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TWO PAPERS ON THE STATISTICS OF PRECISION FREQUENCY GENERATORS

by

James A. Barnes

and

David W. Allan

U. S. DEPARTMENT OF COMMERCE
NATIONAL BUREAU OF STANDARDS
BOULDER LABORATORIES
Boulder, Colorado

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PREFACE

The following two papers represent a treatment of frequency modulation noise that is typically found in many signal generators. The papers are not aimed at pinpointing the noise sources, but rather, at methods of measuring and classifying the particular spectral types of noise that an individual signal generator may exhibit.

The first paper, "Atomic Timekeeping and the Statistics of Precision Signal Generators," is devoted to the consistent treatment of "flicker noise" type of frequency modulation with applications to time standards and measures of frequency stability. The second paper, "Statistics of Atomic Frequency Standards," presents practical approaches to the classification (as to spectral type) of noises that might be found on signal generators in general with a specific application to the maser type of oscillator.

PAPER I

ATOMIC TIMEKEEPING AND THE STATISTICS OF PRECISION SIGNAL GENERATORS

by

J. A. Barnes

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ABSTRACT

Since most systems that generate atomic time employ quartz crystal oscillators to improve reliability, it is essential to determine the effect on the precision of time measurements that these oscillators introduce. A detailed analysis of the calibration procedure shows that the third finite difference of the phase is closely related to the clock errors. It was also found, in agreement with others, that quartz crystal oscillators exhibit a "flicker" or $|\omega|^{-1}$ type of noise modulating the frequency of the oscillator.

The method of finite differences of the phase is shown to be a powerful means of classifying the statistical fluctuations of the phase and frequency for signal generators in general. By employing finite differences it is possible to avoid divergences normally associated with flicker noise spectra. Analysis of several cesium beam frequency standards have shown a complete lack of the $|\omega|^{-1}$ type of noise modulation but an unexplained noise with a spectral variation proportional to $|\omega|^{-1/3}$ for frequency fluctuations was observed for $|\omega|$ less than 10^{-3} to 10^{-4} sec $^{-1}$ depending on the particular standard.

Key words: Atomic frequency standards, time standards, and stability.

Atomic Timekeeping and the Statistics of Precision Signal Generators

James A. Barnes

Introduction

An ordinary clock consists of two basic systems: a periodic phenomenon (pendulum), and a counter (gears, clock face, etc.) to count the periodic events. An atomic clock differs from this only in that the frequency of the periodic phenomenon is, in some sense, controlled by an atomic transition (atomic frequency standard). Since microwave spectroscopic techniques allow frequencies to be measured with a relative precision far better than any other quantity, the desirability of extending this precision to the domain of time measurement has long been recognized.[(1)]

From the standpoint of precision, it would be desirable to run the clock (counter) directly from the atomic frequency standard. However, atomic frequency standards in general are sufficiently complex that reliable operation over very extended periods becomes somewhat doubtful (to say nothing of the cost involved). For this reason, a quartz crystal oscillator is often used as the source of the "periodic" events to run a synchronous clock (or its electronic equivalent). The frequency of this oscillator is then regularly checked by the atomic frequency standard and corrections made.

These corrections can usually take on any of three forms; (a) correction of the oscillator frequency, (b) correction of the indicated time, or (c) an accumulating record of the difference from atomic time of the apparent or indicated time shown by the clock. Both methods (a) and (b) require a calculation of the time difference and it is sufficient to consider only the last method and the errors inherent in it.

A careful consideration of the calibration procedure leads to the development of certain functionals of the phase which have a very important property-- existence of the variance even in the presence of a flicker $\left(\frac{1}{|\omega|}\right)$ type of frequency noise. The simplest of these functionals- the second and third finite differences of the phase- turn out to be stationary, random variables whose auto-covariance function is sufficiently peaked to insure rapid convergence of the variance of a finite sample toward the true (infinite sample) variance. These functionals of the phase have the added features of being closely related to the errors of a clock run from the oscillator as well as being a useful measure of oscillator stability.

With the aid of these functionals, it is possible to classify the statistical fluctuations observed in various signal sources. In agreement with work done by others, [5-8] a flicker noise frequency modulation was observed for all quartz crystal oscillators tested. Contrary to expectations and some published works, [8,15,16] it was found that, of the several cesium beams tested, all were frequency modulated by a noise whose spectral shape was essentially $|\omega|^{-1/3}$ for small $|\omega|$. The source of such a noise spectrum is still unexplained. Similar studies on several commercial rubidium gas cells gave uniform indications of flicker noise modulation of levels comparable to those of the better quartz crystal oscillators.

