the atmosphere could put further restrictions on this parameter, especially at low and very low frequencies.

3. Results of the Computation

The results of the computation are tabulated in tables 1 to 44, and are presented graphically in figures 2 to 17.

The phase, ϕ_c, of the secondary factor, F, in free space, ($\sigma=0$), eq (24), is shown in figure 2. This is also the phase of the secondary factor over an infinitely conducting earth ($\sigma=\infty$). The disturbing influence on the phase, ϕ_c, of the finite conductivity of the earth is also shown at a frequency of 100 kc, for three values of conductivity: $\sigma=5$, 0.005, and 0.0001 mho per meter for both the plane- and the spherical-earth theory.

[15] See figure 16.

[16] See figure 10.

The figures included in this Circular illustrate the effect of frequency, distance along the surface of the earth from the source, conductivity of the earth, and the vertical lapse of the permittivity of the atmosphere on both the phase of the secondary factor, ϕ_c, and the amplitude, $|E|$, of the ground wave. It is of computational interest to note that the value of the phase. ϕ_c, and the value of the amplitude, $|E|$, computed from the plane-earth theory and the spherical-earth theory at moderately decreasing distance, approach each other such that at short distances the two computations become identical. This is illustrated for the phase, ϕ_c, in figure 3. If required, the total phase may be computed quite simply from eq (16). The amplitude of the ground wave, $|E|$, volts per meter, assumes that the source dipole momentum, I_0l, eq (22), (26), and (30), has a value of unity (1 ampere-meter). The effect of the antipode, or the effect of the wave progressing around the earth in the opposite direction ($-\theta$ direction in fig. 1), was found to be negligible as far from the source as 10,000 miles. Also, note that the value of the attenuation is very large for the ground wave (figs. 11, 12, and 13) at such distances from the source. The variation of the phase, ϕ_c, of the secondary factor, F, with the parameter α is shown in figure 4. The corresponding amplitudes, $|E|$, are shown in figure 13. The curves, $\alpha=1$, correspond to a value that would be expected from a homogeneous atmosphere. However, the average atmosphere is not homogeneous, due in large measure to the gravitational field of the earth. The value, $\alpha=0.75$, is a value frequently assumed and corresponds to an average vertical permittivity lapse such that the "effective radius factor," k', eq (38), has a value of 4/3. On the other hand, the value $\alpha=0$ corresponds to the plane-earth theory value. Such a vertical lapse or greater is often considered at very high frequencies to account for the effect of ducting. Further graphical study of this parameter is shown in figures 6 and 10.

Considerable interest has developed in the effect of conductivity and conductivity changes on both the amplitude and phase. The computations of this report assume homogeneous conductivity. The effects of conductivity on the amplitude and phase are shown in figures 3, 7, and 12. Of particular interest is the phase, ϕ_c, of the secondary factor, F, shown in figure 7 for various distances for both the plane- and spherical-earth theory. Between the asymptotes $\sigma=0$ and $\sigma=\infty$ a maximum phase retardation, ϕ_c, is observed. This is in the region of maximum energy absorption by the earth and includes the values of conductivity most frequently observed on the earth. The asymptote, $\sigma=\infty$, is quite rigorous for a highly conductive sphere the size of the earth. However, the value $\sigma=0$ does not represent a dielectric sphere as the terms of the rainbow series S_0, S_1, S_2 . . . , S_k (for $k>-1$) described by Bremmer [6,7] are neglected. This approximation was considered quite reasonable for finite or infinite values of conductivity, σ, as the rainbow terms other than S_{-1} are very highly absorbed by the earth.

The amplitude of the ground wave $|E|$ is slightly enhanced as the value of the conductivity, σ, is increased through values of $\sigma=0.05$, $\sigma=5$ (sea water), and $\sigma=\infty$. This is due to the behavior of the attenuation factors, $\exp[-Im\ \tau_s]$ [7]. The value of $Im\ \tau_0$ for the first term of the residue series (eq (26), $s=0$), which in large measure determines the attenuation is shown graphically in figure 17. A similar curve for the phase parameter, Re τ_0, is shown in figure 19.

4. Physical Significance of the Phase

At a time, t, and a distance, d, from the source, the signal, $f(d,t)$, upon which a measurement is made may be described as follows [28, 29]:

$$f(d,t)=f_0(d,t)+f_1(d,t)+f_2(d,t). \tag{40}$$

In this paper, the continuous wave signal, $f_0(d,t)$ has been evaluated. This continuous wave signal will, in general, be altered by the transient terms, $f_1(d,t)$ and $f_2(d,t)$. The measured phase will not in general correspond to the values computed from $f_0(d,t)$. The transient terms are usually evaluated for specific systems of measurements under study and are considered beyond the scope of this paper. The value of the phase of the continuous signal, $f_0(d,t)$, may, however, under carefully considered conditions have physical significance.

The physical measurement of the phase at some distance from the source of the signal may be interpreted as the time required for the signal to propagate to the observer. The value of the phase, t_c, in this paper, is such a time relative to the free-space propagation time. This measurement can also be described as a velocity measurement if the distance from the source is considered. It has been shown by Sommerfeld [28] and Brillouin [29] that the signal velocity, v_s, is always less than the universal constant, c. At great distance from a source in free space, the signal velocity, v_s, corresponds to the phase velocity, v_c, or the group velocity, v_g. More often, however, the signal velocity and signal time-delay correspond to the group. However, under certain conditions neither the group velocity nor the phase velocity have physical significance. The phase velocity and the group velocity have been implied in this computation and may be explicitly developed.

If the time-dependent factor is included, the phase, ϕ, of the ground wave is as follows:

$$\phi=\phi'+\phi_c=k_1d+\phi_c-\omega t. \tag{41}$$

The group delay can be estimated by Kelvin's principle of stationary phase [25] as follows:

$$\frac{d\phi}{d\omega}=\frac{d}{d\omega}\left\{\frac{\omega}{c}\eta_1 d+\omega t_c-\omega t\right\}=0 \tag{42}$$

$$t=\frac{\eta_1 d}{c}+t_c+\omega\frac{dt_c}{d\omega}. \tag{43}$$

The group delay, t_g, (relative to the primary field) is as follows:

$$t_g=t_c+\omega\frac{dt_c}{d\omega}. \tag{44}$$

The velocity of propagation, v, or the phase velocity, v_c, is as follows:

$$v=v_c=\frac{-\dfrac{\partial\phi}{\partial t}}{\dfrac{\partial\phi}{\partial d}}. \tag{45}$$

The velocity of the group, v_g, is

$$v_g=-\frac{\dfrac{\partial}{\partial t}[\phi'+\phi_g]}{\dfrac{\partial}{\partial d}[\phi'+\phi_g]} \tag{46}$$

$$\phi_g=\omega t_g \tag{47}$$

$$\phi'+\phi_g=k_1d+\phi_c+\omega^2\frac{dt_c}{d\omega}-\omega t \tag{48}$$

$$v_c=\frac{c}{\eta_1+c\dfrac{\partial t_c}{\partial d}} \tag{49}$$

$$v_g=\frac{c}{\eta_1+c\dfrac{\partial}{\partial d}\left[t_c+\omega\dfrac{dt_c}{d\omega}\right]}=\frac{c}{\eta_1+c\dfrac{\partial t_g}{\partial d}}. \tag{50}$$

The index of refraction, η, of the propagation medium is

$$\eta=\frac{c}{v_c}. \tag{51}$$

Close to the source or in free space the velocity of the phase, v_c, the group delay, t_g, and the velocity of the group, v_g, may be computed from eq (24) as follows:

$$\frac{\partial t_c}{\partial d}=-\frac{k_1}{\omega}\left[\frac{k_1^2d^2+1}{k_1^4d^4-k_1^2d^2+1}\right] \tag{52}$$

$$v_c=\frac{c}{\eta_1\left[1-\frac{k_1^2d^2+1}{k_1^4d^4-k_1^2d^2+1}\right]} \tag{53}$$

$$\lim_{d\to\infty} v_c=\frac{c}{\eta_1}=c \qquad (\eta_1=1) \tag{54}$$

$$\lim_{d\to 0} v_c=-\infty \tag{55}$$

$$\omega\frac{dt_c}{d\omega}=\frac{-k_1d(k_1^2d^2+1)}{\omega(k_1^4d^4-k_1^2d^2+1)} \tag{56}$$

$$t_g=t_c-\frac{k_1d(k_1^2d^2+1)}{\omega(k_1^4d^4-k_1^2d^2+1)} \tag{57}$$

$$\lim_{d\to 0} t_g=t_c=\frac{\pi}{\omega} \tag{58}$$

$$\frac{\partial t_g}{\partial d}=-\frac{k_1}{\omega}\,\frac{3k_1^2d^2+1}{k_1^4d^4-k_1^2d^2+1}-\frac{2k_1^2d^2(k_1^2d^2+1)(2k_1^2d^2-1)}{(k_1^4d^4-k_1^2d^2+1)^2} \tag{59}$$

$$v_g=\frac{c}{\eta_1\left\{1-\left[\frac{3k_1^2d^2+1}{k_1^4d^4-k_1^2d^2+1}-\frac{2k_1^2d^2(k_1^2d^2+1)(2k_1^2d^2-1)}{(k_1^4d^4-k_1^2d^2+1)^2}\right]\right\}} \tag{60}$$

$$\lim_{d\to 0} v_g=-\infty \tag{61}$$

$$\lim_{d\to\infty} v_g=\frac{c}{\eta_1}=c \qquad (\eta_1=1). \tag{62}$$

At intermediate distances from the source, the group delay, group velocity, and phase velocity may be evaluated by graphical construction of the factors:

$$\omega\frac{dt_c}{d\omega},\; c\,\frac{\partial t_c}{\partial d},\; \text{and } c\,\frac{\partial t_g}{\partial d}.$$

At great distance from the source, close to the earth, the group velocity and group delay may be estimated from the first term of the residue series as follows:

$$\phi=k_1a+(k_1a)^{1/3}\,\mathrm{Re}\,\tau_0\alpha^{2/3}\,\frac{d}{a}+\frac{\alpha d}{2a}+\frac{\pi}{4}-\arg f(\tau_0,\delta_e)-\omega t \tag{63}$$

$$f(\tau_0,\delta_e)=2\tau_0-\frac{1}{\delta_e^2} \tag{64}$$

$$t_g=(k_1a)^{1/3}\alpha^{2/3}\theta\left[\frac{\mathrm{Re}\,\tau_0}{3\omega}+\frac{d}{d\omega}\,[\mathrm{Re}\,\tau_0]\right]+\frac{d}{d\omega}\left[-\arg f(\tau_0,\delta_e)\right] \tag{65}$$

$$t_g=5.0986(10^{-9})d+1.2796\times 10^{-6} \tag{66}$$

$$(\sigma=0.005,\ \epsilon_2=15,\ \alpha=0.75,\ f=100\text{ kc},\ d>2{,}000),$$

where d is expressed in miles.

Figures 18, 19, 20, and 21 illustrate the development of group delay, phase delay, group velocity, phase velocity, and index of refraction of the propagation medium. Physical significance cannot be attributed to the immediate region of the source because velocities, v, in excess of the constant c, or indices, η_1, less than 1, would be contrary to the principle of relativity.

5. Bibliography

[1] K. A. Norton, The calculation of the ground wave field intensity over a finitely conducting spherical earth, Proc. IRE **29,** No. 12, p. 623–639 (December 1941).

[2] K. A. Norton, The physical reality of space and surface waves in the radiation field of radio antennas, Pt. III, Proc. IRE **25,** No. 9, 1192–1202 (September 1937).

[3] K. A. Norton, The propagation of radio waves over the surface of the earth and in the upper atmosphere, Proc. IRE **25,** No. 9, 1203–1236 (September 1937).

[4] K. A. Norton, The propagation of radio waves over the surface of the earth and in the upper atmosphere, Pt I, Proc. IRE **24,** No. 10, 1337–1387 (October 1936).

[5] K. A. Norton, Propagation in the FM and broadcast band, Advances in Electronics, vol. I (Academic Press Inc., New York 10, N. Y., 1948).

[6] Balth van der Pol and H. Bremmer, The diffraction of electromagnetic waves from an electrical point source round a finitely conducting sphere, with applications to radiotelegraphy and the theory of the rainbow, Natuurkundig Laboratorium der H. V. Philips Gloeilampen Fabriken, Eindhoven, Holland; Phil. Mag. [7] **24,** Pt. I, 141 (July 1937); pt. II, 825 (Nov. 1937); Supp. J. Science **25,** 817 (June 1938).

[7] H. Bremmer, Terrestrial radio waves; theory of propagation (Elsevier Publ. Co., New York, N. Y., 1949).

[8] G. N. Watson, The diffraction of electric waves by the earth and the transmission of electric waves round the earth, Proc. Roy. Soc. (London) [A] XCV, 83 (1918); and [A] XCV, 546 (1919).

[9] W. H. Wise, Condenser antenna radiation, Proc. IRE **19,** No. 9, 1684 (1931).

[10] W. H. Wise, Asymptotic dipole radiation formulas, Bell System Tech. J. VIII, No. 4, 662 (October 1929).

[11] A. Sommerfeld, Uber die Ausbreitung der Wellen in der drahtlosen Telegraphie, Ann. Physik, Vierte Folge, Band 28, No. 4 (1909).

[12] The Staff of the Computation Laboratory, Tables of modified Hankel functions of order one-third and of their derivatives (Harvard Univ. Press, Cambridge, Mass., 1945).

[13] J. C. P. Miller, The Airy integral, giving tables of solutions of differential equation $y''=xy$, Math Tables Part—vol. B, British Assoc. for Adv. of Sci., Table III, B–43 (Univ. Press, Cambridge, 1946).

[14] E. F. Florman, NBS Tech. News Bul. **39,** No. 1, p. 1–3 (January 1955).

[15] B. van der Pol and K. F. Niessen, The propagation of electromagnetic waves over a level ground, Ann. Physik [5] **6,** No. 3, 273–294 (1930).

[16] J. R. Wait, Radiation from a vertical electric dipole over a stratified ground, pt. I, Trans. Inst. Radio Engrs., PGAP–1, No. 1, p. 9–11 (July 1953); also, J. R. Wait and W. C. G. Fraser, Radiation from a vertical electric dipole over a stratified ground, pt. II, Trans. Inst. Radio Engrs., PGAP–2, No. 4, p. 144–146 (October 1954).

[17] Physical constants and conversion factors, 4th ed. (Office of Naval Research, Dept. of the Navy, Washington, D. C., Sept. 1953).

[18] H. von Hoerschelmann, Uber der Wirkungsweise des geknickten Marconischen Senders in der drahtlosen Telegraphie, Jahrbuch der drahtlosen Telegraphie and Telephonie **5,** 14–34 (Barth, Leipzig, 1912).

[19] G. Kirchoff, Zur Theorie der Lichtstrahlen, Ann. Physik und Chemie, Newe Folge, Band XVIII, p. 663 (Leipzig, 1883).

[20] H. Hertz, Die Kräfte elektrischer Schwingungen behandelt nach der Maxwell'schen Theorie, Ann. Physik und Chemie, Newe Folge, Band XXXVI, p. 1 (Leipzig, 1889).

[21] J. R. Wait and L. L. Campbell, Transmission curves for ground wave propagation at low radio frequencies, Radio Physics Laboratory, Rept. No. R–1 (Defense Research Telecommunications Dept., Ottawa, Canada, April 1953).

[22] A. B. Schneider, Phase variations with range of the ground wave signal from C. W. transmitters in the 70–130 kc/s band, J. British Inst. Radio Engrs. **12,** No. 3, 181–194 (March 1955). (Decca.)

[23] G. Millington and J. C. Thackray, Ground wave propagation curves for frequencies from 150 kc/s to 10 Mc/s, Marconi Review, No. 110, 3d Quarter, vol. XVI (1953).

[24] H. A. Lorentz, The theory of electrons and its applications to the phenomena of light and radiant heat (G. E. Stechert and Co., New York, N. Y., 1906, 1923).

[25] G. N. Watson, A treatise on the theory of Bessel functions, p. 229 (Macmillan Co., New York, N. Y., 1948).

[26] J. R. Wait and H. H. Howe, Amplitude and phase curves for ground wave propagation in the band 200 cycles per second to 500 kilocycles, NBS Circular 574 (1956).

[27] B. G. Pressy, G. E. Ashwell, and C. S. Fowler, The measurement of the phase velocity of ground wave propagation at low frequencies over a land path, Proc. Inst. Elec. Engrs. **100**, pt. III, 73–84 (1953).

[28] A. Sommerfeld, Über die Fortpflanzung des Lichtes in dispergierenden Medien, Ann. Physik, Vierte Folge, Band 44, No. 10, p. 177–202 (1914).

[29] L. Brillouin, Über die Fortpflanzung des Lichtes in dispergierenden Medien, Ann. Physik, Vierte Folge, Band 44, No. 10, p. 203–240 (1914).

