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Sunspot cycle simulation using random noise

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Abstract—The square of a narrowband Gaussian process is used to simulate sunspot cycles at computer speeds. The method is appealing because: (i) the model is extremely simple yet its physical basis, a simple resonance, is a widely occurring natural phenomenon, and (ii) the model recreates practically all of the features of the observed sunspot record. In particular, secular cycles and recurring extensive minima are characteristic of narrowband Gaussian processes. Additionally, the model lends itself to limited prediction of sunspot cycles.

Since the discovery of the cyclic behavior of sunspots by Schwabe in 1843, many authors have referred to the sunspot record as an example of naturally occurring periodic behavior. Yule (1927) characterized the sunspot numbers as a "disturbed harmonic function" which he likened to the motion of a pendulum which boys are pelting with peas. Time series analysis texts (Anderson, 1971; Koopmans, 1974; Bloomfield, 1976) and statistical works (Spencer-Smith, 1944; Moran, 1954) commonly cite the sunspot number series as a function which is more or less periodic. The noisy, but nearly periodic, character of the sunspot record has led the authors to a very simple model of solar activity which mimics the observed sunspot numbers to a surprising degree. The observed annual mean sunspot numbers (Eddy, 1976) and simulated annual mean sunspot numbers (produced using methods described in this paper) are shown in Fig. 1.

While the annual mean sunspot numbers display a more or less periodic behavior, they are of necessity always positive. These two facts suggest a model based on narrowband noise (the periodic part) which is squared (to insure positivity). If one assumes the noise part to be Gaussian, then the square of the Gaussian noise is distributed as a Chi-square distribution with one degree of freedom. Indeed, the annual mean sunspot numbers since 1650 are reasonably well characterized by such a distribution (see Fig. 2).

In order to model the actual annual mean sunspot numbers more closely, two cosmetic features have been added: (a) a broadband ("white") noise, and (b) a rise/fall correction to simulate the rapid rise and slower fall observed in the larger sunspot cycles. This completes the model which was used to produce the simulated data plotted in Fig. 1. Of course, the rise/fall correction slightly distorts the distribution function from a perfect Chi-square distribution and, in fact, the simulated distribution is closer to the actually observed distribution than is the Chi-square distribution as shown in Fig. 2.

For computer simulation, the narrowband noise plus the broadband noise is generated by an Auto Regressive, Moving Average (ARMA) model (Box and Jenkins, 1970). The output of the ARMA model is squared and a rise/fall correction applied. The simulation equations are as follows:

$Z_n = a_n = 0$ for $n < 1$ (initial conditions)

$Z_n = \phi_1 Z_{n-1} + \phi_2 Z_{n-2} + a_n - \theta_1 a_{n-1} - \theta_2 a_{n-2}$ (ARMA model)

where* $\phi_1 = 1.90693$ $\theta_1 = 0.78512$

$\phi_2 = -0.98751$ $\theta_2 = -0.40662$

Z_n are the output of the ARMA model, and

a_n are random, normal deviates with zero mean and standard deviation $\sigma_a = 0.4$.

$X_n = Z_n^2$ (square of Z_n)

$Y_n = X_n + \alpha(X_{n-1} - X_{n-2})^2$ (rise/fall correction)

where $\alpha = 0.03$, and the Y_n simulate annual mean sunspot numbers.

In terms of "physical interpretations", the ARMA model corresponds to the filtering of "white", Gaussian noise by a filter with a bandwidth of about 0.002 cycles/year centered at 1/22 cycles per year. This corresponds to a "Q" of about 23. Squaring the output effectively doubles the frequency to 1/11 cycles per year.

Using the ARMA coefficients, the model was run to produce thousands of years of simulated sunspot numbers. The cycles shown in Fig. 3 are representative of this simulated data.

Certain portions of Fig. 3 look very familiar and would rapidly be identified as sunspot cycles by an uncritical observer. Throughout hundreds of thousands of years of simulated cycles, there are frequent spans of data that are very reminiscent of the actually observed sunspot record. Indeed, one can find, without much trouble, patterns in cycle-to-cycle amplitude almost exactly like those described in the literature and sometimes used for sunspot cycle forecasts.