In Chapter I, the effects, on the precision of a time scale due entirely to the calibration procedure of the quartz crystal oscillator and the oscillator's inherent frequency instability, are considered.

In Chapter II the experimental results of Chapter I are used as the basis for a theoretical model of oscillator frequency fluctuations and the results are compared to other experimenters.

In Chapter III the statistics of an atomic frequency standard of the passive type (e. g., Cs-beam or Rb-gas cell) are considered. Also in Chapter III the composite clock system is treated. Chapter IV is devoted to a brief discussion of stability measures for signal sources.

Chapter V is devoted to an explaination of the NBS-A time system and an evaluation of its precision.

I. QUARTZ CRYSTAL OSCILLATOR PHASE FLUCTUATIONS

Typical Gross Behavior

Figure 1 shows a typical aging curve for a fairly good quartz crystal oscillator. The oscillator had been operating for a few months prior to the date shown in the graph. On May 1, 1963, the frequency of the oscillator was reset in order to maintain relatively small corrections.

Least square fits of straight lines to the two parts of Figure 1, yield aging rates of 0.536×10^{-10} per day and 0.515×10^{-10} per day, respectively. This difference in aging rates could be explained by an acceleration of the frequency of about -9×10^{-15} per day per day. This acceleration of the frequency is sufficiently small over periods of a few days when compared to other sources of error that it can be safely ignored. Thus the frequency of conventional quartz oscillators can be written in the form

$$\Omega(t) = \Omega_o \left[1 + \alpha t + \dot{\varepsilon}(t)\right] \tag{1}$$

where α is the aging rate, $\dot{\varepsilon}(t)$ is a variation of the frequency probably caused by noise processes in the oscillator itself, and t can be considered to be some rather gross measure of the time (since α and $\dot{\varepsilon}$ are quite small corrections).

Actual Calibration Procedure of a Clock System

The fundamental equation for atomic timekeeping is[(2)]

$$\Omega = \frac{d\phi}{dt}, \tag{2}$$

where Ω is the instantaneous frequency of the oscillator as measured by an atomic frequency standard, $d\phi$ is the differential phase change, and dt is an increment of time as generated by this clock system. Since <u>time</u> is to be generated by this system, and ϕ and Ω are the directly measured