6. Figures

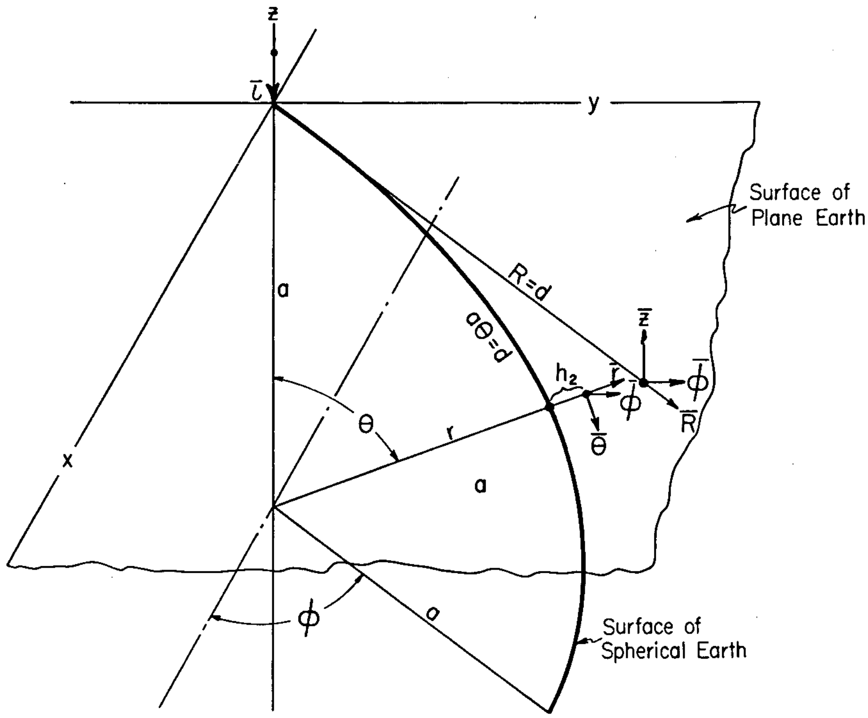

FIGURE 1. *Coordinate systems.*

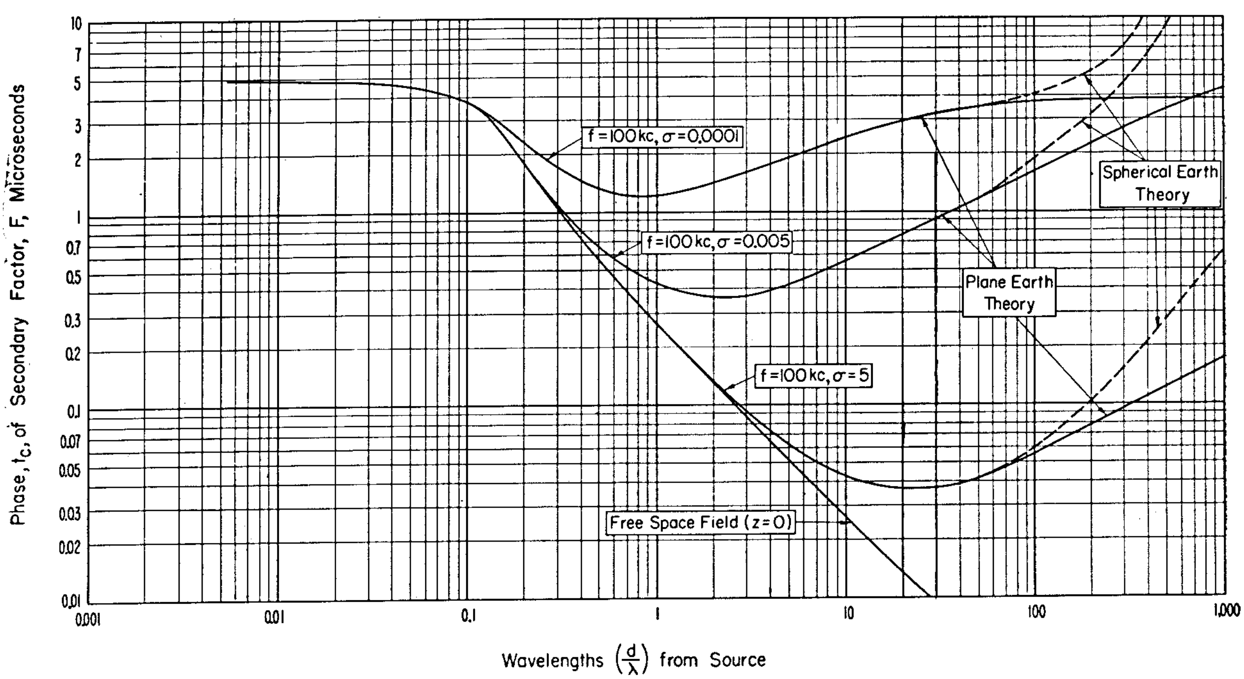

FIGURE 2. *Phase of secondary factor.*

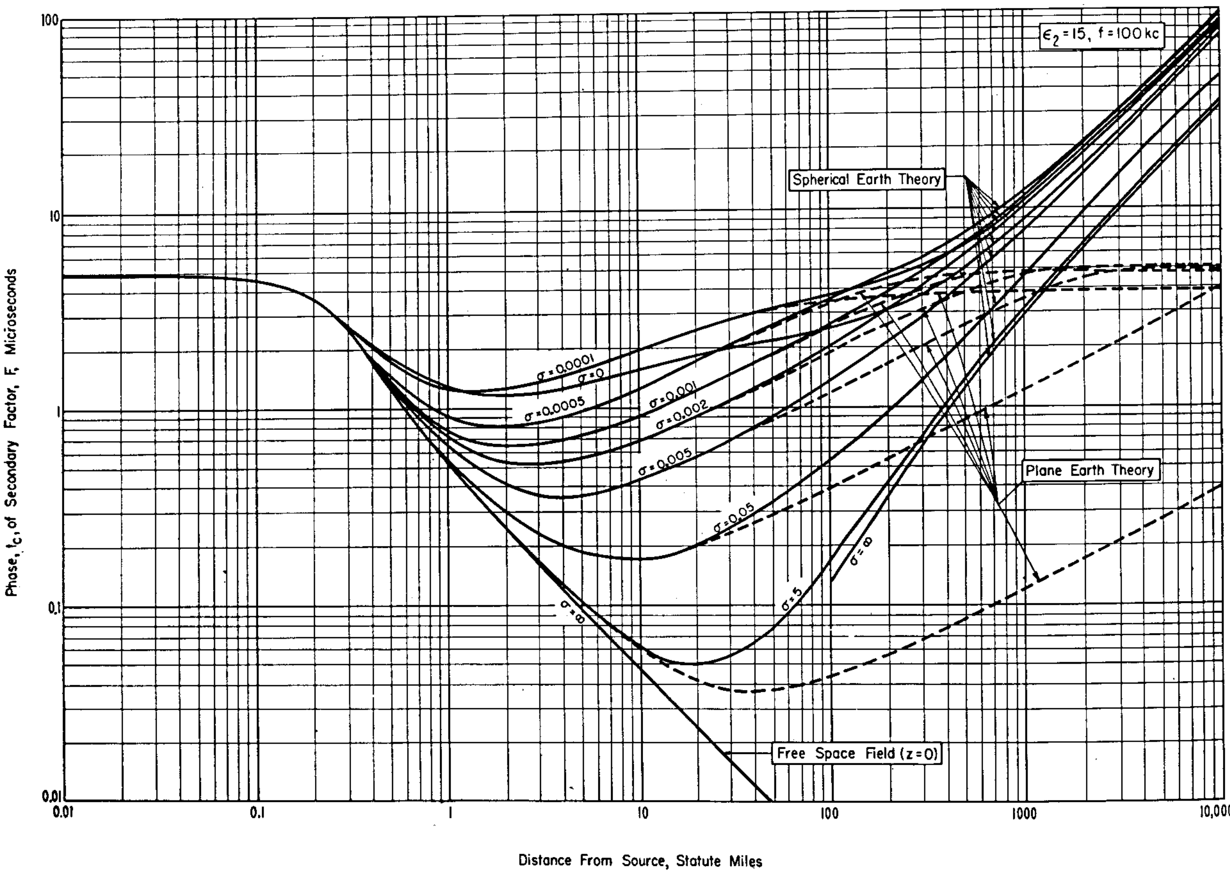

FIGURE 3. *Phase of secondary factor with distance from source for various conductivities.*

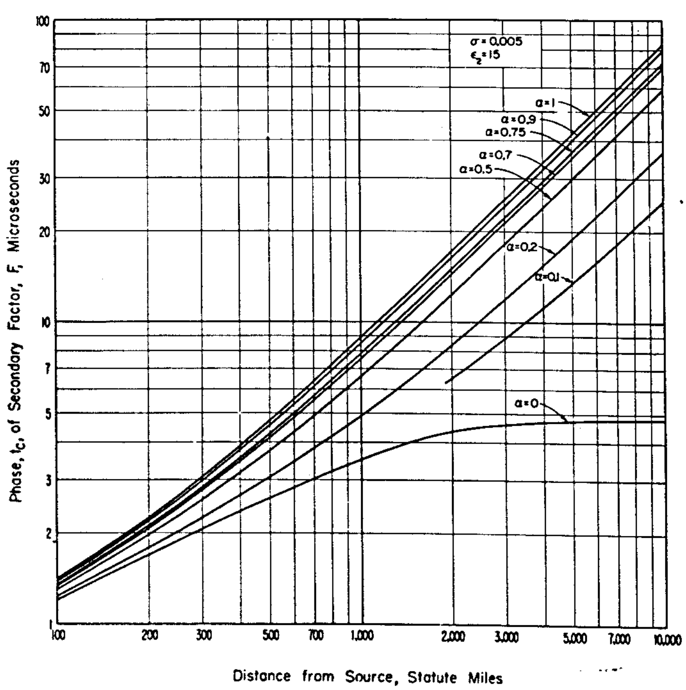

FIGURE 4. *Phase variation of secondary factor with distance for various α factors.*

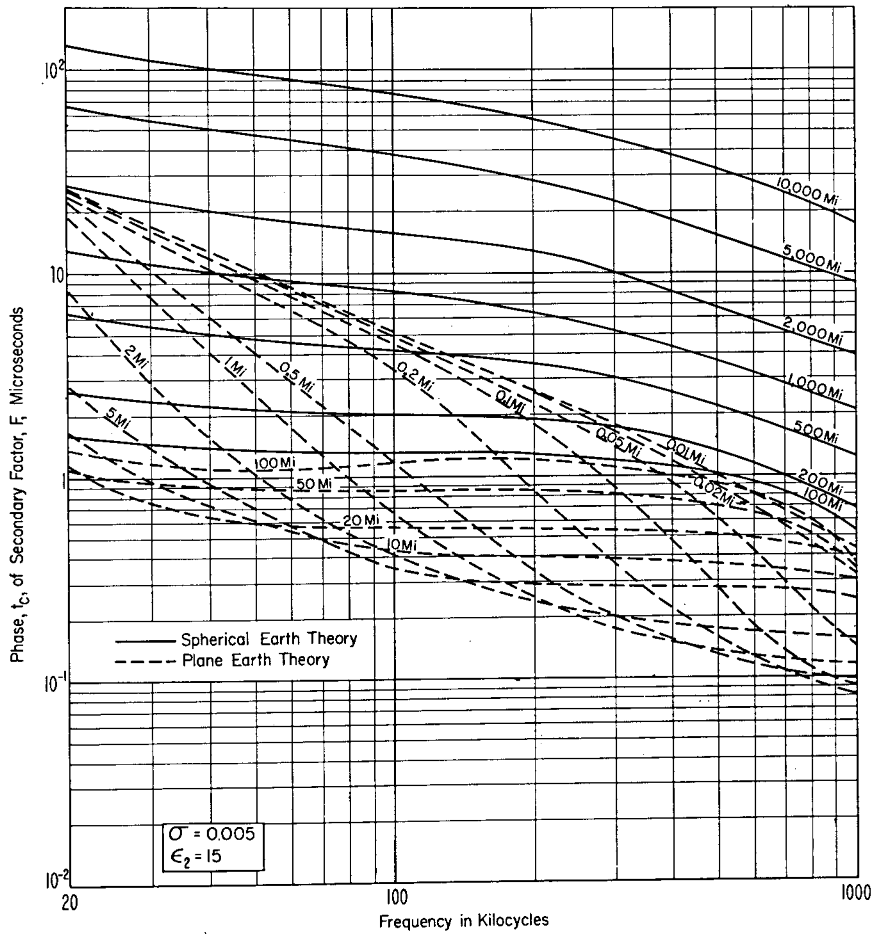

FIGURE 5. *Phase variation of secondary factor with frequency for various distances.*

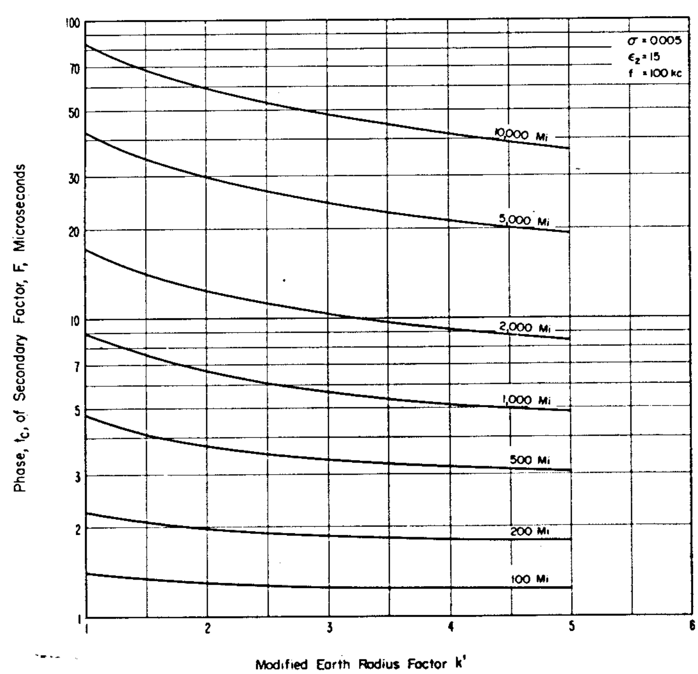

FIGURE 6. *Effect of modified earth radius factor k′ on phase of secondary factor at various distances from source.*

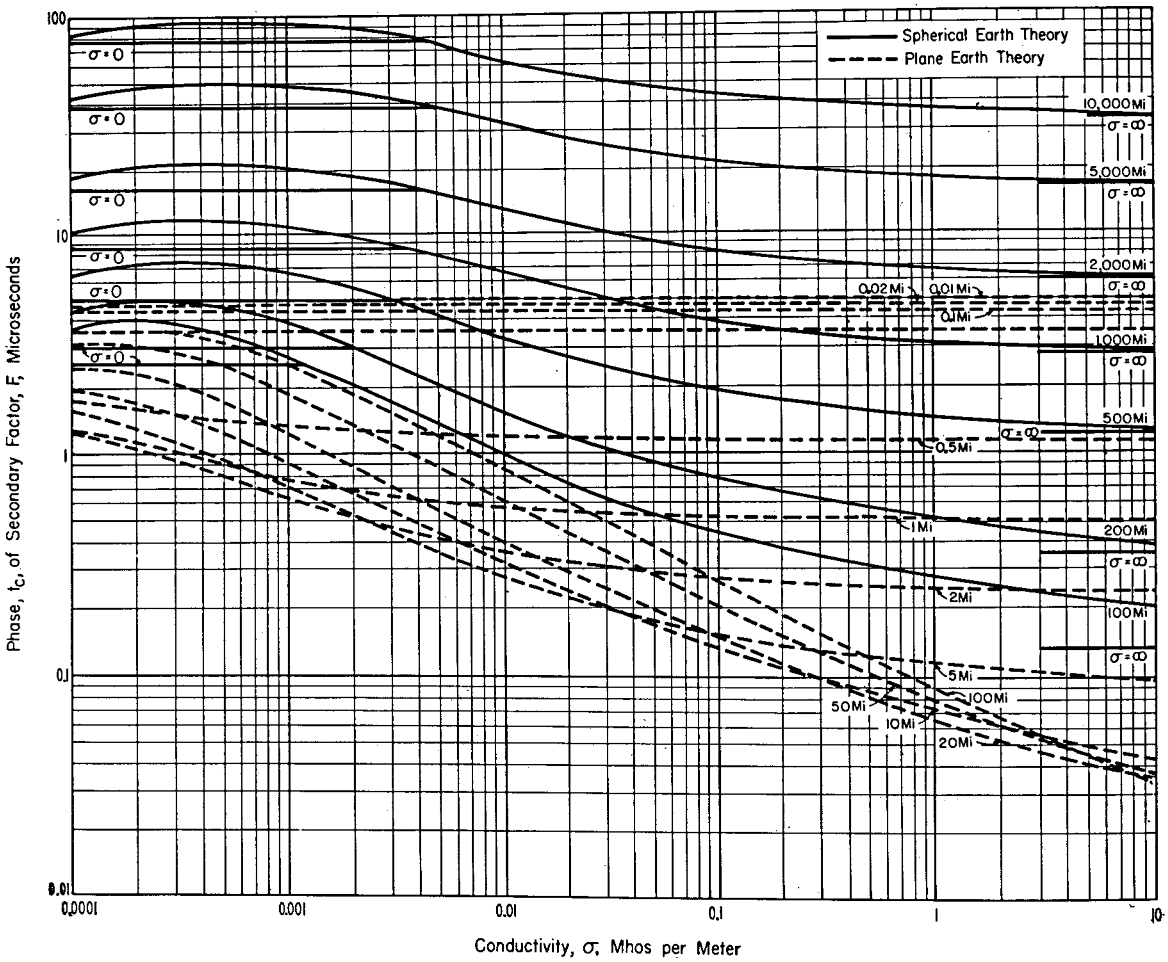

FIGURE 7. *Effect of conductivity on phase of secondary factor at various distances from source.*

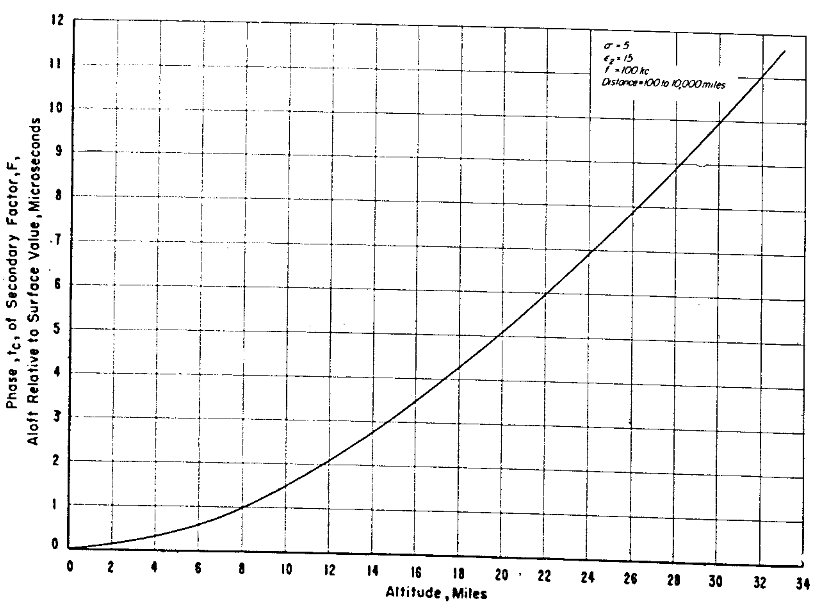

FIGURE 8. *Variation of phase of the secondary factor with altitude.*

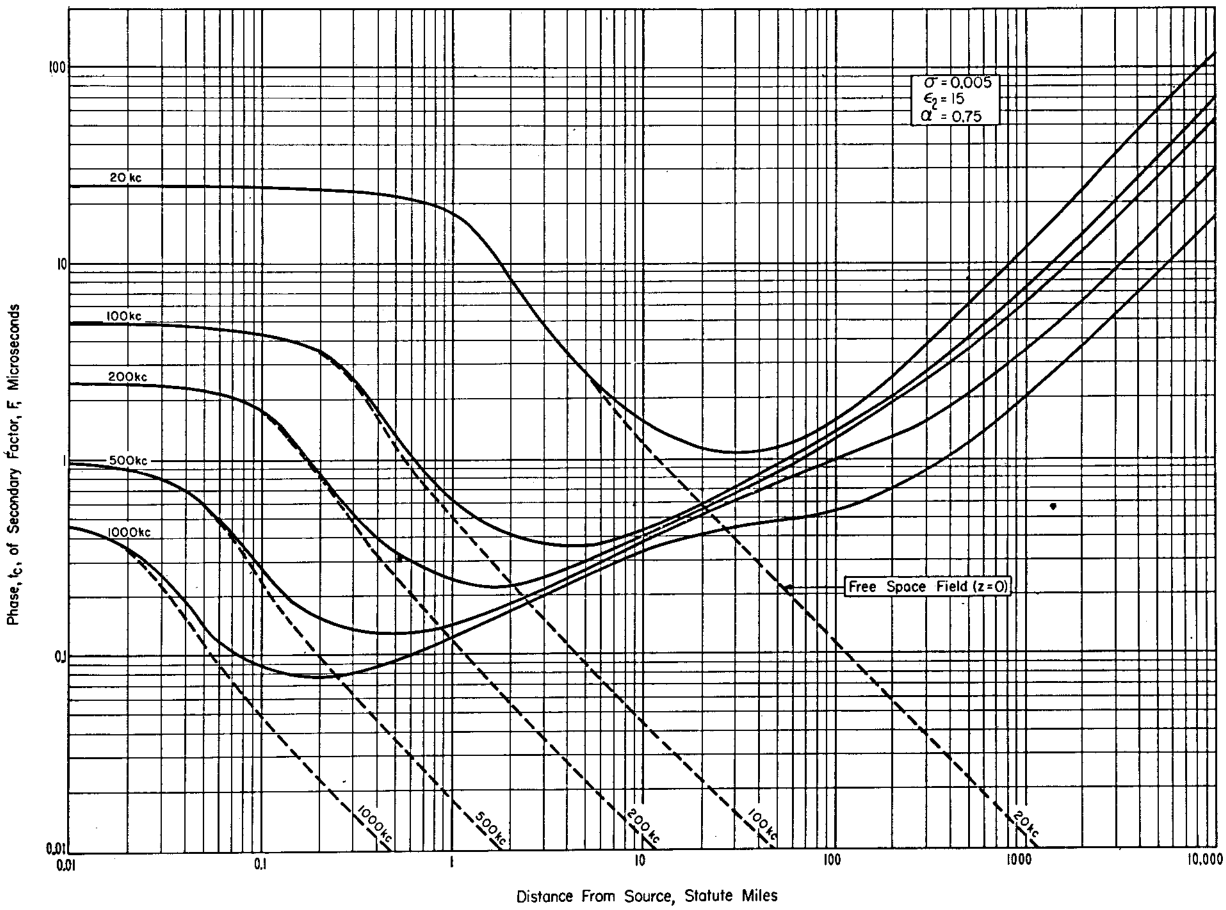

FIGURE 9. *Phase variation of the secondary factor with distance for frequencies from 20 to 1,000 kc.*