A striking feature of the simulated sunspot data is the occasional (yet fairly regular) occurrence of extensive sunspot minima (referred to in this paper as Eddy minima). If such a minimum is defined (arbitrarily) as a period of at least 50 years during which the annual mean sunspot number does not exceed 20, then these minima are observed to occur in the simulated series at the rate of twice per thousand years, on the average. Some extremely long minima show up in the simulations. For example, an Eddy minimum spanning more than 500 years is shown in the bottom row of Fig. 3.

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*Since the coefficients ϕ_1, ϕ_2, θ_1, and θ_2 interact with each other, the number of significant digits given here is very large relative to the confidencc intervals of the "physical" parameters. Dropping digits can materially alter the model beyond what one might normally expect, because roots of the "operator" equation are changed significantly. This is often an annoying feature of digital filters and does not imply exactness in the overall model.

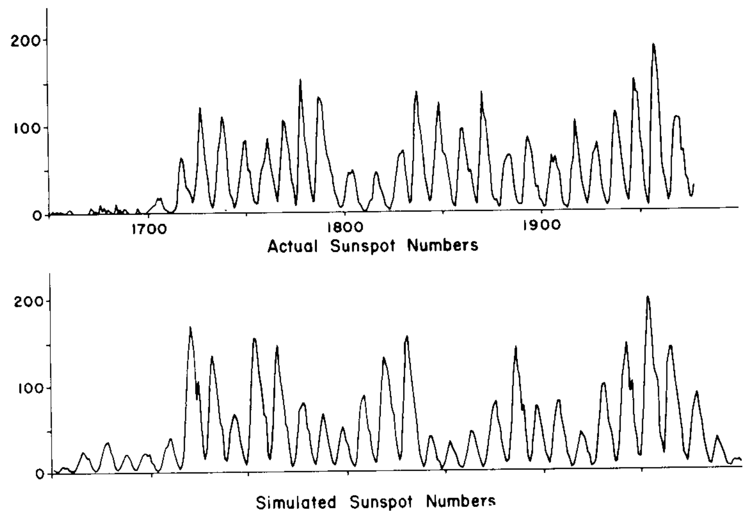

Fig. 1. Actual and simulated annual, mean sunspot numbers.

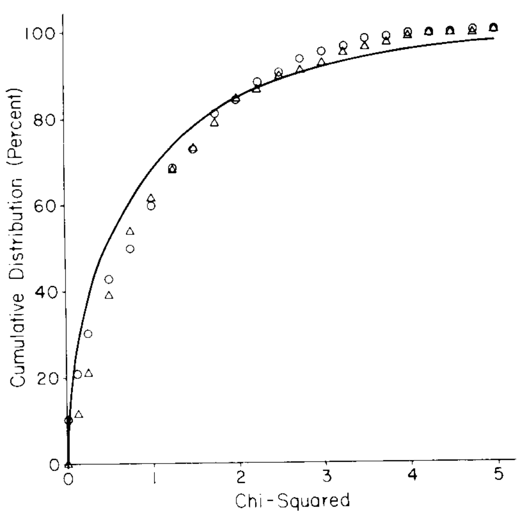

Fig. 2. Cumulative distributions for observed annual mean sunspot numbers (circles) and simulated annual mean sunspot numbers (triangles) divided by their means for 328 years. The solid curve is the Chi-square distribution for one degree of freedom.