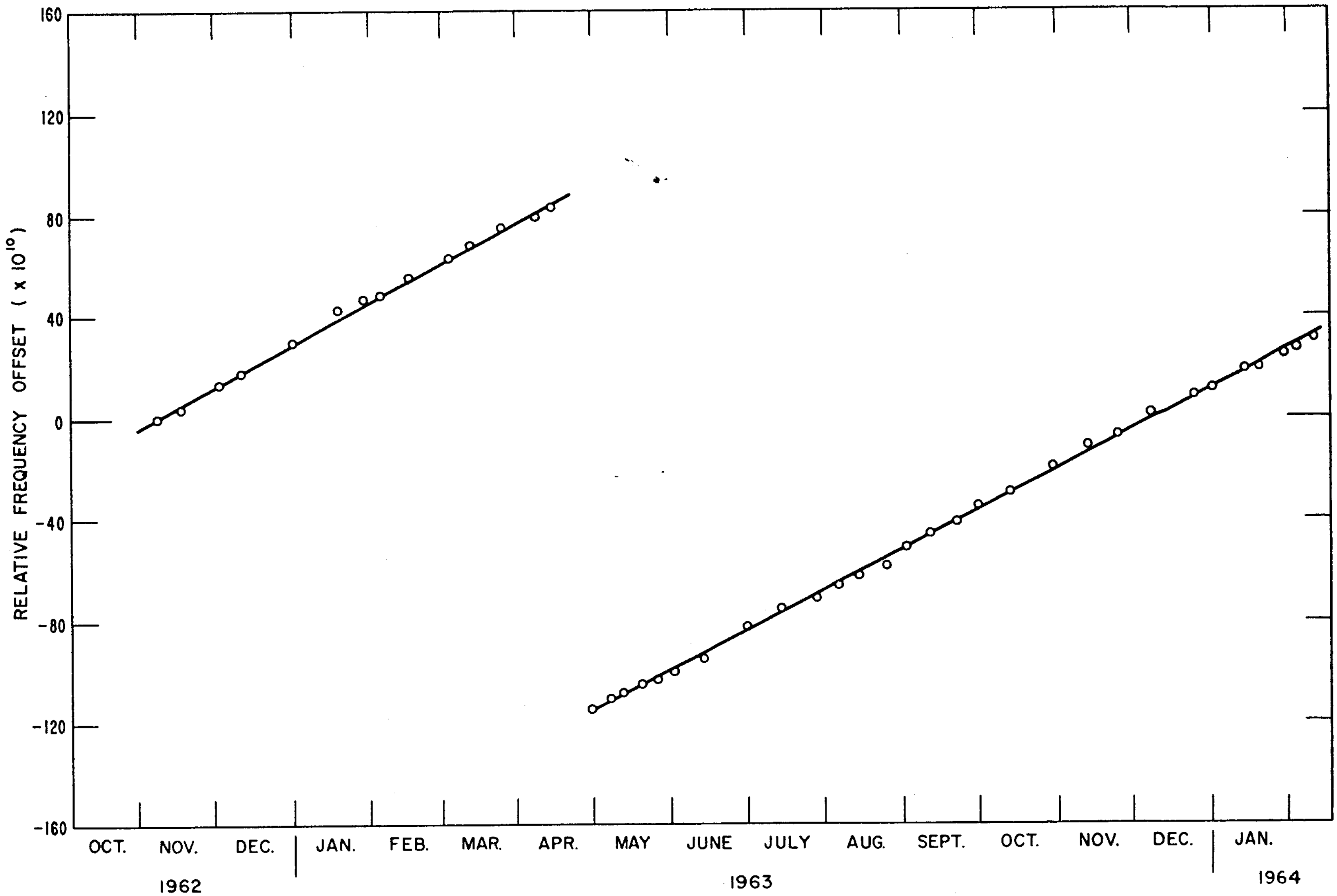

FIG. I FREQUENCY AGING OF A GOOD QUARTZ CRYSTAL OSCILLATOR quantities, it is of convenience to assume that $\Omega = \Omega(\phi)$ and write the solution of Equation (2) in the form,

$$\Delta t = \int_{\phi_1}^{\phi_2} \frac{d\phi}{\Omega(\phi)} \quad . \tag{3}$$

If one divides the output phase of the oscillator by Ω_o, and defines the apparent or indicated time, t_A, to be

$$t_A \equiv \frac{\phi}{\Omega_o} \quad , \tag{4}$$

Equations (1), (3), and (4) can be combined to give

$$\Delta t = \int_{t_{A1}}^{t_{A2}} \frac{dt_A}{1+\alpha t_A + \dot{\varepsilon}(t_A)} \quad . \tag{5}$$

As it is indicated in Figure 1, it is possible to maintain the magnitude of the relative frequency offset, $|\alpha t_A + \dot{\varepsilon}|$, within fixed bounds of 10^{-8}. Expanding Equation (5) to first order in this relative frequency offset yields

$$\Delta t \simeq \Delta t_A - \int_{t_{A1}}^{t_{A2}} [\alpha t_A + \dot{\varepsilon}(t_A)]\, dt_A \quad . \tag{6}$$

Equation (6) should then be valid to about one part in 10^{16}.

Normally the frequency of the oscillator is measured over some period of time (usually a few minutes) at regular intervals (usually a few days). At this point it is desirable to restrict the discussion to the case where the calibration is periodic (i. e. , period T, determined by t_A) and then generalize to other situations later. One period of the calibration is as follows:

t_A	start of calibration interval
$t_A+\frac{1}{2}(T-\tau)$	start of frequency measurement ($\tau<T$)
$t_A+\frac{1}{2}(T+\tau)$	end of frequency measurement
t_A+T	end of calibration interval.

If $\dot{\varepsilon}$ were constant in time, the frequency measured during the interval $t_A+\frac{1}{2}(T-\tau)$ to $t_A+\frac{1}{2}(T+\tau)$ would be just the average frequency during the complete measurement interval, T, since the oscillator would have an exactly linear drift in frequency. Also, if $\dot{\varepsilon}$ were constant, Equation (6) could be written as

$$\Delta t = \Delta t_A - T \left\langle \frac{\delta\Omega}{\Omega_o} \right\rangle_\tau , \tag{7}$$

where $\left\langle \frac{\delta\Omega}{\Omega_o} \right\rangle_\tau$ is the average relative frequency offset during the interval $t_A+\frac{1}{2}(T-\tau)$ to $t_A+\frac{1}{2}(T+\tau)$.