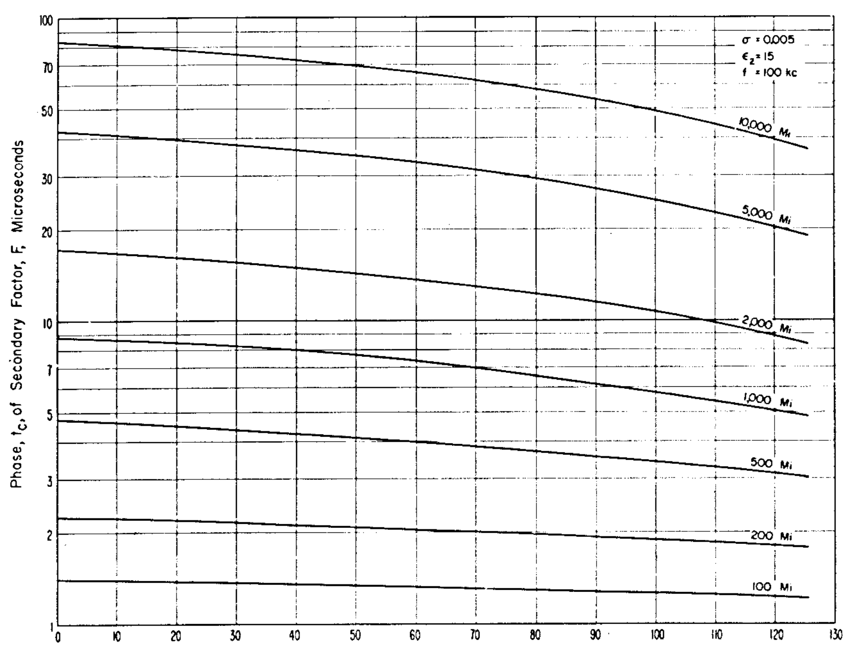

FIGURE 10. *Phase variation of secondary factor with vertical lapse of atmosphere at various distances from source.*