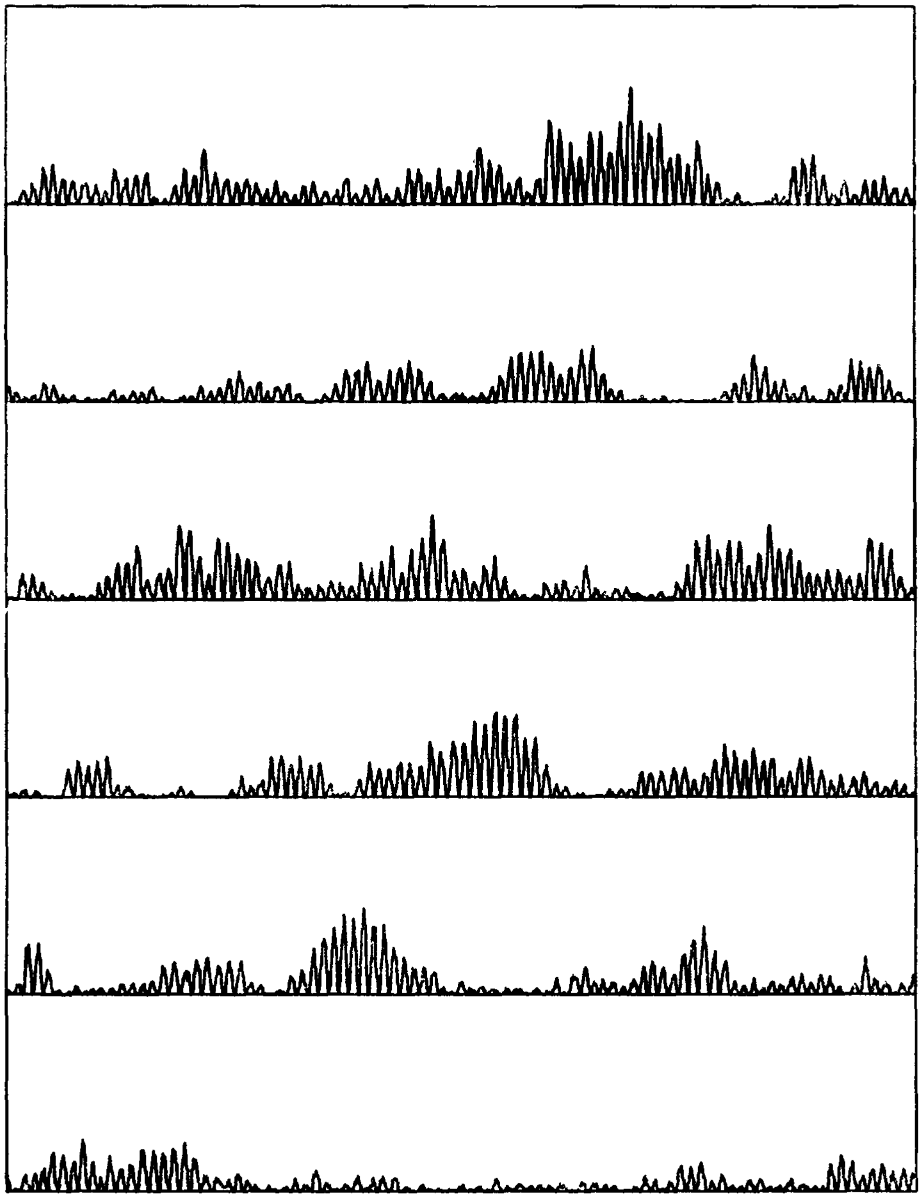

Fig. 3. 6000 consecutive years of simulated annual mean sunspot numbers. Each line is 1000 years long and the lines are separated by a relative sunspot number of 400.

The model simulates (a) the approximately eleven-year period, (b) the variability of period, (c) the relative variability of amplitude (including Eddy minima), (d) the average amplitude, (e) the short term (year-to-year) fluctuations, (f) the rapid rise and slow decay, (g) the observed distribution of values, and (h) the general appearance of sunspot cycles. Further, the model is extremely simple. A computer program in Basic which simulates the sunspot cycles is shown below:

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100 X = RND (-2) : A = 1.90693 : B = - .98751
110 C = .78512 : D = - .40662 : E = .4 : F = .03 : G = 0
130 FOR N = 1 TO 300
140 IF G = 1 THEN GOTO 180
150 X = RND (X) : Y = SQR ( - 2 * LOG (X))
160 X = RND (X) : K = Y * E * COS (6.28318 * X)
170 G = 1 : GOTO 190
180 G = 0 : K = Y * E * SIN (6.28318 * X)
190 H = A * I + B * J + K - C * L - D * M
200 M = L : L = K : S = I * I - J * J
210 T = H * H + F * S * S
220 PRINT T
230 J = I : I = H
240 NEXT N
250 STOP
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