Even though, in general, $\dot{\varepsilon}$ is not constant, ε and $\dot{\varepsilon}$ are not knowable, and thus one is usually reduced to using Equation (7) anyway. The problem, then, is to determine how much error is introduced by using Equation (7).

The time error, δt, accumulated over an interval T committed by using Equation (7) can be expressed in the form,

$$\delta t = \int_{t_A}^{t_A+T} [\alpha t_A + \dot{\varepsilon}]\, dt_A - T \left\langle \frac{\delta\Omega}{\Omega_o} \right\rangle_\tau , \tag{8}$$

where the quantity $\langle \frac{\delta\Omega}{\Omega_o} \rangle_\tau$ is given by

$$\langle \frac{\delta\Omega}{\Omega_o} \rangle_\tau = \frac{1}{\tau} \int_{t_A + \frac{1}{2}(T-\tau)}^{t_A + \frac{1}{2}(T+\tau)} [\alpha t_A + \dot{\varepsilon}] \, dt_A \ . \tag{9}$$

Equations (8) and (9) can be combined to give

$$\delta t = \varepsilon(t_A + T) - \varepsilon(t_A) - \frac{T}{\tau} \left[\varepsilon(t_A + \frac{T+\tau}{2}) - \varepsilon(t_A + \frac{T-\tau}{2}) \right] . \tag{10}$$

It is this equation which relates the random phase fluctuations with the corresponding errors in the time determination.

Meaningful Quantities

It is again of value to further restrict the discussion to a particular situation and generalize at a later point. In particular, let $T = 3\tau$ then Equation (10) becomes

$$\delta t = \varepsilon(t_A + 3\tau) - \varepsilon(t_A) - 3 \left[\varepsilon(t_A + 2\tau) - \varepsilon(t_A + \tau) \right] . \tag{11}$$

It is now possible to define the discrete variable, ε_n, by the relation

$$\varepsilon_n \equiv \varepsilon(t_A + n\tau), \quad n = 0, 1, 2 \ldots$$

and rewrite Equation (11) in the simpler form (see Table I)

$$\delta t = \Delta^3 \varepsilon_n \tag{12}$$

where $\Delta^3 \varepsilon_n$ is the third finite difference of the discrete variable, ε_n.

TABLE I

FINITE DIFFERENCES

$$\varepsilon_n \equiv \varepsilon(t_A + n\tau)$$

Discrete variable		Definition
ε_n	ε_n	$\varepsilon(t_A + n\tau)$
$\Delta\varepsilon_n$	$\varepsilon_{n+1} - \varepsilon_n$	$\varepsilon(t_A + \tau) - \varepsilon(t_A)$
$\Delta^2\varepsilon_n$	$\Delta\varepsilon_{n+1} - \Delta\varepsilon_n$	$\varepsilon(t_A + 2\tau) - 2\varepsilon(t_A + \tau) + \varepsilon(t_A)$
$\Delta^3\varepsilon_n$	$\Delta^2\varepsilon_{n+1} - \Delta^2\varepsilon_n$	$\varepsilon(t_A + 3\tau) - 3[\varepsilon(t_A + 2\tau) - \varepsilon(t_A + \tau)] - \varepsilon(t_A)$
$\Delta^4\varepsilon_n$	$\Delta^3\varepsilon_{n+1} - \Delta^3\varepsilon_n$	$\varepsilon(t_A + 4\tau) - 4[\varepsilon(t_A + 3\tau) + \varepsilon(t_A + \tau)]$ $+ 6\varepsilon(t_A + 2\tau) + \varepsilon(t_A)$

Similarly, Equation (2) may be integrated directly using Equation (1) to obtain

$$\phi(t_A) = \Omega_o \left[t_A + \frac{\alpha}{2} t_A^2 + \varepsilon(t_A) \right] + \phi(o) \; . \tag{13}$$

Thus, by defining another discrete variable, ϕ_n, the third difference of Equation (13) yields the relation

$$\Delta^3 \phi_n = \Omega_o \Delta^3 \varepsilon_n \tag{14}$$

or equivalently

$$\delta t = \left(\frac{1}{\Omega_o} \right) \Delta^3 \phi_n \; . \tag{15}$$

It is now possible to set up a table of meaningful quantities for time measurement (see Table II).