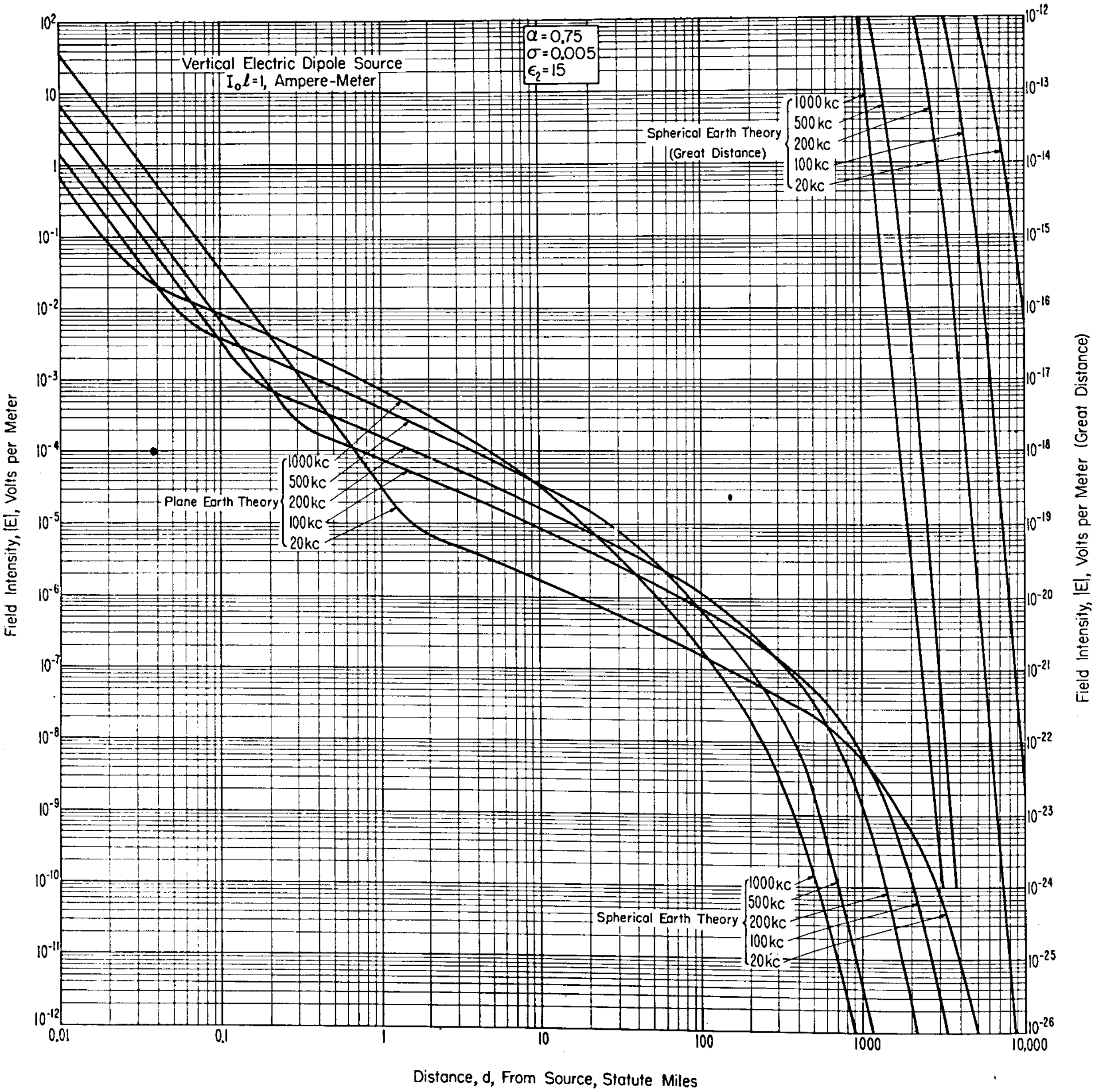

FIGURE 11. *Amplitude of ground wave at various frequencies.*

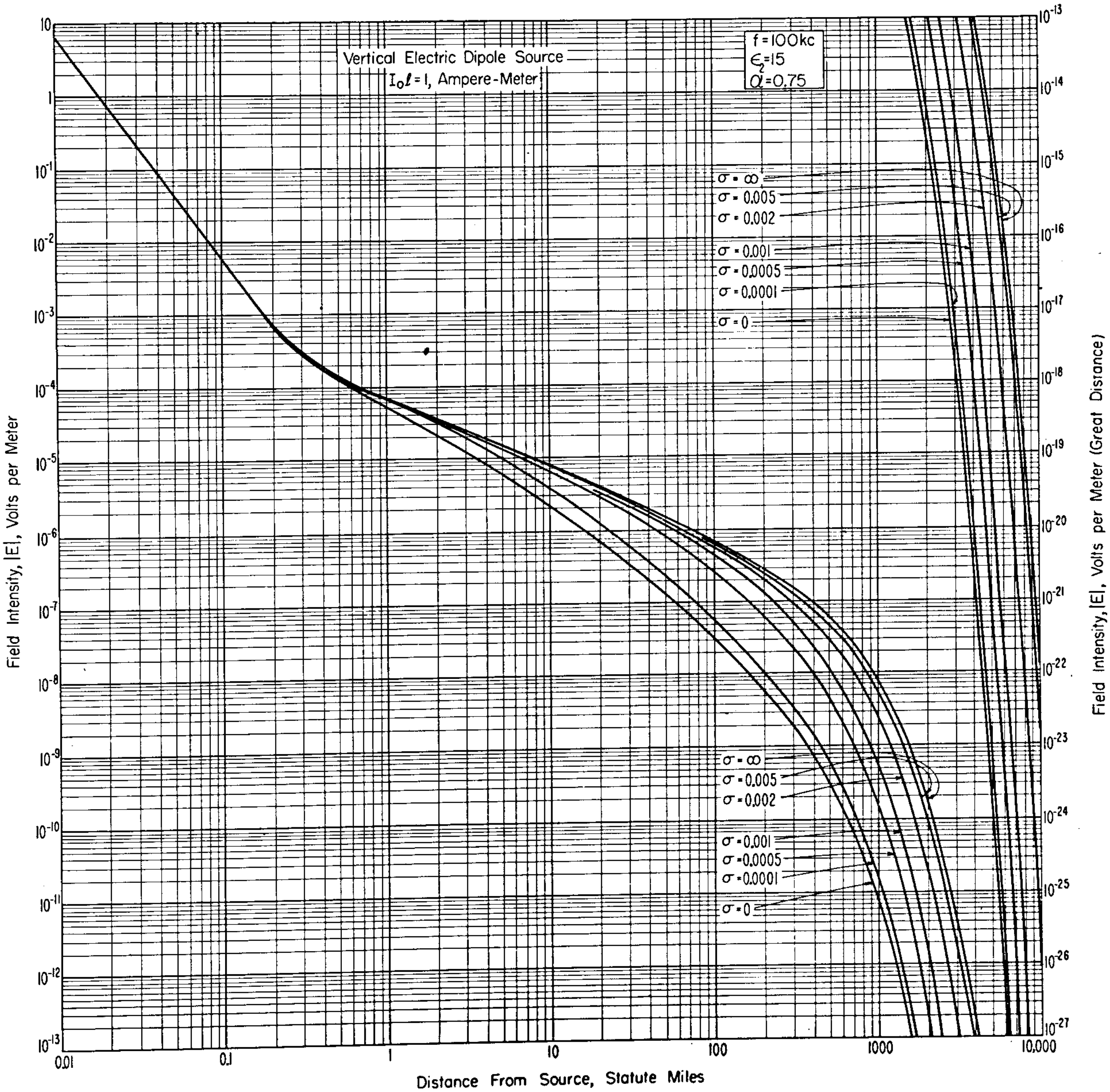

FIGURE 12. *Amplitude of ground wave for various conductivities.*

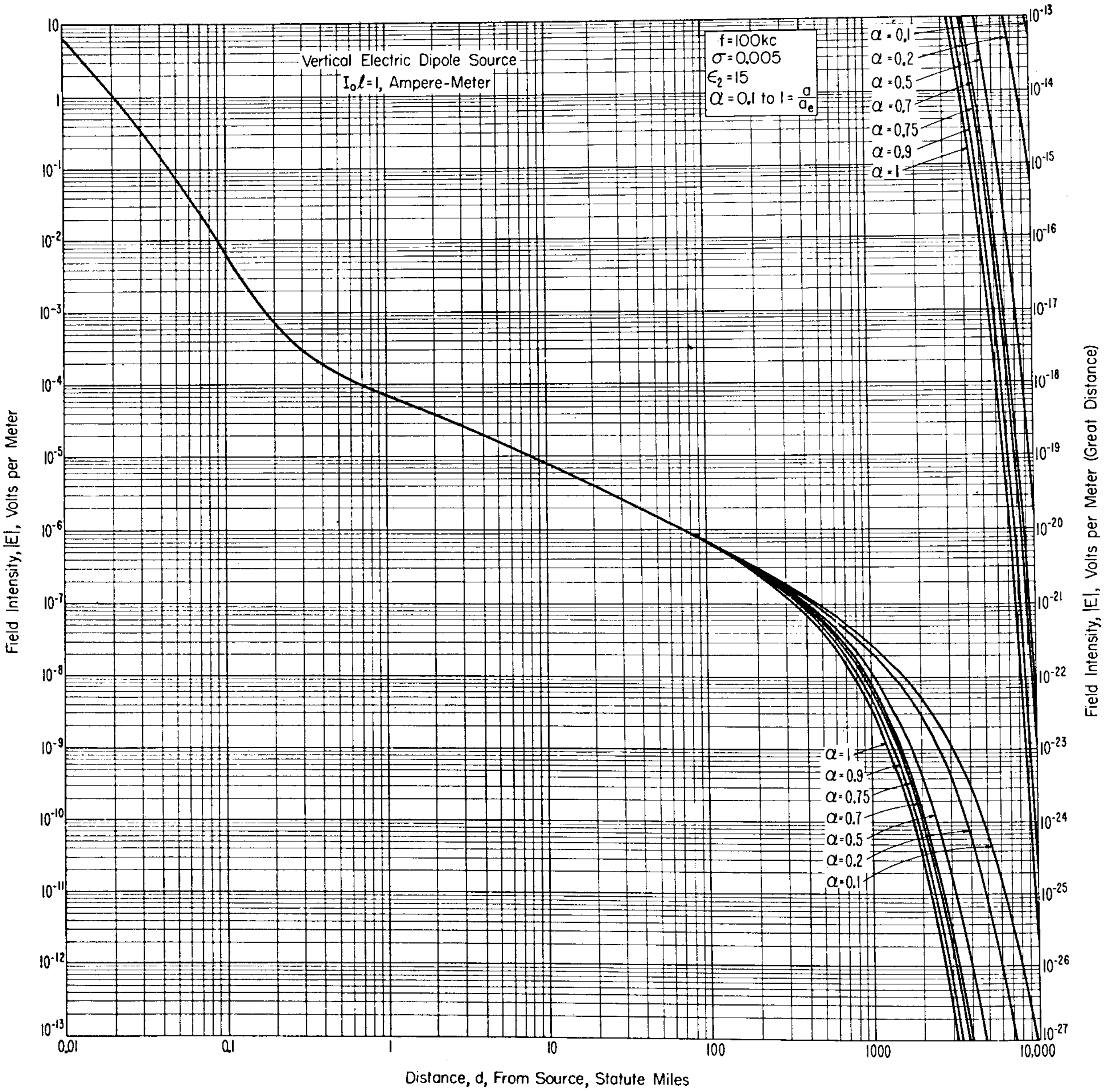

FIGURE 13. *Amplitude of ground wave for various vertical lapse factors.*

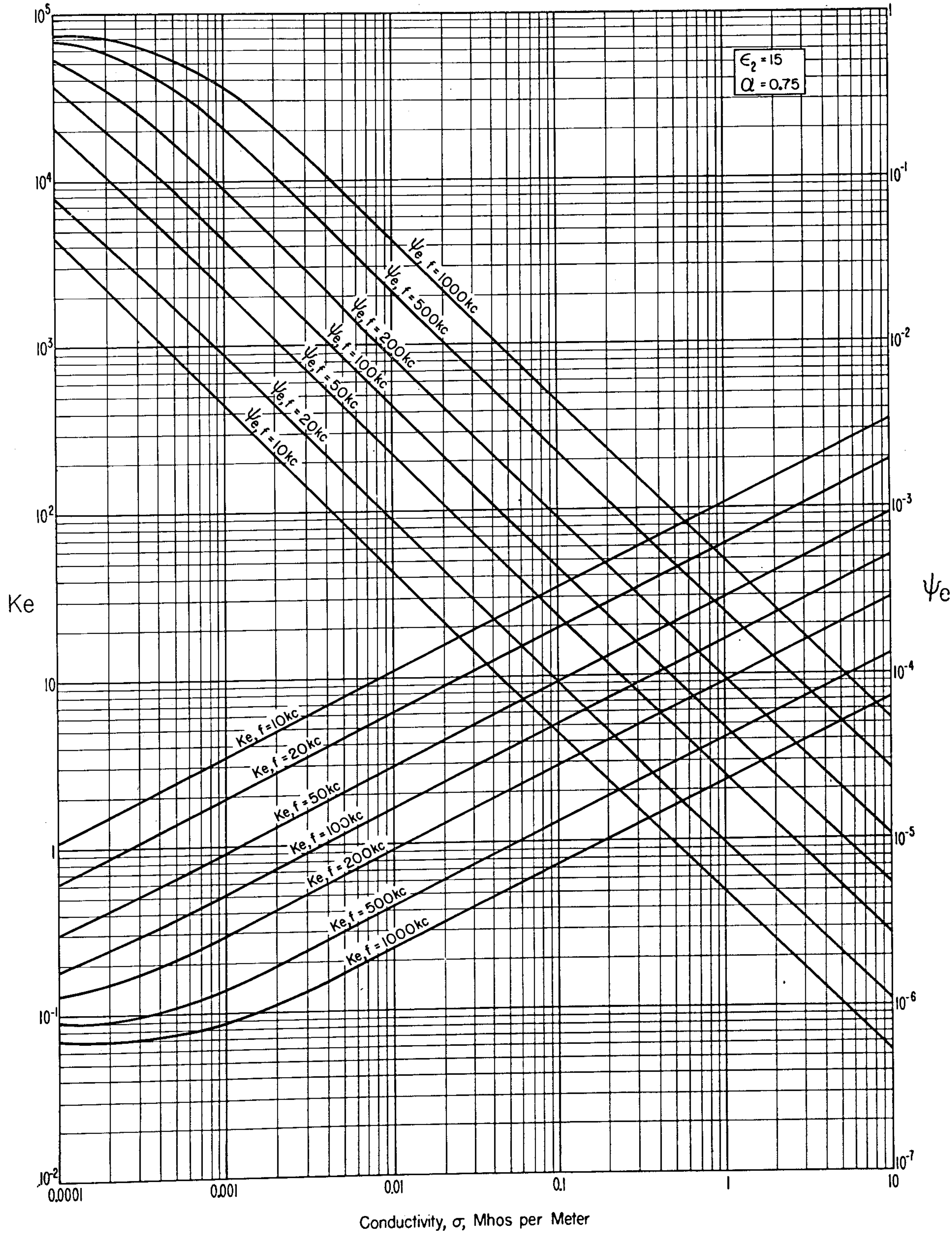

FIGURE 14. *Variation of K_e and ψ_e with conductivity for various frequencies.*

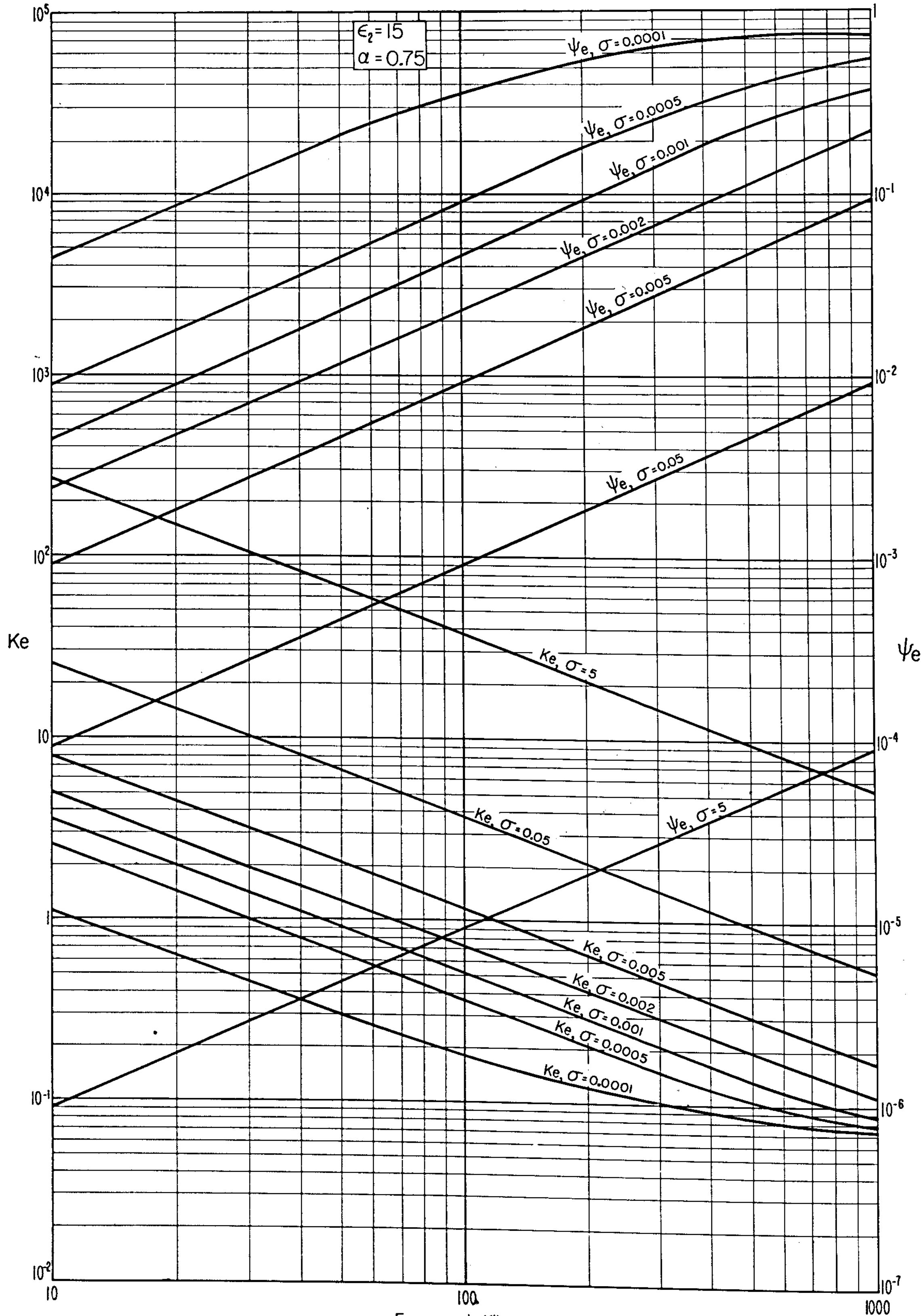

FIGURE 15. *Variation of K_e and ψ_e with frequency for various conductivities.*

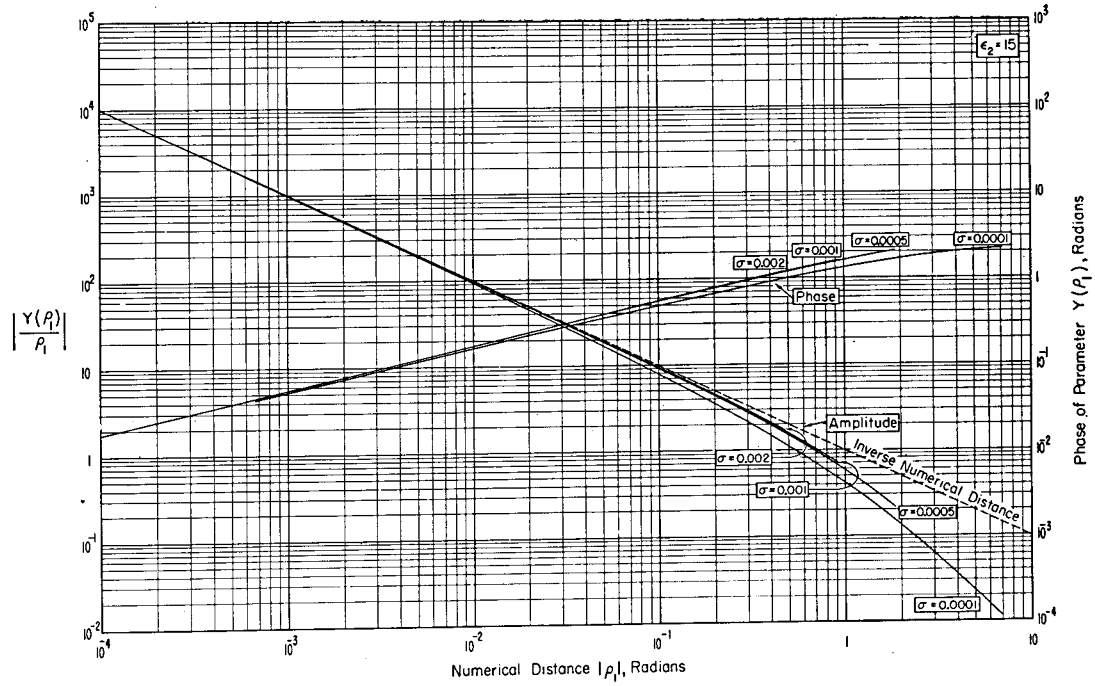

FIGURE 16. *Variation of amplitude and phase of y (ρ_1) with numerical distance ρ_1 at various conductivities.*

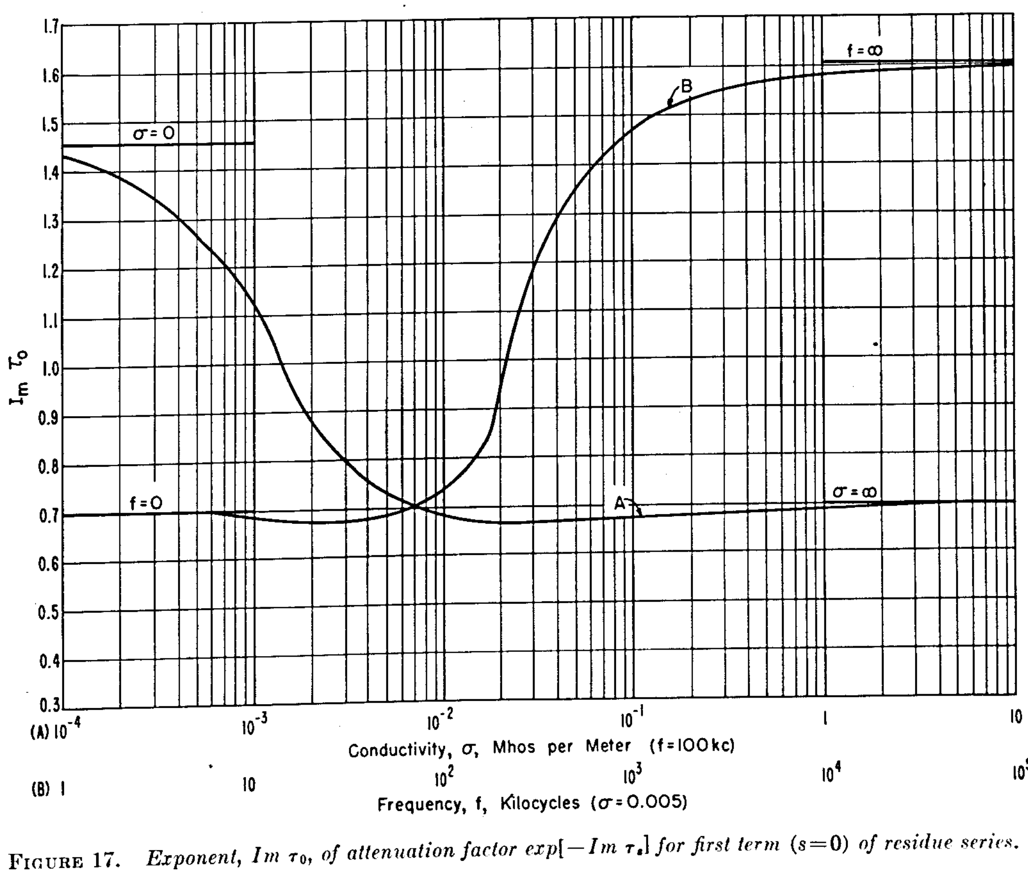

FIGURE 17. *Exponent, $Im\ \tau_0$, of attenuation factor $exp[-Im\ \tau_s]$ for first term ($s=0$) of residue series.*

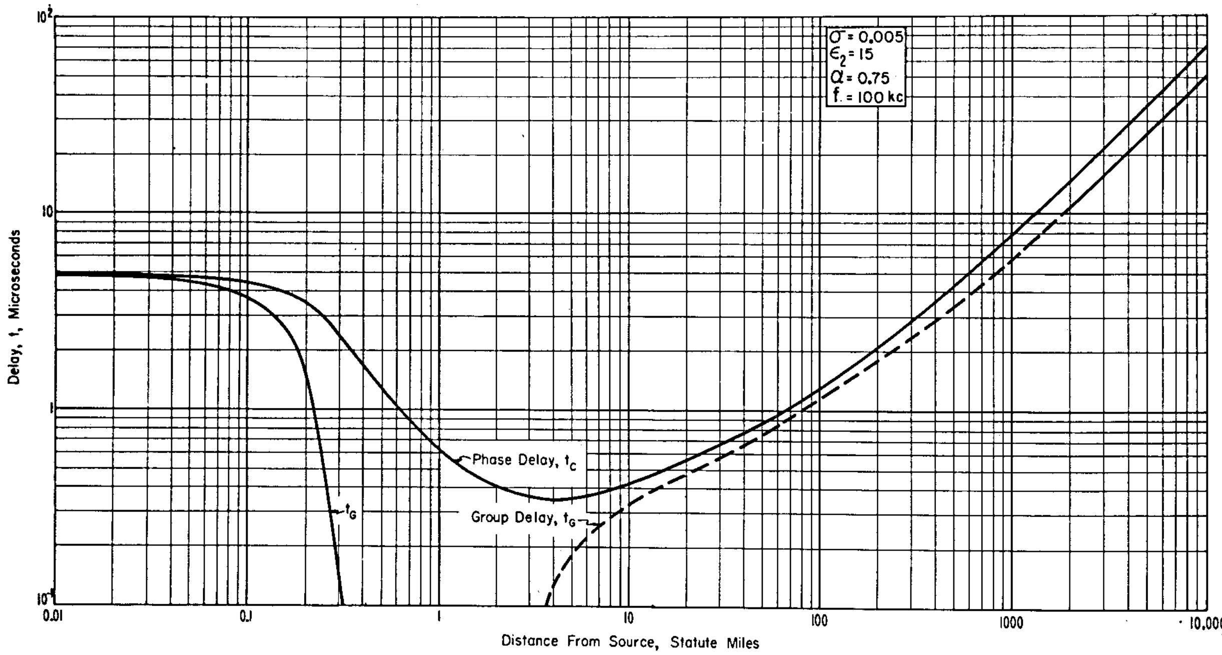

FIGURE 18. *Group delay*

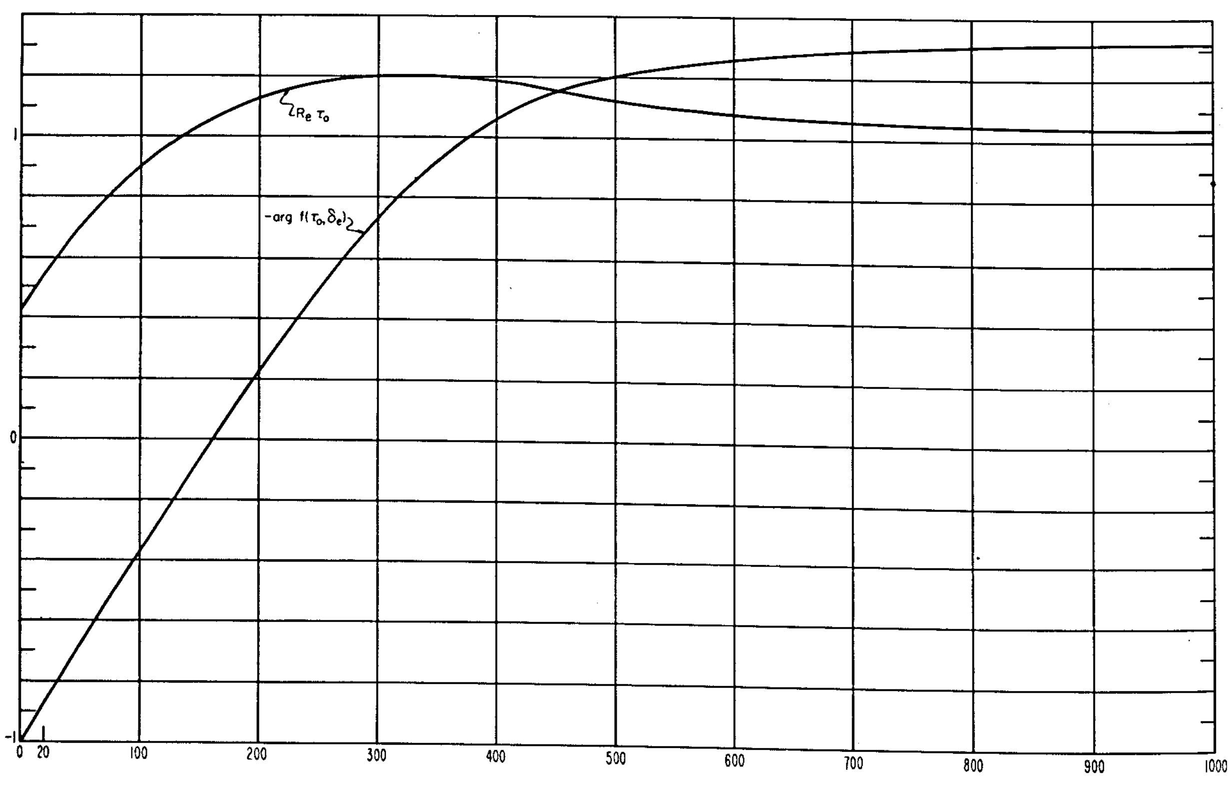

FIGURE 19. *Phase parameters Re τ_0 and $-arg\ f(\tau_0, \delta_e)$ at various frequencies.*