Experimental Determination of Phase Fluctuations

If one measures the phase difference between two oscillators, Equation (15) applies to both, and hence the difference phase, $\Theta_n = \phi_n^{(1)} - \phi_n^{(2)}$, is related to the difference time, $\delta t_{12} = \delta t_1 - \delta t_2$, by the relations,

$$\delta t_{12} = \frac{1}{\Omega_o} \Delta^3 \Theta_n = \Delta^3 \varepsilon_n^{(1)} - \Delta^3 \varepsilon_n^{(2)} \; . \tag{16}$$

Provided that the cross-correlation coefficient, $\langle (\Delta^3 \varepsilon_n^{(1)})(\Delta^3 \varepsilon_n^{(2)}) \rangle$, is zero (i.e., the $\varepsilon_n^{(i)}$ are non-correlated), the variance of δt_{12} becomes

$$\sigma_{12}^2 \equiv \langle (\delta t_{12})^2 \rangle = \langle (\Delta^3 \varepsilon_n^{(1)})^2 \rangle + \langle (\Delta^3 \varepsilon_n^{(2)})^2 \rangle \; , \tag{17}$$

TABLE II

MEANINGFUL QUANTITIES

QUANTITY		DISCUSSION
Phase,	ϕ_n	Must exist for all times of interest and be measurable
First difference of the phase,	$\Delta\phi_n$	Proportional to the average frequency in the interval τ. Must exist for all time.
Second difference of the phase,	$\Delta^2\phi_n$	Related to the drift of the oscillator frequency. Must exist for all time.
Variance of the second difference	$\langle[\Delta^2\phi_n - \langle\Delta^2\phi_n\rangle]^2\rangle$	(It is possible to construct a time scale even if this does not exist. However, experiment indicates that it is probably finite.)
Third difference of the phase,	$\Delta^3\phi_n$	Proportional to the clock error inthe time interval $T = 3\tau$. Must exist if clock is to be of value.
Mean square third difference,	$\langle(\Delta^3\phi_n)^2\rangle$	Proportional to the precision of time interval measurements. Must exist if clock is to be of value.

Where all averages are defined by the relation:

$$\langle f(t)\rangle = \lim_{T\to\infty} \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)\,dt \; .$$

since it is assumed that $\langle \Delta^3 \varepsilon_n \rangle = 0$. The assumption that the cross-correlation coefficients vanish is equivalent to postulating an absence of linear coupling between the oscillators either electrically or through their environment. It is of value to develop a scheme which is capable of classifying individual oscillators rather than treating ensembles of assumed identical members. Thus this development is restricted to time averages of individual oscillators rather than ensemble averages.

If three oscillators are used, it is possible to independently measure the three quantities σ_{12}, σ_{13}, and σ_{23}. Thus there exist three independent equations:

$$\left.\begin{aligned} \sigma_{12}^2 &= \sigma_1^2 + \sigma_2^2 \\ \sigma_{13}^2 &= \sigma_1^2 + \sigma_3^2 \\ \sigma_{23}^2 &= \sigma_2^2 + \sigma_3^2 \end{aligned}\right\} . \tag{18}$$

While the three equations,

$$\delta t_{ij} = \Delta^3 \varepsilon_n^{(i)} - \Delta^3 \varepsilon_n^{(j)} \qquad \begin{cases} i, j = 1, 2, 3 \\ i < j \end{cases} ,$$

are not linearly independent, the standard deviations σ_{ij}^2 given by Equations (18), in fact, form linearly independent equations (subject only to certain conditions analogous to the triangle inequalities). Thus the systems of Equations (18) are solvable for the three quantities $\sigma_i^2 = \langle (\delta t_i)^2 \rangle$. It is thus possible to estimate the statistical behavior of each individual oscillator.

The block diagram of the phase meter which was used is shown in Figure 2. The basic system was alligned with an electrical-to-mechanical angle tolerance of about ± 0.25% of one complete cycle, and since the phase meter is operated at 10 Mc/s, this implies the possibility of measuring the time difference to ± 0.25 nanoseconds. The output shaft in turn drives a digital encoder with one hundred counts per revolution and a total accumulation (before starting over) of one million. Thus the digital information is accurate to within one nanosecond and accumulates up to one millisecond. Since measurements are made relatively often, it is easy for a computer to spot when the digital encoder has passed one millisecond on the data and this restriction on the data format in no way hampers the total length of the data handled.

Since the oscillator is assumed to have a linear drift in frequency with time, the variance (about the mean) of the second difference of the phase should depend only on the $\varepsilon_n^{(i)}$. For the evaluation of clocks, it is the mean square time error which is important and thus the mean square (rather than variance) of the third difference of the phase is the quantity of importance. Calculations of these quantities from the phase, Θ_n, were accomplished on a digital computer. Normally the phase difference is printed every hour for at least 200 hours and the computer program computes the $\Delta^2\Theta_n$ and $\Delta^3\Theta_n$ for τ equal to 2, 4, 6, 8, etc., hours. Thus it is possible to plot the square root of the variance of the second difference and the root mean square third difference of the phase as a function of the variable τ. A typical plot of these two quantities appears in Figure 3.