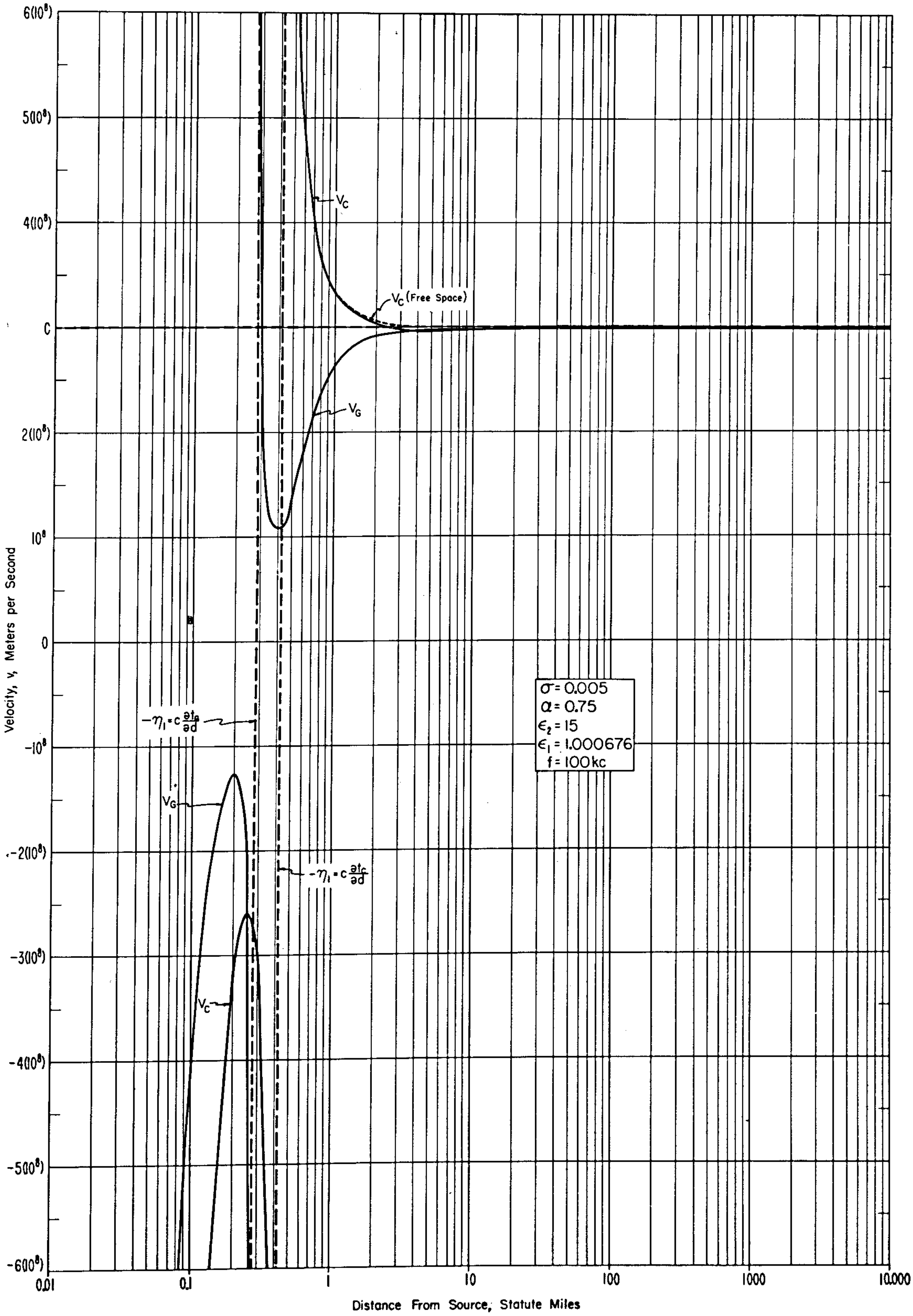

FIGURE 20. *Phase velocity and group velocity of ground wave.*

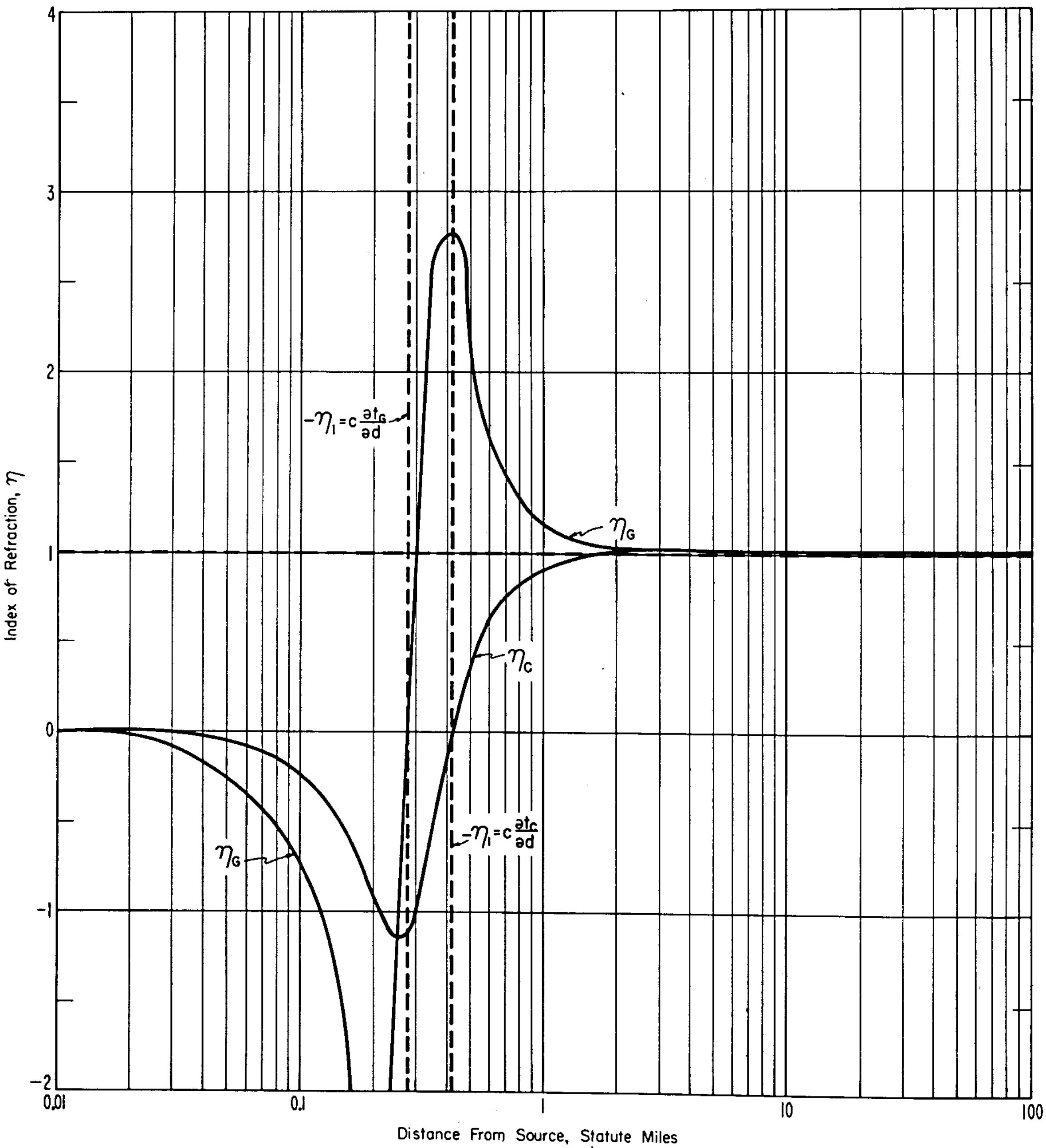

FIGURE 21. *Index of refraction of ground-wave propagation medium.*

7. Tables

I. Tables for 20 kc

A. Variation of ϕ_c, t_c and $|E|$ with σ for d from 0.01 to 10,000 statute miles, $h_2 = 0$, $\epsilon_2 = 15$, and $\alpha = 0.75$.

σ	Table No.	
	Spherical Earth Theory	Plane Earth Theory
0.005	1	2
∞		3

II. Tables for 100 kc

A. Variation of ϕ_c, t_c and $|E|$ with σ for d from 0.01 to 10,000 statute miles, $h_2 = 0$, $\epsilon_2 = 15$, and $\alpha = 0.75$

σ	Table No.	
	Spherical Earth Theory	Plane Earth Theory
0	4	5
0.0001	6	7
0.0005	8	9
0.001	10	11
0.002	12	13
0.005	14	15
0.05	16	17
5	18	19
5 (ϵ=80)		20
∞	21	22

B. Variation of ϕ_c, t_c and $|E_r|$ with α for d from 100 to 10,000 statute miles, $h_2 = 0$, $\epsilon_2 = 15$, and $\alpha = 0.005$, (spherical earth theory).

α	Table No.
0.1	23
0.2	24
0.5	25
0.7	26
0.9	27
1	28

C. Variation of t_{h_2} t_c with h_2 for d from 500 to 10,000 statute miles, $\sigma = 5$, and $\alpha = 0.75$.

h_2	Table No.
5.5 to 33.0 statute miles	29

III Tables for 200 kc

A. Variation of ϕ_c, t_c and $|E|$ with σ for d from 0.01 to 10,000 statute miles.

σ	Table No.	
	Spherical Earth Theory	Plane Earth Theory
0.005	30	31
∞		32

IV Tables for 500 kc

A. Variation of ϕ_c, t_c and $|E|$ with σ for d from 0.01 to 10,000 statute miles, $h_2 = 0$, $\epsilon_2 = 15$, and $\alpha = 0.75$.

σ	Table No.	
	Spherical Earth Theory	Plane Earth Theory
0.005	33	34
∞		35

V Tables for 1000 kc

A. Variation of ϕ_c, t_c and $|E|$ with σ for d from 0.01 to 10,000 statute miles, $h_2 = 0$, $\epsilon_2 = 15$, and $\alpha = 0.75$.

σ	Table No.	
	Spherical Earth Theory	Plane Earth Theory
0.005	36	37
∞		38

VI. Constants for ϕ_c, t_c and $|E|$ calculations

Constant	σ	f	Table No.
$\frac{\sigma\mu_o c^2}{\omega}$	0.0001 to 5	10 to 1000 kc	39
K_e	0.0001 to 5	10 to 1000 kc	40
ψ_e	0.0001 to 5	10 to 1000 kc	41
$f(\sigma, \epsilon)$	0.0001 to 5	100 kc	42
$f(\sigma, \epsilon)$	0.005	20 to 1000 kc	42
$-\frac{1}{ik_1 d} + \frac{1}{(ik_1 d)^2}$	d(0.01 to 200 mi.)	100 kc	43
$\tau_{s,0}$, $\tau_{s,\infty}$	s(0 to 50)		44

TABLE 1.

Spherical Earth Theory

$h_2 = 0$
$\epsilon_2 = 15$
$\sigma = 0.005$
$\alpha = 0.75$
$f = 20$ kc

d(miles)	ϕ_c(radians)	t_c(microseconds)	$\lvert E_r \rvert$	$\lvert E_r \rvert$ (decibels)
100	0.19582	1.5583	$1.5089(10^{-7})$	$-1.36(10^2)$
200	.33525	2.6678	$7.0579(10^{-8})$	$-1.43(10^2)$
500	.77024	6.1293	$2.1315(10^{-8})$	$-1.53(10^2)$
1000	1.5679	12.477	$5.5710(10^{-9})$	$-1.65(10^2)$
2000	3.2219	25.639	$5.6272(10^{-10})$	$-1.85(10^2)$
5000	8.1936	65.203	$1.0710(10^{-12})$	$-2.39(10^2)$
10,000	16.480	131.14	$5.5778(10^{-17})$	$-3.25(10^2)$

TABLE 2

Plane Earth Theory

$h_2 = 0$
$\epsilon_2 = 15$
$\sigma = 0.005$
$\alpha = 0.75$
$f = 20$ kc

d(miles)	ϕ_c(radians)	t_c(microseconds)	$\lvert E_z \rvert$	$\lvert E_z \rvert$ (decibels)
0.01	3.1348	24.946	$3.4294(10)$	$3.07(10)$
0.02	3.1281	24.893	4.2864	$1.26(10)$
0.05	3.1078	24.731	$2.7420(10^{-1})$	$-1.12(10)$
0.1	3.0739	24.461	$3.4217(10^{-2})$	$-2.93(10)$
0.2	3.0049	23.912	$4.2484(10^{-3})$	$-4.74(10)$
0.5	2.7765	22.095	$2.6028(10^{-4})$	$-7.17(10)$
1	2.2448	17.863	$2.9930(10^{-5})$	$-9.05(10)$
2	1.0371	8.2529	$6.9192(10^{-6})$	$-1.03(10^2)$
5	0.34820	2.7709	$3.0294(10^{-6})$	$-1.10(10^2)$
10	0.19889	1.5827	$1.5567(10^{-6})$	$-1.16(10^2)$
20	0.14296	1.1376	$7.8326(10^{-7})$	$-1.22(10^2)$
50	0.13787	1.0971	$3.1380(10^{-7})$	$-1.30(10^2)$
100	0.16727	1.3311	$1.5711(10^{-7})$	$-1.36(10^2)$

TABLE 3

Plane Earth Theory

$h_2 = 0$
$\epsilon_2 = 15$
$\sigma = \infty$
$\alpha = 0.75$
$f = 20$ kc

d(miles)	ϕ_c(radians)	t_c(microseconds)	$\lvert E_z \rvert$	$\lvert E_z \rvert$ (decibels)
0.01	3.1348	24.946	$3.4294(10)$	$3.07(10)$
0.02	3.1281	24.893	4.2864	$1.26(10)$
0.05	3.1078	24.731	$2.7420(10^{-1})$	$-1.12(10)$
0.1	3.0739	24.461	$3.4217(10^{-2})$	$-2.93(10)$
0.2	3.0050	23.913	$4.2483(10^{-3})$	$-4.74(10)$
0.5	2.7778	22.105	$2.6015(10^{-4})$	$-7.17(10)$
1	2.2498	17.904	$2.9739(10^{-5})$	$-9.05(10)$
2	1.0240	8.1489	$6.7731(10^{-6})$	$-1.03(10^2)$
5	0.31416	2.5000	$2.9956(10^{-6})$	$-1.10(10^2)$
10	0.15037	1.1966	$1.5448(10^{-6})$	$-1.16(10^2)$
20	0.074367	0.59179	$7.7870(10^{-7})$	$-1.22(10^2)$
50	0.029655	0.23599	$3.1220(10^{-7})$	$-1.30(10^2)$
100	0.014821	0.11794	$1.5615(10^{-7})$	$-1.36(10^2)$

TABLE 4

Spherical Earth Theory

$h_2 = 0$
$\epsilon_2 = 15$
$\sigma = 0$
$\alpha = 0.75$
$f = 100$ kc

d(miles)	ϕ_c(radians)	t_c(microseconds)	$\lvert E_r \rvert$	$\lvert E_r \rvert$ (decibels)
100	1.5806	2.5155	$3.2900(10^{-8})$	$-1.50(10^2)$
200	1.9179	3.0524	$6.7367(10^{-9})$	$-1.63(10^2)$
500	3.1041	4.9403	$3.7535(10^{-10})$	$-1.88(10^2)$
1000	5.3895	8.5776	$7.2839(10^{-12})$	$-2.23(10^2)$
2000	10.060	16.012	$3.8661(10^{-15})$	$-2.88(10^2)$
5000	24.071	38.311	$1.0665(10^{-24})$	$-4.79(10^2)$
10,000	47.423	75.476	$2.2218(10^{-40})$	$-7.93(10^2)$

TABLE 5

Plane Earth Theory

$h_2 = 0$
$\epsilon_2 = 15$
$\sigma = 0$
$\alpha = 0.75$
$f = 100$ kc

d(miles)	ϕ_c(radians)	t_c(microseconds)	$\lvert E_z \rvert$	$\lvert E_z \rvert$ (decibels)
0.01	3.1078	4.9462	6.8558	$1.67(10)$
0.02	3.0737	4.8920	$8.5589(10^{-1})$	-1.35
0.05	2.9682	4.7241	$5.4354(10^{-2})$	$-2.53(10)$
0.1	2.7742	4.4153	$6.6647(10^{-3})$	$-4.35(10)$
0.2	2.2935	3.6503	$8.4143(10^{-4})$	$-6.15(10)$
0.5	1.1714	1.8644	$1.3178(10^{-4})$	$-7.76(10)$
1	0.80651	1.2836	$5.6107(10^{-5})$	$-8.50(10)$
2	0.72816	1.1589	$2.3348(10^{-5})$	$-9.26(10)$
5	0.81749	1.3011	$6.6253(10^{-6})$	$-1.04(10^2)$
10	0.96243	1.5318	$2.3208(10^{-6})$	$-1.13(10^2)$
20	1.1431	1.8193	$7.3296(10^{-7})$	$-1.23(10^2)$
50	1.34	2.13	$1.39(10^{-7})$	$-1.37(10^2)$
100	1.4426	2.2960	$3.6401(10^{-8})$	$-1.49(10^2)$

TABLE 6

Spherical Earth Theory

$h_2 = 0$
$\epsilon_2 = 15$
$\sigma = 0.0001$
$\alpha = 0.75$
$f = 100$ kc

d(miles)	ϕ_c(radians)	t_c(microseconds)	$\lvert E_r \rvert$	$\lvert E_r \rvert$ (decibels)
100	2.3637	3.7620	$5.8757(10^{-8})$	$-1.45(10^2)$
200	2.7781	4.4215	$1.1450(10^{-8})$	$-1.59(10^2)$
500	4.0746	6.4850	$6.3957(10^{-10})$	$-1.84(10^2)$
1000	6.5385	10.406	$1.3160(10^{-11})$	$-2.18(10^2)$
2000	11.573	18.420	$7.8959(10^{-15})$	$-2.82(10^2)$
5000	26.677	42.458	$3.1442(10^{-24})$	$-4.70(10^2)$
10,000	51.849	82.521	$1.2076(10^{-29})$	$-5.78(10^2)$

TABLE 7

Plane Earth Theory

$h_2 = 0$
$\epsilon_2 = 15$
$\sigma = 0.0001$
$\alpha = 0.75$
$f = 100$ kc

d(miles)	ϕ_c(radians)	t_c(microseconds)	$\lvert E_z \rvert$	$\lvert E_z \rvert$ (decibels)
0.01	3.1077	4.9461	6.8554	1.67(10)
0.02	3.0735	4.8917	$8.5566(10^{-1})$	-1.35
0.05	2.9666	4.7215	$5.4250(10^{-2})$	-2.53(10)
0.1	2.7646	4.4001	$6.6147(10^{-3})$	-4.36(10)
0.2	2.2363	3.5591	$8.3455(10^{-4})$	-6.16(10)
0.5	1.0877	1.7312	$1.5364(10^{-4})$	-7.63(10)
1	0.79238	1.2611	$7.0917(10^{-5})$	-8.30(10)
2	0.77545	1.2342	$3.1816(10^{-5})$	-8.99(10)
5	0.96570	1.5370	$1.0199(10^{-5})$	-9.98(10)
10	1.2209	1.9431	$3.9541(10^{-6})$	$-1.08(10^2)$
20	1.5475	2.4629	$1.3605(10^{-6})$	$-1.17(10^2)$
50	1.987	3.162	$2.608(10^{-7})$	$-1.32(10^2)$
100	2.22	3.53	$6.38(10^{-8})$	$-1\ 44(10^2)$

TABLE 8

Spherical Earth Theory

$h_2 = 0$
$\epsilon_2 = 15$
$\sigma = 0.0005$
$\alpha = 0.75$
$f = 100$ kc

d(miles)	ϕ_c(radians)	t_c(microseconds)	$\lvert E_r \rvert$	$\lvert E_r \rvert$ (decibels)
100	2.1839	3.4758	$2.9713(10^{-7})$	$-1.31(10^2)$
200	2.9527	4.6994	$7.0152(10^{-8})$	$-1.43(10^2)$
500	4.6331	7.3738	$4.5826(10^{-9})$	$-1.67(10^2)$
1000	7.4308	11.826	$1.3696(10^{-10})$	$-1.97(10^2)$
2000	13.140	20.912	$1.8518(10^{-13})$	$-2.55(10^2)$
5000	30.271	48.177	$8.4226(10^{-22})$	$-4.21(10^2)$
10,000	58.823	93.619	$1.8740(10^{-35})$	$-6.95(10^2)$

TABLE 9

Plane Earth Theory

$h_2 = 0$
$\epsilon_2 = 15$
$\sigma = 0.0005$
$\alpha = 0.75$
$f = 100$ kc

d(miles)	ϕ_c(radians)	t_c(microseconds)	$\lvert E_z \rvert$	$\lvert E_z \rvert$ (decibels)
0.01	3.1078	4.9462	6.8551	1.67(10)
0.02	3.0737	4.8920	$8.5546(10^{-1})$	-1.36
0.05	2.9679	4.7235	$5.4139(10^{-2})$	-2.53(10)
0.1	2.7688	4.4066	$6.5373(10^{-3})$	-4.37(10)
0.2	2.2211	3.5351	$7.8391(10^{-4})$	-6.21(10)
0.5	0.89768	1.4287	$1.5338(10^{-4})$	-7.63(10)
1	0.56133	0.89338	$7.8570)10^{-5})$	-8.21(10)
2	0.49519	0.78812	$3.8824(10^{-5})$	-8.82(10)
5	0.59426	0.94579	$1.4756(10^{-5})$	-9.66(10)
10	0.77494	1.2334	$6.9090(10^{-6})$	$-1.03(10^2)$
20	1.0853	1.7272	$3.3130(10^{-6})$	$-1.10(10^2)$
50	1.5685	2.4964	$9.4151(10^{-7})$	$-1.21(10^2)$
100	2.0707	3.2957	$3.1683(10^{-7})$	$-1.30(10^2)$

TABLE 10

Spherical Earth Theory

$h_2 = 0$
$\epsilon_2 = 15$
$\sigma = 0.001$
$\alpha = 0.75$
$f = 100$ kc

d(miles)	ϕ_c(radians)	t_c(microseconds)	$\lvert E_r \rvert$	$\lvert E_r \rvert$ (decibels)
100	1.6956	2.6987	$4.7476(10^{-7})$	$-1.26(10^2)$
200	2.4183	3.8489	$1.4441(10^{-7})$	$-1.37(10^2)$
500	4.1335	6.5787	$1.3254(10^{-8})$	$-1.58(10^2)$
1000	6.8789	10.948	$5.5691(10^{-10})$	$-1.85(10^2)$
2000	12.431	19.785	$1.5035(10^{-12})$	$-2.36(10^2)$
5000	29.096	46.307	$5.4666(10^{-20})$	$-3.85(10^2)$
10,000	56.869	90.510	$3.8874(10^{-32})$	$-6.28(10^2)$

TABLE 11

Plane Earth Theory

$h_2 = 0$
$\epsilon_2 = 15$
$\sigma = 0.001$
$\alpha = 0.75$
$f = 100$ kc

d(miles)	ϕ_c(radians)	t_c(microseconds)	$\lvert E_z \rvert$	$\lvert E_z \rvert$ (decibels)
0.01	3.1078	4.9464	6.8550	1.67(10)
0.02	3.0738	4.8921	$8.5544(10^{-1})$	-1.36
0.05	2.9684	4.7244	$5.4124(10^{-2})$	-2.53(10)
0.1	2.7714	4.4108	$6.5241(10^{-3})$	-4.37(10)
0.2	2.2270	3.5444	$7.7042(10^{-4})$	-6.23(10)
0.5	0.84818	1.3499	$1.4946(10^{-4})$	-7.65(10)
1	0.48800	0.77668	$7.8226(10^{-5})$	-8.21(10)
2	0.39536	0.62924	$3.9316(10^{-5})$	-8.81(10)
5	0.44303	0.70510	$1.5368(10^{-5})$	-9.63(10)
10	0.56809	0.90414	$7.4426(10^{-6})$	$-1.03(10^2)$
20	0.76968	1.2250	$3.5257(10^{-6})$	$-1.09(10^2)$
50	1.1752	1.8704	$1.2262(10^{-6})$	$-1.18(10^2)$
100	1.6073	2.5580	$4.9584(10^{-7})$	$-1.26(10^2)$

TABLE 12

Spherical Earth Theory

$h_2 = 0$
$\epsilon_2 = 15$
$\sigma = 0.002$
$\alpha = 0.75$
$f = 100$ kc

d(miles)	ϕ_c(radians)	t_c(microseconds)	$\lvert E_r \rvert$	$\lvert E_r \rvert$ (decibels)
100	1.2774	2.0331	$5.8580(10^{-7})$	$-1.25(10^2)$
200	1.9129	3.0444	$2.1510(10^{-7})$	$-1.33(10^2)$
500	3.5925	5.7176	$3.1275(10^{-8})$	$-1.50(10^2)$
1000	6.2792	9.9936	$2.4725(10^{-9})$	$-1.72(10^2)$
2000	11.713	18.643	$2.2437(10^{-11})$	$-2.13(10^2)$
5000	28.019	44.593	$3.1146(10^{-17})$	$-3.30(10^2)$
10,000	55.194	87.844	$9.5877(10^{-27})$	$-5.20(10^2)$

TABLE 13

Plane Earth Theory

$h_2 = 0$
$\epsilon_2 = 15$
$\sigma = 0.002$
$\alpha = 0.75$
$f = 100$ kc

d(miles)	ϕ_c (radians)	t_c(microseconds)	$\lvert E_z \rvert$	$\lvert E_z \rvert$ (decibels)
0.01	3.1078	4.9462	6.8550	1.67(10)
0.02	3.0738	4.8921	$8.5543(10^{-1})$	-1.36
0.05	2.9692	4.7256	$5.4118(10^{-2})$	-2.53(10)
0.10	2.7733	4.4138	$6.5167(10^{-3})$	-4.37(10)
0.20	2.2329	3.5537	$7.6170(10^{-4})$	-6.24(10)
0.50	0.81505	1.2972	$1.4612(10^{-4})$	-7.67(10)
1	0.43629	0.69438	$7.7511(10^{-5})$	-8.22(10)
2	0.32474	0.51684	$3.9342(10^{-5})$	-8.81(10)
5	0.33154	0.52766	$1.5589(10^{-5})$	-9.61(10)
10	0.41311	0.65748	$7.6706(10^{-6})$	$-1.02(10^2)$
20	0.55471	0.88285	$3.7348(10^{-6})$	$-1.09(10^2)$
50	0.85170	1.3555	$1.3932(10^{-6})$	$-1.17(10^2)$
100	1.1821	1.8814	$6.2575(10^{-7})$	$-1.24(10^2)$

TABLE 14

Spherical Earth Theory

$h_2 = 0$
$\epsilon_2 = 15$
$\sigma = 0.005$
$\alpha = 0.75$
$f = 100$ kc

d(miles)	ϕ_c(radians)	t_c(microseconds)	$\lvert E_r \rvert$	$\lvert E_r \rvert$ (decibels)
100	0.85471	1.3603	$6.6847(10^{-7})$	$-1.23(10^2)$
200	1.3308	2.1180	$2.7382(10^{-7})$	$-1.31(10^2)$
500	2.6649	4.2414	$5.2565(10^{-8})$	$-1.46(10^2)$
1000	4.9028	7.8031	$6.0373(10^{-9})$	$-1.64(10^2)$
2000	9.3956	14.954	$1.1468(10^{-10})$	$-1.99(10^2)$
5000	22.875	36.406	$1.4600(10^{-15})$	$-2.97(10^2)$
10,000	45.340	72.161	$1.8059(10^{-23})$	$-4.55(10^2)$

TABLE 15

Plane Earth Theory

$h_2 = 0$
$\epsilon_2 = 15$
$\sigma = 0.005$
$\alpha = 0.75$
$f = 100$ kc

d(miles)	ϕ_c (radians)	t_c(microseconds)	$\lvert E_z \rvert$	$\lvert E_z \rvert$ (decibels)
0.01	3.1078	4.9462	6.8550	1.67(10)
0.02				
0.05				
0.1	2.7775	4.4205	$6.5103(10^{-3})$	-4.37(10)
0.2	2.2389	3.5633	$7.5457(10^{-4})$	-6.24(10)
0.5	0.78686	1.2522	$1.4292(10^{-4})$	-7.69(10)
1	0.39097	0.62225	$7.6655(10^{-5})$	-8.23(10)
2	0.25947	0.41296	$3.9164(10^{-5})$	-8.81(10)
5	0.22184	0.35307	$1.5627(10^{-5})$	-9.61(10)
10	0.27273	0.43407	$7.7806(10^{-6})$	$-1.02(10^2)$
20	0.35792	0.56964	$3.8491(10^{-6})$	$-1.08(10^2)$
50	0.54638	0.86959	$1.4977(10^{-6})$	$-1.16(10^2)$
100	0.74891	1.1919	$7.2817(10^{-7})$	$-1.23(10^2)$

TABLE 16

Spherical Earth Theory

$h_2 = 0$
$\epsilon_2 = 15$
$\sigma = 0.05$
$\alpha = 0.75$
$f = 100$ kc

d(miles)	ϕ_c(radians)	t_c(microseconds)	$\lvert E_r \rvert$	$\lvert E_r \rvert$ (decibels)
100	0.33119	0.52711	$7.1911(10^{-7})$	$-1.23(10^2)$
200	0.58296	0.92781	$3.1228(10^{-7})$	$-1.30(10^2)$
500	1.4030	2.2330	$6.8105(10^{-8})$	$-1.43(10^2)$
1000	2.8483	4.5332	$9.1205(10^{-9})$	$-1.61(10^2)$
2000	5.7533	9.1566	$2.3177(10^{-10})$	$-1.93(10^2)$
5000	14.468	23.026	$7.0652(10^{-15})$	$-2.83(10^2)$
10,000	28.992	46.143	$3.7451(10^{-22})$	$-4.29(10^2)$

TABLE 17

Plane Earth Theory

$h_2 = 0$
$\epsilon_2 = 15$
$\sigma = 0.05$
$\alpha = 0.75$
$f = 100$ kc

d(miles)	ϕ_c(radians)	t_c(microseconds)	$\lvert E_z \rvert$	$\lvert E_z \rvert$ (decibels)
0.01	3.1078	4.9463	6.8550	1.67(10)
0.02				
0.05				
0.1	2.7769	4.4196	$6.5060(10^{-3})$	-4.37(10)
0.2	2.2463	3.5751	$7.4685(10^{-4})$	-6.25(10)
0.5	0.75487	1.2014	$1.3903(10^{-4})$	-7.71(10)
1	0.33830	0.53842	$7.5471(10^{-5})$	-8.24(10)
2	0.18476	0.29406	$3.8813(10^{-5})$	-8.82(10)
5	0.11378	0.18108	$1.5633(10^{-5})$	-9.61(10)
10	0.10651	0.16951	$7.8159(10^{-6})$	$-1.02(10^2)$
20	0.12348	0.19652	$3.9034(10^{-6})$	$-1.08(10^2)$
50	0.17764	0.28272	$1.5567(10^{-6})$	$-1.16(10^2)$
100	0.24568	0.39101	$7.7495(10^{-7})$	$-1.22(10^2)$

TABLE 18

Spherical Earth Theory

$h_2 = 0$
$\epsilon_2 = 15$
$\sigma = 5$
$\alpha = 0.75$
$f = 100$ kc

d(miles)	ϕ_c(radians)	t_c(microseconds)	$\lvert E_r \rvert$	$\lvert E_r \rvert$ (decibels)
100	0.11026	0.17549	$7.2338(10^{-7})$	$-1.23(10^2)$
200	0.26421	0.42051	$3.1454(10^{-7})$	$-1.30(10^2)$
500	0.85319	1.3579	$6.7505(10^{-8})$	$-1.43(10^2)$
1000	1.9359	3.0811	$8.5871(10^{-9})$	$-1.61(10^2)$
2000	4.1134	6.5466	$1.9584(10^{-10})$	$-1.94(10^2)$
5000	10.645	16.942	$4.3161(10^{-15})$	$-2.87(10^2)$
10,000	21.532	34.269	$1.3324(10^{-22})$	$-4.38(10^2)$

TABLE 19

Plane Earth Theory

$h_2 = 0$
$\epsilon_2 = 15$
$\sigma = 5$
$\alpha = 0.75$
$f = 100$ kc

d(miles)	ϕ_c(radians)	t_c(microseconds)	$\lvert E_z \rvert$	$\lvert E_z \rvert$ (decibels)
0.01	3.1078	4.9463	6.8550	1.67(10)
0.02				
0.05				
0.1	2.7777	4.4209	$6.5040(10^{-3})$	-4.37(10)
0.2	2.2495	3.5802	$7.4382(10^{-4})$	-6.26(10)
0.5	0.74186	1.1807	$1.3740(10^{-4})$	-7.72(10)
1	0.31657	0.50384	$7.4949(10^{-5})$	-8.25(10)
2	0.15381	0.24479	$3.8640(10^{-5})$	-8.83(10)
5	0.064847	0.10321	$1.5594(10^{-5})$	-9.61(10)
10	0.037337	0.059424	$7.8067(10^{-6})$	$-1.02(10^{2})$
20	0.025684	0.040878	$3.9043(10^{-6})$	$-1.08(10^{2})$
50	0.023104	0.036771	$1.5617(10^{-6})$	$-1.16(10^{2})$
100	0.02725809	0.043383	$7.8083(10^{-7})$	$-1.22(10^{2})$

TABLE 20

Plane Earth Theory

$h_2 = 0$
$\epsilon_2 = 80$
$\sigma = 5$
$\alpha = 0.75$
$f = 100$ kc

d(miles)	ϕ_c(radians)	t_c(microseconds)	$\lvert E_z \rvert$	$\lvert E_z \rvert$ (decibels)
0.01	3.1078	4.9463	6.8550	1.67(10)
0.02				
0.05				
0.1	2.7777	4.4209	$6.5040(10^{-3})$	-4.37(10)
0.2	2.2495	3.5802	$7.4382(10^{-4})$	-6.26(10)
0.5	0.74186	1.1807	$1.3740(10^{-4})$	-7.72(10)
1	0.31657	0.50384	$7.4949(10^{-5})$	-8.25(10)
2	0.15381	0.24479	$3.8640(10^{-5})$	-8.83(10)
5	0.064846	0.10321	$1.5594(10^{-5})$	-9.61(10)
10	0.037237	0.059265	$7.8031(10^{-6})$	$-1.02(10^{2})$
20	0.025684	0.040878	$3.9043(10^{-6})$	$-1.08(10^{2})$
50	0.023104	0.036769	$1.5617(10^{-6})$	$-1.16(10^{2})$
100	0.027253	0.043375	$7.8085(10^{-7})$	$-1.22(10^{2})$

TABLE 21

Spherical Earth Theory

$h_2 = 0$
$\epsilon_2 = 15$
$\sigma = \infty$
$\alpha = 0.75$
$f = 100$ kc

d(miles)	ϕ_c(radians)	t_c(microseconds)	$\lvert E_r \rvert$	$\lvert E_r \rvert$ (decibels)
100	0.085609	0.13625	$7.2290(10^{-7})$	$-1.23(10^{2})$
200	0.22856	0.36376	$3.1403(10^{-7})$	$-1.30(10^{2})$
500	0.79170	1.2600	$6.7062(10^{-8})$	$-1.43(10^{2})$
1000	1.8339	2.9188	$8.4369(10^{-9})$	$-1.61(10^{2})$
2000	3.9299	6.2546	$1.8809(10^{-10})$	$-1.95(10^{2})$
5000	10.217	16.262	$3.8723(10^{-15})$	$-2.88(10^{2})$
10,000	20.697	32.940	$1.0671(10^{-22})$	$-4.39(10^{2})$

TABLE 22

Plane Earth Theory

$h_2 = 0$
$\epsilon_2 = 15$
$\sigma = \infty$
$\alpha = 0.75$
$f = 100$ kc

d(miles)	ϕ_c(radians)	t_c(microseconds)	$\lvert E_z \rvert$	$\lvert E_z \rvert$ (decibels)
0.01	3.1078	4.9463	6.8550	1.67(10)
0.02	3.0739	4.8923	$8.5542(10^{-1})$	-1.35
0.05	2.9697	4.7264	$5.4108(10^{-2})$	-2.53(10)
0.1	2.7778	4.4210	$6.5038(10^{-3})$	-4.37(10)
0.2	2.2498	3.5807	$7.4349(10^{-4})$	-6.26(10)
0.5	0.74042	1.1784	$1.3722(10^{-4})$	-7.73(10)
1	0.31416	0.50000	$7.4890(10^{-5})$	-8.25(10)
2	0.15037	0.23933	$3.8620(10^{-5})$	-8.83(10)
5	0.059415	0.094562	$1.5589(10^{-5})$	-9.61(10)
10	0.029655	0.047198	$7.8049(10^{-5})$	$-1.02(10^{2})$
20	0.014821	0.023589	$3.9038(10^{-6})$	$-1.08(10^{2})$
50	0.0059277	0.0094343	$1.5616(10^{-6})$	$-1.16(10^{2})$
100	0.0029638	0.0047170	$7.8083(10^{-7})$	$-1.22(10^{2})$

TABLE 23

Spherical Earth Theory

$h_2 = 0$
$\epsilon_2 = 15$
$\sigma = 0.005$
$\alpha = 0.1$
$f = 100$ kc

d(miles)	ϕ_c(radians)	t_c(microseconds)	$\lvert E_r \rvert$	$\lvert E_r \rvert$ (decibels)
2000	4.1049	6.5331	$5.0369(10^{-9})$	$-1.66(10^{2})$
5000	8.4637	13.470	$6.7123(10^{-11})$	$-2.03(10^{2})$
10,000	15.758	25.079	$7.9376(10^{-14})$	$-2.62(10^{2})$

TABLE 24

Spherical Earth Theory

$h_2 = 0$
$\epsilon_2 = 15$
$\sigma = 0.005$
$\alpha = 0.2$
$f = 100$ kc

d(miles)	ϕ_c(radians)	t_c(microseconds)	$\lvert E_r \rvert$	$\lvert E_r \rvert$ (decibels)
100	0.77633	1.2356	$6.9872(10^{-7})$	$-1.23(10^{2})$
200	1.1340	1.8047	$3.1129(10^{-7})$	$-1.30(10^{2})$
500	1.9270	3.0669	$8.6152(10^{-8})$	$-1.41(10^{2})$
1000	3.0631	4.8751	$2.2217(10^{-8})$	$-1.53(10^{2})$
2000	5.2805	8.4042	$2.5686(10^{-9})$	$-1.72(10^{2})$
5000	11.956	19.029	$7.7363(10^{-12})$	$-2.22(10^{2})$
10,000	23.085	36.742	$7.4435(10^{-16})$	$-3.03(10^{2})$

TABLE 25

Spherical Earth Theory

$h_2 = 0$
$\epsilon_2 = 15$
$\sigma = 0.005$
$\alpha = 0.5$
$f = 100$ kc

d(miles)	ϕ_c(radians)	t_c(microseconds)	$\lvert E_r \rvert$	$\lvert E_r \rvert$ (decibels)
100	0.82322	1.3102	$6.8417(10^{-7})$	$-1.23(10^2)$
200	1.2435	1.9791	$2.9148(10^{-7})$	$-1.31(10^2)$
500	2.3473	3.7358	$6.6128(10^{-8})$	$-1.44(10^2)$
1000	4.1522	6.6084	$1.0818(10^{-8})$	$-1.59(10^2)$
2000	7.7858	12.391	$4.3201(10^{-10})$	$-1.87(10^2)$
5000	18.692	29.750	$5.0061(10^{-14})$	$-2.66(10^2)$
10,000	36.870	58.680	$2.2215(10^{-20})$	$-3.93(10^2)$

TABLE 26

Spherical Earth Theory

$h_2 = 0$
$\epsilon_2 = 15$
$\sigma = 0.005$
$\alpha = 0.7$
$f = 100$ kc

d(miles)	ϕ_c(radians)	t_c(microseconds)	$\lvert E_r \rvert$	$\lvert E_r \rvert$(decibels)
100	0.84832	1.3501	$6.7161(10^{-7})$	$-1.23(10^2)$
200	1.3135	2.0905	$2.7724(10^{-7})$	$-1.31(10^2)$
500	2.6033	4.1432	$5.5030(10^{-8})$	$-1.45(10^2)$
1000	4.7606	7.5768	$6.7699(10^{-9})$	$-1.63(10^2)$
2000	9.9043	14.474	$1.4812(10^{-10})$	$-1.97(10^2)$
5000	22.096	35.167	$2.8660(10^{-15})$	$-2.91(10^2)$
10,000	43.766	69.656	$6.9346(10^{-23})$	$-4.43(10^2)$

TABLE 27

Spherical Earth Theory

$h_2 = 0$
$\epsilon_2 = 15$
$\sigma = 0.005$
$\alpha = 0.9$
$f = 100$ kc

d(miles)	ϕ_c(radians)	t_c(microseconds)	$\lvert E_r \rvert$	$\lvert E_r \rvert$ (decibels)
100	0.87344	1.3901	$6.5922(10^{-7})$	$-1.24(10^2)$
200	1.3821	2.1997	$2.6374(10^{-7})$	$-1.32(10^2)$
500	2.8447	4.5275	$4.5832(10^{-8})$	$-1.47(10^2)$
1000	5.3101	8.4512	$4.3086(10^{-9})$	$-1.67(10^2)$
2000	10.253	16.318	$5.4493(10^{-11})$	$-2.05(10^2)$
5000	25.080	39.917	$2.0858(10^{-16})$	$-3.14(10^2)$
10,000	49.793	79.249	$3.9010)10^{-25})$	$-4.88(10^2)$

TABLE 28

Spherical Earth Theory

$h_2 = 0$
$\epsilon_2 = 15$
$\sigma = 0.005$
$\alpha = 1$
$f = 100$ kc

d(miles)	ϕ_c(radians)	t_c(microseconds)	$\lvert E_r \rvert$	$\lvert E_r \rvert$ (decibels)
100	0.88590	1.4100	$6.5320(10^{-7})$	$-1.24(10^2)$
200	1.4160	2.2537	$2.5723(10^{-7})$	$-1.32(10^2)$
500	2.9606	4.7119	$4.1850(10^{-8})$	$-1.48(10^2)$
1000	5.5677	8.8613	$3.4591(10^{-9})$	$-1.69(10^2)$
2000	10.791	17.174	$3.3770(10^{-11})$	$-2.09(10^2)$
5000	26.461	42.114	$6.0353(10^{-17})$	$-3.24(10^2)$
10,000	52.577	83.680	$3.5924(10^{-26})$	$-5.09(10^2)$

TABLE 29

Spherical Earth Theory

$t_{h_2} - t_c$

for d (500 - 10,000 miles)

$\epsilon_2 = 15$
$\sigma = 5$
$\alpha = 0.75$
$f = 100$ kc

h_2(miles)	500	1000	2000	5000	10,000
5.5	0.49858	0.49879	0.49897	0.49861	0.49897
11.0	1.8152	1.8088	1.8100	1.8098	1.8097
16.5	3.7221	3.6771	3.6806	3.6806	3.6799
22.0	5.9454	5.9863	5.9866	5.9866	5.9865
27.5	8.8032	8.6338	8.4730	8.6455	8.6458
33.0		11.626	11.638	11.638	11.638

TABLE 30

Spherical Earth Theory

$h_2 = 0$
$\epsilon_2 = 15$
$\sigma = 0.005$
$\alpha = 0.75$
$f = 200$ kc

d(miles)	ϕ_c(radians)	t_c(microseconds)	$\lvert E_r \rvert$	$\lvert E_r \rvert$ (decibels)
100	1.6096	1.2809	$1.0156(10^{-6})$	$-1.20(10^2)$
200	2.4160	1.9226	$3.2272(10^{-7})$	$-1.30(10^2)$
500	4.5622	3.6305	$3.0609(10^{-8})$	$-1.50(10^2)$
1000	8.1059	6.4505	$1.1283(10^{-9})$	$-1.79(10^2)$
2000	15.209	12.103	$2.2145(10^{-12})$	$-2.33(10^2)$
5000	36.517	29.060	$3.1101(10^{-20})$	$-3.90(10^2)$
10,000	72.032	57.321	$4.5308(10^{-33})$	$-6.47(10^2)$

TABLE 31

Plane Earth Theory

$h_2 = 0$
$\epsilon_2 = 15$
$\sigma = 0.005$
$\alpha = 0.75$
$f = 200$ kc

d(miles)	ϕ_c(radians)	t_c(microseconds)	$\|E_z\|$	$\|E_z\|$ (decibels)
0.01	3.0738	2.4461	3.4217	1.07(10)
0.02	3.0094	2.3948	$4.2486(10^{-1})$	-7.43
0.05	2.7738	2.2073	$2.6060(10^{-2})$	-3.17(10)
0.1	2.2345	1.7782	$3.0384(10^{-3})$	-5.03(10)
0.2	1.0678	0.84973	$7.2333(10^{-4})$	-6.28(10)
0.5	0.42321	0.33678	$3.0914(10^{-4})$	-7.02(10)
1	0.30501	0.24272	$1.5718(10^{-4})$	-7.61(10)
2	0.29236	0.23266	$7.8327(10^{-5})$	-8.21(10)
5	0.37297	0.29680	$3.0846(10^{-5})$	-9.02(10)
10	0.49863	0.39680	$1.5099(10^{-5})$	-9.64(10)
20	0.68803	0.54752	$7.2712(10^{-6})$	$-1.03(10^2)$
50	1.0648	0.84731	$2.6214(10^{-6})$	$-1.12(10^2)$
100	1.4717	1.1712	$1.1117(10^{-6})$	$-1.19(10^2)$

TABLE 32

Plane Earth Theory

$h_2 = 0$
$\epsilon_2 = 15$
$\sigma = \infty$
$\alpha = 0.75$
$f = 200$ kc

d(miles)	ϕ_c(radians)	t_c(microseconds)	$\|E_z\|$	$\|E_z\|$ (decibels)
0.01	3.0739	2.4461	3.4217	1.07(10)
0.02	3.0050	2.3913	$4.2483(10^{-1})$	-7.44
0.05	2.7778	2.2105	$2.6015(10^{-2})$	-3.17(10)
0.1	2.2498	1.7904	$2.9739(10^{-3})$	-5.05(10)
0.2	1.0240	0.81489	$6.7731(10^{-4})$	-6.34(10)
0.5	0.31416	0.25000	$2.9956(10^{-4})$	-7.05(10)
1	0.15037	0.11966	$1.5448(10^{-4})$	-7.62(10)
2	0.074367	0.059179	$7.7870(10^{-5})$	-8.22(10)
5	0.029655	0.023599	$3.1220(10^{-5})$	-9.01(10)
10	0.014821	0.011794	$1.5615(10^{-5})$	-9.61(10)
20	0.0074098	0.0058965	$7.8082(10^{-6})$	$-1.02(10^2)$
50	0.0029638	0.0023585	$3.1233(10^{-6})$	$-1.10(10^2)$
100	0.0014819	0.0011793	$1.5617(10^{-6})$	$-1.16(10^2)$

TABLE 33

Spherical Earth Theory

$h_2 = 0$
$\epsilon_2 = 15$
$\sigma = 0.005$
$\alpha = 0.75$
$f = 500$ kc

d(miles)	ϕ_c(radians)	t_c(microseconds)	$\|E_r\|$	$\|E_r\|$ (decibels)
100	3.1443	1.0008	$5.7781(10^{-7})$	$-1.25(10^2)$
200	4.0613	1.2928	$8.1241(10^{-8})$	$-1.42(10^2)$
500	6.7974	2.1637	$1.4903(10^{-9})$	$-1.77(10^2)$
1000	11.601	3.6928	$3.5359(10^{-12})$	$-2.29(10^2)$
2000	21.217	6.7536	$2.7858(10^{-17})$	$-3.31(10^2)$
5000	50.064	15.936	$2.5322(10^{-32})$	$-6.32(10^2)$
10,000	98.143	31.240	$3.8484(10^{-58})$	

TABLE 34

Plane Earth Theory

$h_2 = 0$
$\epsilon_2 = 15$
$\sigma = 0.005$
$\alpha = 0.75$
$f = 500$ kc

d(miles)	ϕ_c(radians)	t_c(microseconds)	$\|E_z\|$	$\|E_z\|$ (decibels)
0.01	2.9684	0.94488	1.3531	2.63
0.02	2.7714	0.88215	$1.6310(10^{-1})$	-1.58(10)
0.05	1.8851	0.60006	$1.0477(10^{-2})$	-3.96(10)
0.1	0.84822	0.27000	$3.7364(10^{-3})$	-4.86(10)
0.2	0.48803	0.15534	$1.9556(10^{-3})$	-5.42(10)
0.5	0.39291	0.12507	$7.8351(10^{-4})$	-6.21(10)
1	0.44305	0.14103	$3.8417(10^{-4})$	-6.83(10)
2	0.56842	0.18093	$1.8599(10^{-4})$	-7.46(10)
5	0.85266	0.27141	$6.8786(10^{-5})$	-8.32(10)
10	1.1752	0.37409	$3.0651(10^{-5})$	-9.03(10)
20	1.6030	0.51024	$1.4411(10^{-5})$	-9.68(10)
50	2.3230	0.73943	$2.8035(10^{-6})$	$-1.11(10^2)$
100	2.8191	0.89735	$6.4959(10^{-7})$	$-1.24(10^2)$

TABLE 35

Plane Earth Theory

$h_2 = 0$
$\epsilon_2 = 15$
$\sigma = \infty$
$\alpha = 0.75$
$f = 500$ kc

d(miles)	ϕ_c(radians)	t_c(microseconds)	$\|E_z\|$	$\|E_z\|$ (decibels)
0.01	2.9697	0.94527	1.3527	2.63
0.02	2.7778	0.88420	$1.6259(10^{-1})$	-1.58(10)
0.05	1.9003	0.60489	$9.7834(10^{-3})$	-4.02(10)
0.1	0.74042	0.23568	$3.4306(10^{-3})$	-4.93(10)
0.2	0.31416	$1.00000(10^{-1})$	$1.8722(10^{-3})$	-5.46(10)
0.5	0.11967	$3.8091(10^{-2})$	$7.7541(10^{-4})$	-6.22(10)
1	0.059415	$1.8912(10^{-2})$	$3.8973(10^{-4})$	-6.82(10)
2	0.029655	$9.4396(10^{-3})$	$1.9512(10^{-4})$	-7.42(10)
5	0.011856	$3.7740(10^{-3})$	$7.8078(10^{-5})$	-8.21(10)
10	0.0059277	$1.8869(10^{-3})$	$3.9041(10^{-5})$	-8.82(10)
20	0.0029638	$9.4341(10^{-4})$	$1.9521(10^{-5})$	-9.42(10)
50	0.0011855	$3.7736(10^{-4})$	$7.8084(10^{-6})$	$-1.02(10^2)$
100	0.00059276	$1.8868(10^{-4})$	$3.9042(10^{-6})$	$-1.08(10^2)$

TABLE 36

Spherical Earth Theory

$h_2 = 0$
$\epsilon_2 = 15$
$\sigma = 0.005$
$\alpha = 0.75$
$f = 1000$ kc

d(miles)	ϕ_c(radians)	t_c(microseconds)	$\|E_r\|$	$\|E_r\|$ (decibels)
100	3.4947	0.55621	$1.6786(10^{-7})$	$-1.36(10^2)$
200	4.4153	0.70272	$1.9355(10^{-8})$	$-1.54(10^2)$
500	7.6717	1.2210	$1.1002(10^{-10})$	$-1.99(10^2)$
1000	13.244	2.1079	$3.0686(10^{-13})$	$-2.50(10^2)$
2000	24.388	3.8815	$3.3809(10^{-20})$	$-3.89(10^2)$
5000	57.820	9.2023	$8.4025(10^{-42})$	
10,000	113.54	18.070	$6.8276(10^{-76})$	

TABLE 37

Plane Earth Theory

$h_2 = 0$
$\epsilon_2 = 15$
$\sigma = 0.005$
$\alpha = 0.75$
$f = 1000$ kc

d(miles)	ϕ_c(radians)	t_c(microseconds)	$\lvert E_z \rvert$	$\lvert E_z \rvert$ (decibels)
0.01	2.7688	0.44066	$6.5373(10^{-1})$	-3.69
0.02	2.2211	0.35351	$7.8391(10^{-2})$	-2.21(10)
0.05	0.89769	0.14287	$1.5338(10^{-2})$	-3.63(10)
0.1	0.56134	0.089341	$7.8570(10^{-3})$	-4.21(10)
0.2	0.49521	0.078816	$3.8824(10^{-3})$	-4.82(10)
0.5	0.59429	0.094585	$1.4756(10^{-3})$	-5.66(10)
1	0.77502	0.12335	$6.9087(10^{-4})$	-6.32(10)
2	1.0486	0.16688	$3.0992(10^{-4})$	-7.02(10)
5	1.5686	0.24965	$9.4145(10^{-5})$	-8.05(10)
10	2.0708	0.32958	$3.1679(10^{-5})$	-9.00(10)
20	2.5744	0.40973	$8.2343(10^{-6})$	$-1.02(10^{2})$
50	2.9156	0.46404	$1.0279(10^{-6})$	$-1.20(10^{2})$
100	2.9484	0.46925	$2.3066(10^{-7})$	$-1.33(10^{2})$

TABLE 38

Plane Earth Theory

$h_2 = 0$
$\epsilon_2 = 15$
$\sigma = \infty$
$\alpha = 0.75$
$f = 1000$ kc

d(miles)	ϕ_c(radians)	t_c(microseconds)	$\lvert E_z \rvert$	$\lvert E_z \rvert$ (decibels)
0.01	2.7778	0.44210	$6.5038(10^{-1})$	-3.74
0.02	2.2498	0.35807	$7.4349(10^{-2})$	-2.26(10)
0.05	0.74042	0.11784	$1.3722(10^{-2})$	-3.73(10)
0.1	0.31416	0.050000	$7.4890(10^{-3})$	-4.25(10)
0.2	0.15037	0.023933	$3.8620(10^{-3})$	-4.83(10)
0.5	0.059415	0.0094562	$1.5589(10^{-3})$	-5.61(10)
1	0.029655	0.0047198	$7.8049(10^{-4})$	-6.22(10)
2	0.014821	0.0023589	$3.9038(10^{-4})$	-6.82(10)
5	0.0059277	0.00094343	$1.5616(10^{-4})$	-7.61(10)
10	0.0029638	0.00047171	$7.8083(10^{-5})$	-8.21(10)
20	0.0014189	0.00023585	$3.9042(10^{-5})$	-8.82(10)
50	0.00059276	0.000094341	$1.5617(10^{-5})$	-9.61(10)
100	0.00029638	0.000047170	$7.8084(10^{-6})$	$-1.02(10^{2})$

TABLE 39

$\epsilon_2 = 15$

$$\frac{\sigma \mu_o c^2}{\omega}$$

for f (10 - 1000 kc)

σ	10	20	50	100	200	500	1000
0.0001	$1.79754(10^{2})$	8.98771(10)	3.59508(10)	1.79754(10)	8.98771	3.59508	1.79754
0.0005	$8.98771(10^{2})$	$4.49386(10^{2})$	$1.79754(10^{2})$	8.98771(10)	4.49386(10)	1.79754(10)	8.98771
0.001	$1.79754(10^{3})$	$8.98771(10^{2})$	$3.59508(10^{2})$	$1.79754(10^{2})$	8.98771(10)	3.59508(10)	1.79754(10)
0.002	$3.59508(10^{3})$	$1.79754(10^{3})$	$7.19017(10^{2})$	$3.59508(10^{2})$	$1.79754(10^{2})$	7.19017(10)	3.59508(10)
0.005	$8.98771(10^{3})$	$4.49386(10^{3})$	$1.79754(10^{3})$	$8.98771(10^{2})$	$4.49386(10^{2})$	$1.79754(10^{2})$	8.98771(10)
0.05	$8.98771(10^{4})$	$4.49386(10^{4})$	$1.79754(10^{4})$	$8.98771(10^{3})$	$4.49386(10^{3})$	$1.79754(10^{3})$	$8.98771(10^{2})$
5	$8.98771(10^{6})$	$4.49386(10^{6})$	$1.79754(10^{6})$	$8.98771(10^{5})$	$4.49386(10^{5})$	$1.79754(10^{5})$	$8.98771(10^{4})$

TABLE 40

K_e

σ	10 kc	20 kc	50 kc	100 kc	200 kc	500 kc	1000 kc
0.0001	1.10809	$6.25504(10^{-1})$	$3.02532(10^{-1})$	$1.82779(10^{-1})$	$1.30283(10^{-1})$	$9.08437(10^{-2})$	$7.14624(10^{-2})$
0.0005	2.47312	1.38832	$6.48016(10^{-1})$	$3.65797(10^{-1})$	$2.09846(10^{-1})$	$1.09821(10^{-1})$	$7.61899(10^{-2})$
0.001	3.49731	1.96291	$9.15090(10^{-1})$	$5.14331(10^{-1})$	$2.90333(10^{-1})$	$1.40423(10^{-1})$	$8.71652(10^{-2})$
0.002	4.94587	2.77582	1.29366	$7.26307(10^{-1})$	$4.08225(10^{-1})$	$1.92151(10^{-1})$	$1.11454(10^{-1})$
0.005	7.82007	4.38888	2.04524	1.14792	$6.44400(10^{-1})$	$3.00782(10^{-1})$	$1.69788(10^{-1})$
0.05	$2.47292(10^{1})$	$1.38788(10^{1})$	6.46749	3.62976	2.03714	$9.49316(10^{-1})$	$5.32817(10^{-1})$
5	$2.47292(10^{2})$	$1.38788(10^{2})$	$6.46749(10^{1})$	$3.62975(10^{1})$	$2.03713(10^{1})$	9.49297	5.32775

TABLE 41

ψ_e

σ	10 kc	20 kc	50 kc	100 kc	200 kc	500 kc	1000 kc
0.0001	$4.43927(10^{-2})$	$8.81109(10^{-2})$	$2.09608(10^{-1})$	$3.64569(10^{-1})$	$5.30950(10^{-1})$	$6.75849(10^{-1})$	$7.29982(10^{-1})$
0.0005	$8.90050(10^{-3})$	$1.77955(10^{-2})$	$4.43927(10^{-2})$	$8.81109(10^{-2})$	$1.71164(10^{-1})$	$3.64569(10^{-1})$	$5.30950(10^{-1})$
0.001	$4.45060(10^{-3})$	$8.90050(10^{-3})$	$2.22385(10^{-2})$	$4.43927(10^{-2})$	$8.81109(10^{-2})$	$2.09608(10^{-1})$	$3.64569(10^{-1})$
0.002	$2.22536(10^{-3})$	$4.45060(10^{-3})$	$1.11250(10^{-2})$	$2.22385(10^{-2})$	$4.43927(10^{-2})$	$1.09521(10^{-1})$	$2.09608(10^{-1})$
0.005	$8.90141(10^{-4})$	$1.78028(10^{-3})$	$4.45060(10^{-3})$	$8.90050(10^{-3})$	$1.77955(10^{-2})$	$4.43927(10^{-2})$	$8.81109(10^{-2})$
0.05	$8.90142(10^{-5})$	$1.78028(10^{-4})$	$4.45071(10^{-4})$	$8.90141(10^{-4})$	$1.78028(10^{-3})$	$4.45060(10^{-3})$	$8.90050(10^{-3})$
5	$8.90142(10^{-7})$	$1.78028(10^{-6})$	$4.45071(10^{-6})$	$8.90142(10^{-6})$	$1.78028(10^{-5})$	$4.45071(10^{-5})$	$8.90142(10^{-5})$

TABLE 42

$f(\sigma, \epsilon)$

$\epsilon = 15$
$f = 100$ kc

σ	$R_e[f(\sigma,\epsilon)]$	$I_m[f(\sigma,\epsilon)]$
0.0001	$9.722882(10^{-1})$	$3.101972(10^{-2})$
0.0005	$9.980782(10^{-1})$	$1.079296(10^{-2})$
0.001	$9.995083(10^{-1})$	$5.523321(10^{-3})$
0.002	$9.998764(10^{-1})$	$2.777978(10^{-3})$
0.005	$9.999802(10^{-1})$	$1.113032(10^{-3})$
0.05	9.999980	$1.113382(10^{-4})$
5 ($\epsilon_2 = 15$)	1.000000	$1.113385(10^{-6})$
5 ($\epsilon_2 = 80$)	1.000000	$1.113383(10^{-6})$

$\epsilon_2 = 15$
$\sigma = 0.005$

f	$R_e[f(\sigma,\epsilon)]$	$I_m[f(\sigma,\epsilon)]$
20	$9.999992(10^{-1})$	$2.226737(10^{-4})$
100	$9.999802(10^{-1})$	$1.113032(10^{-3})$
200	$9.999208(10^{-1})$	$2.223957(10^{-3})$
500	$9.995083(10^{-1})$	$5.523319(10^{-3})$
1000	$9.980782(10^{-1})$	$1.079297(10^{-2})$

TABLE 43

$$\frac{1}{(ik_1 d)} + \frac{1}{(ik_1 d)^2}$$

$f = 100$ kc

d	$\frac{1}{(ikd)^2}$	$\frac{1}{(ikd)}$
0.01	$-8.7840826(10^2)$	$i\,2.9637953(10)$
0.02	$-2.1960207(10^2)$	$i\,1.4818976(10)$
0.05	$-3.5136330(10)$	$i\,5.9275906$
0.1	-8.7840826	$i\,2.9637953$
0.2	-2.1960207	$i\,1.4818976$
0.5	$-3.5136330(10^{-1})$	$i\,5.9275906(10^{-1})$
1	$-8.7840826(10^{-2})$	$i\,2.9637953(10^{-1})$
2	$-2.1960207(10^{-2})$	$i\,1.4818976(10^{-1})$
5	$-3.5136330(10^{-3})$	$i\,5.9275906(10^{-2})$
10	$-8.7840826(10^{-4})$	$i\,2.9637953(10^{-2})$
20	$-2.1960207(10^{-4})$	$i\,1.4818976(10^{-2})$
50	$-3.5136330(10^{-5})$	$i\,5.9275906(10^{-3})$
100	$-8.7840826(10^{-6})$	$i\,2.9637953(10^{-3})$
200	$-2.1960207(10^{-6})$	$i\,1.4818976(10^{-3})$

TABLE 44

s	$\tau_{s,0}$	$\tau_{s,\infty}$	s	$\tau_{s,0}$	$\tau_{s,\infty}$
0	1.85575708	0.808616516	25	19.45383898	19.20085366
1	3.24460762	2.57809613	26	19.95428298	19.70453341
2	4.38167124	3.82571528	27	20.44852842	20.20185516
3	5.38661378	4.89182029	28	20.93687144	20.69312830
4	6.30526301	5.85130097	29	21.41958427	21.17863681
5	7.16128272	6.73731638	30	21.89691791	21.65864212
6	7.96889165	7.56829093	31	22.36910440	22.13338559
7	8.73747153	8.35580960	32	22.83635881	22.60309063
8	9.47362183	9.10775848	33	23.29888096	23.06796458
9	10.18220685	9.82981304	34	23.75585692	23.52820029
10	10.86694205	10.52623016	35	24.21046034	23.98397750
11	11.53074627	11.20030653	36	24.65985356	24.43546415
12	12.17596542	11.85466121	37	25.10518866	24.88281736
13	12.80452070	12.49141870	38	25.54660838	25.32618449
14	13.41801060	13.11233258	39	25.98424688	25.76570393
15	14.01778319	13.71887155	40	26.41823048	26.20150586
16	14.60498862	14.31228141	41	26.84867830	26.63371297
17	15.18061824	14.89363039	42	27.27570281	27.05244101
18	15.74553413	15.46384328	43	27.69941041	27.48779937
19	16.30049193	16.02372745	44	28.11990179	27.90989158
20	16.84615869	16.57399308	45	28.53727244	28.32881568
21	17.38312698	17.11526902	46	28.95161299	28.74466471
22	17.91192624	17.64811556	47	29.36300957	29.15752704
23	18.43303197	18.17303452	48	29.77154409	29.56748664
24	18.94687327	18.69047771	49	30.17729458	29.97462349

8. Appendix I. Computation Formulas

The wave number k is given as follows: For the atmosphere at the surface of the earth:

$$k_1=\frac{\omega}{c}\sqrt{\epsilon_1}=\omega\sqrt{\kappa\mu_0}=\frac{\omega}{c}\,\eta_1 \quad \text{(radians per meter)} \tag{67}$$

or for a medium of finite conductivity, σ, such as the earth:

$$k_2{}^2=\omega^2\kappa\mu_0+i\omega\sigma\mu_0 \quad \text{(radians per meter)}^2 \tag{68}$$

$$\kappa=\frac{\epsilon}{c^2\mu_0} \quad \text{(farads per meter)} \tag{69}$$

$$k_2{}^2=\frac{\omega^2}{c^2}\left[\epsilon_2+i\,\frac{\sigma\mu_0c^2}{\omega}\right]. \tag{70}$$

The field of the vertical electric dipole source was found to be completely described by the following scalar components:

8.1. Spherical Earth Theory [17]

$$E_r=\frac{-1}{r\sin\theta}\frac{\partial}{\partial\theta}\left[\sin\theta\,\frac{\partial\Pi}{\partial\theta}\right]. \tag{71}$$

$$E_\theta=\frac{1}{r}\,\frac{\partial^2}{\partial\theta\partial r}\,[r\Pi] \tag{72}$$

$$H_\phi=\frac{k^2}{\mu_0\,\frac{\partial}{\partial t}}\,\frac{\partial\Pi}{\partial\theta}. \tag{73}$$

8.2. Plane-Earth Theory

$$E_z=k^2\Pi+\frac{\partial^2}{\partial z^2}\,\Pi \tag{74}$$

$$E_r=\frac{\partial^2\Pi}{\partial z\partial R} \tag{75}$$

$$H_\phi=\frac{k^2}{\mu_0 i\omega}\,\frac{\partial\Pi}{\partial R}. \tag{76}$$

In the spherical-earth theory, the quantity Π was computed from the following formula [6, 7]:

$$\Pi=2\left[\frac{I_0l}{4\pi\kappa\omega}\right]\exp\,[ik_1a\theta]\left[\frac{2\pi(k_1a)^{1/3}\alpha^{11/3}}{a^4\sin\alpha\theta}\right]^{1/2}\sum_{s=0}^{\infty}\frac{f_s(h_1)f_s(h_2)}{[2\tau_s-1/\delta_e{}^2]}\exp\left\{i\left[(k_1a)^{1/3}\tau_s\alpha^{2/3}\theta+\frac{\theta\alpha}{2}+\frac{\pi}{4}\right]\right\}$$

(volts×meters). (77)

$$E_r=2\left[\frac{I_0l}{4\pi\kappa}\right]\exp\,[ik_1a\theta]\left[\frac{2\pi(k_1a)^{13/3}\alpha^{5/3}}{\omega^2a^6\sin\alpha\theta}\right]^{1/2}\sum_{s=0}^{\infty}\frac{f_s(h_1)f_s(h_2)}{[2\tau-1/\delta_e^2]}\exp\left\{i\left[(k_1a)^{1/3}\tau_s\alpha^{2/3}\theta+\frac{\theta\alpha}{2}+\frac{\pi}{4}\right]\right\}$$

(volts per meter). (78)

For $\sin\alpha\theta\sim\alpha\theta\sim\alpha(d/a)$; $\theta=(d/a)$, $r=a+h_2$,

$$E_r=2E_{pr}F_r \quad \text{(volts per meter)}. \tag{79}$$

[17] See figure 1.

The following computational formulas may be used to evaluate F_r (eq (27)):

$$\delta_e=\frac{i\,\dfrac{k_2^2}{k_1^2}\,\alpha^{1/3}}{(k_1a)^{1/3}\left[\dfrac{k_2^2}{k_1^2}-1\right]^{1/2}} \tag{80}$$

$$\delta_e=K_e\,\exp\left\{i\left[\frac{3\pi}{4}-\psi_e\right]\right\} \tag{81}$$

$$K_e=\frac{\left[\dfrac{c\alpha}{\epsilon_1^2\omega a}\right]^{1/3}\left[\epsilon_2^2+\dfrac{\sigma^2\mu_0^2c^4}{\omega^2}\right]^{1/2}}{\left[(\epsilon_2-\epsilon_1)^2+\dfrac{\sigma^2\mu_0^2c^4}{\omega^2}\right]^{1/4}} \tag{82}$$

$$\psi_e=\tan^{-1}\left[\frac{\omega\epsilon_2}{\mu_0c^2\sigma}\right]-\frac{1}{2}\tan^{-1}\left[\frac{\omega(\epsilon_2-\epsilon_1)}{\mu_0c^2\sigma}\right] \tag{83}$$

$$f_s(h_2)=\left[\frac{(k_1a)^{2/3}\,\dfrac{2h_2}{a}\,\alpha^{1/3}-2\tau_s}{-2\tau_s}\right]^{1/2}\frac{H_{1/3}^{(1)}\left\{\dfrac{1}{3}\left[(k_1a)^{2/3}\,\dfrac{2h_2\alpha^{1/3}}{a}-2\tau_s\right]^{3/2}\right\}}{H_{1/3}^{(1)}\left\{\dfrac{1}{3}\,(-2\tau_s)^{3/2}\right\}}. \tag{84}$$

The functions $H_{1/3}^{(1)}(z)$ are Hankel functions [6, 7]. It is often more convenient to use modified Hankel functions $h(z)$ for which values with complex argument $(z=x+iy)$ have been tabulated [12]:

$$f_s(h_2)=\frac{h\left\{\left(\dfrac{1}{2}\right)^{2/3}\left[(k_1a)^{2/3}\,\dfrac{2h_2\alpha^{1/3}}{a}-2\tau_s\right]\right\}}{h\{-(2)^{1/3}\tau_s\}}. \tag{85}$$

τ_s was calculated from Riccati's differential equation [6, 7] as follows:

$$\frac{d\delta_e}{d\tau_s}-2\delta_e^2\tau_s+1=0$$

$$\frac{d\tau_s}{d\delta_e}=\frac{1}{2\tau_s\delta_e^2-1}$$

$$\frac{d\tau_s}{d\left(\dfrac{1}{\delta_e}\right)}=-\delta_e^2\,\frac{d\tau_s}{d\delta_e}=\frac{1}{\dfrac{1}{\delta_e^2}-2\tau_s} \tag{86}$$

$$\tau_s=\sum_{n=0}^{\infty}\frac{1}{n!}\left[\frac{d^n\tau_s}{d\delta_e^n}\right]_{\substack{\delta_e=0\\ \tau_s=\tau_{s,0}}}\cdot\delta_e^n \tag{87}$$

$$|\delta^2\tau|<\frac{1}{2},\qquad K_e\ll 1$$

$$\tau_s=\sum_{n=0}^{\infty}\frac{1}{n!}\left[\frac{d^n\tau_s}{d\left(\dfrac{1}{\delta_e}\right)^n}\right]_{\substack{\delta_e=\infty\\ \tau_s=\tau_{s,\infty}}}\cdot\frac{1}{\delta_e^n} \tag{88}$$

$$|\delta^2\tau|>\frac{1}{2}\qquad K_e\gg 1$$

$$\tau_s=\tau_{s,0}-\delta-\frac{2}{3}\tau_{s,0}\delta^3+\frac{1}{2}\delta^4-\frac{4}{5}\tau_{s,0}^2\delta^5+\frac{14}{9}\tau_{s,0}\delta^6-\frac{1}{7}(5+8\tau_{s,0}^3)\delta^7+\frac{58}{15}\tau_{s,0}^2\delta^8$$

$$-\left(\frac{328}{81}\tau_{s,0}+\frac{16}{9}\tau_{s,0}^4\right)\delta^9+\left(\frac{423}{315}+\frac{1552}{175}\tau_{s,0}^3\right)\delta^{10}-\left(\frac{7576}{495}\tau_{s,0}^2+\frac{32}{11}\tau_{s,0}^5\right)\delta^{11}+\ldots \qquad (89)$$

$$|\delta^2\tau|>\frac{1}{2} \qquad K_e\gg 1$$

$$\tau_s=\tau_{s,\infty}-\left[\frac{1}{2\tau_{s,\infty}}\right]\frac{1}{\delta}-\left[\frac{1}{8\tau_{s,\infty}^3}\right]\frac{1}{\delta^2}-\left[\frac{1}{12\tau_{s,\infty}^2}+\frac{1}{16\tau_{s,\infty}^5}\right]\frac{1}{\delta^3}-\left[\frac{7}{96\tau_{s,\infty}^4}+\frac{5}{128\tau_{s,\infty}^7}\right]\frac{1}{\delta^4}$$

$$-\left[\frac{1}{40\tau_{s,\infty}^3}+\frac{21}{320\tau_{s,\infty}^6}+\frac{7}{256\tau_{s,\infty}^9}\right]\frac{1}{\delta^5}-\left[\frac{29}{720\tau_{s,\infty}^5}+\frac{77}{1280\tau_{s,\infty}^8}+\frac{21}{1024\tau_{s,\infty}^{11}}\right]\frac{1}{\delta^6}$$

$$-\left[\frac{1}{112\tau_{s,\infty}^4}+\frac{19}{360\tau_{s,\infty}^7}+\frac{143}{2560\tau_{s,\infty}^{10}}+\frac{33}{2048\tau_{s,\infty}^{13}}\right]\frac{1}{\delta^7}-\left[\frac{97}{4480\tau_{s,\infty}^6}+\frac{163}{2560\tau_{s,\infty}^9}+\frac{429}{8192\tau_{s,\infty}^{12}}+\frac{429}{32768\tau_{s,\infty}^{15}}\right]\frac{1}{\delta^8}$$

$$-\left[\frac{1}{288\tau_{s,\infty}^5}+\frac{13661}{362880\tau_{s,\infty}^8}+\frac{6769}{92160\tau_{s,\infty}^{11}}+\frac{2431}{49152\tau_{s,\infty}^{14}}+\frac{715}{65536\tau_{s,\infty}^{17}}\right]\frac{1}{\delta^9}$$

$$-\left[\frac{2309}{201600\tau_{s,\infty}^7}+\frac{820573}{14515200\tau_{s,\infty}^{10}}+\frac{37961}{460800\tau_{s,\infty}^{13}}+\frac{46189}{983040\tau_{s,\infty}^{16}}+\frac{2431}{262144\tau_{s,\infty}^{19}}\right]\frac{1}{\delta^{10}}+\ldots \qquad (90)$$

$$|\delta^2\tau|<\frac{1}{2} \qquad K_e\ll 1.$$

Values of $\tau_{s,0}$ and $\tau_{s,\infty}$ are given in table 44 [13].

In the plane-earth theory, the vertical electric field intensity at the surface of the earth was found to be as follows:

$$E_z=2E_{pr}F_z \quad \text{(volts per meter)}. \qquad (91)$$

The following computational formulas may be used to evaluate F_z (eq (31)):

$$y(\rho_1)=i\sqrt{\pi\rho_1}\,e^{-\rho_1}+\sum_{n=0}^{\infty}\frac{(-1)^n2^n(n!)}{(2n)!}(2\rho_1)^n \qquad |\rho_1|\ll 1 \qquad (92)$$

$$y(\rho_1)=-\sum_{n=1}^{\infty}\frac{(2n)!}{2^n(n!)}\frac{1}{(2\rho_1)^n} \qquad |\rho_1|\gg 1. \qquad (93)$$

ρ_1 is the numerical distance described first by Sommerfeld [11]:

$$\rho_1=\frac{(k_1a)^{1/3}\dfrac{d}{a}\,\alpha^{2/3}}{2i\,\delta_e^2}=\frac{ik_1^3}{2k_2^2}\left[1-\frac{k_1^2}{k_2^2}\right]d \qquad (\alpha=1) \quad \text{(radians)}. \qquad (94)$$

The numerical distance, ρ_1, is related to the spherical-earth theory parameters K_e and ψ_e and can be computed therefrom if it is convenient.

$$f(\sigma,\epsilon)=1-\frac{\epsilon_1\epsilon_2}{\epsilon_2^2+\left[\frac{\sigma\mu_0c^2}{\omega}\right]^2}+\frac{\epsilon_1\left\{\epsilon_2^2-\left[\frac{\sigma\mu_0c^2}{\omega}\right]^2\right\}}{\left\{\epsilon_2^2+\left[\frac{\sigma\mu_0c^2}{\omega}\right]^2\right\}^2}+i\frac{\left[\frac{\sigma\mu_0c^2}{\omega}\right]\left\{\epsilon_1\epsilon_2^2+\epsilon_1\left[\frac{\sigma\mu_0c^2}{\omega}\right]^2-2\epsilon_1\epsilon_2\right\}}{\left\{\epsilon_2^2+\left[\frac{\sigma\mu_0c^2}{\omega}\right]^2\right\}^2}. \qquad (95)$$

9. Appendix II. Glossary

d=the distance, meters, along the surface of a plane or spherical earth.

$d_{\text{miles}}=d\times 0.62136996\times 10^{-3}$.

ρ_1=the numerical distance of Sommerfeld [11].

δ_e=the conductivity and permittivity parameter, for a vertical dipole source,

$$\delta_e=K_e \exp\left\{i\left[\frac{3\pi}{4}-\psi_e\right]\right\}.$$

K_e=the modulus or amplitude of δ_e.

ψ_e=the argument or phase of δ_e.

ϵ=the permittivity, esu.

ϵ_1=the permittivity of the air at the surface of the earth, esu. A value of 1.000676 was assumed.

ϵ_2=the permittivity of the earth, esu. A value of 15 was assumed.

κ=the permittivity, *mks*, farads per meter

$$\kappa=\frac{\epsilon}{\mu_0 c^2}.$$

μ_0=a universal constant, henrys per meter, the permeability of free space. A value of $4\pi\times 10^{-7}$ was assumed.

c=a universal constant, meters per second. A value of 2.997951×10^{8} was assumed [14]. A value of 2.997529×10^{8} is frequently assumed [17].

σ=the conductivity of the medium, specifically, the earth, mhos per meter.

a=the radius of the spherical earth, meters. A value of 6.36739×10^{6} was assumed.

a_e=the "effective radius" of the spherical earth, meters.

k'=the "effective radius factor" of the spherical earth, meters.

α=the parameter associated with the vertical lapse of the permittivity of the atmosphere,

$$\alpha=\frac{a}{a_e}=\frac{1}{k'}.$$

f=the frequency, cycles per second.

ω=the frequency, radians per second.

h_1=the altitude of the source above the surface of the earth, meters.

h_2=the altitude, meters.

$h_{2\,\text{miles}}=h_2\times 0.62136996\times 10^{-3}$.

ϕ=the phase, radians.

t=the time, seconds (microseconds).

ϕ'=the phase in free space or over infinitely conducting ($\sigma=\infty$) earth, $\phi'=k_1d$, radians.

ϕ_c=the phase of the secondary factor, F, radians. $F=|F|\exp[i\phi_c]$.

t_c=the phase of the secondary factor, microseconds.

$[t_{ch_2}-t_c]$=the phase of the secondary factor, F, aloft, relative to the surface value.

$|E|$=the amplitude of the electric field intensity.

$\frac{d}{\lambda}$=the distance in wavelengths from the source.

λ=the wavelength, meters.

k=the wave number, radians per meter. $k=\omega/c$ in free space.

k_1=the wave number of the atmosphere at the surface of the earth, radians per meter,

$$k_1=\frac{\omega}{c}\sqrt{\epsilon_1}=\frac{\omega}{c}\,\eta_1.$$

η=the index of refraction.

η_1=the index of refraction of the atmosphere at the surface of the earth, $\eta_1=\sqrt{\epsilon_1}$.

η_2=the index of refraction of the atmosphere at some point aloft.

k_2=the wave number of the earth, radians per meter,

$$k_2=\frac{\omega}{c}\left[\epsilon_2+\frac{i\sigma\mu_0c^2}{\omega}\right]^{1/2}.$$

$\overline{\Pi}$=the Hertz vector, volts $\times$ meters.

$\overline{\Omega}$=the source vector, $\overline{\Omega}=\overline{\Omega}(t)$, $[\overline{\Omega}]=\overline{\Omega}(t-\eta d/c)$, $\bar{\imath}=\partial\overline{\Omega}/\partial t$ amperes per square meter.

$\overline{E}$=the electric intensity, volts per meter.

$\bar{\imath}$=the source, amperes per square meter.

E_r=the scalar vertical electric field intensity over spherical earth, volts per meter.

E_z'=the free space vertical electric field intensity, cylindrical coordinates, volts per meter.

E_z=the vertical electric field intensity over plane earth, volts per meter.

F=the secondary factor, dimensionless,

$$F=\frac{E}{2E_{pr}}\quad\left|\quad\begin{array}{l}\sigma=0,\quad F=\frac{1}{2}\\ \sigma=\infty,\quad F=1.\end{array}\right.$$

F_r=the secondary factor, computed from spherical-earth theory.

F_z=the secondary factor, computed from plane-earth theory or free space, cylindrical coordinates.

F_z'=the secondary factor, free space.

E_{pr}=the field intensity in free space, volts per meter; $E_{pr}=E_z'$ in cylindrical coordinates.

I_0=the amplitude of the source current, amperes.

$\overline{H}$=the magnetic field intensity, amperes per meter.

$f_s(h_1)$=the "height gain" factor of the source. A value of unity (1) was assumed, i. e., it was assumed that the source (transmitter) was on the surface of the earth.

$f_s(h_2)$=the "height gain" factor of the observer (receiver).

θ=the angular distance from the source over spherical earth, radians, $\theta=d/a$.

r=the distance from the center of spherical earth, meters. $r=a+h_2$.

$|E|$ (decibels)=20 $\log_{10}|E|$.

t_g=group delay, secondary factor, seconds (microseconds).

v_c=phase velocity, meters per second.

v_g=group velocity, meters per second.

v_s=signal velocity, meters per second.

∇^2=the operator of Laplace.

BOULDER, March 14, 1956.

U. S. GOVERNMENT PRINTING OFFICE: 1956