Since many revolutions of the phase meter are normally encountered between data points, the combined effects of non-perfect

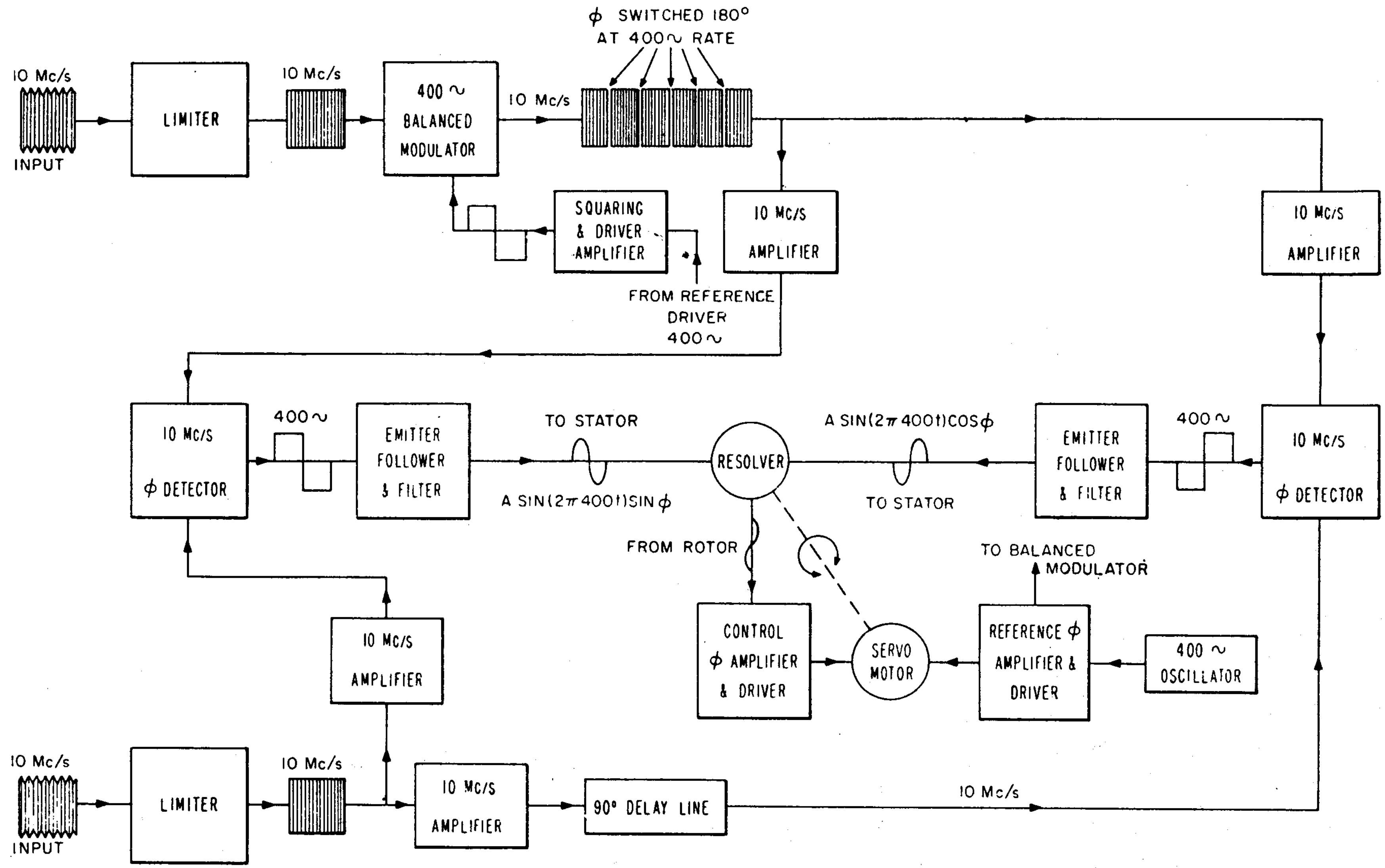

FIG. 2 BLOCK DIAGRAM OF 10 Mc/s, ELECTRICAL–TO–MECHANICAL PHASE CONVERTER

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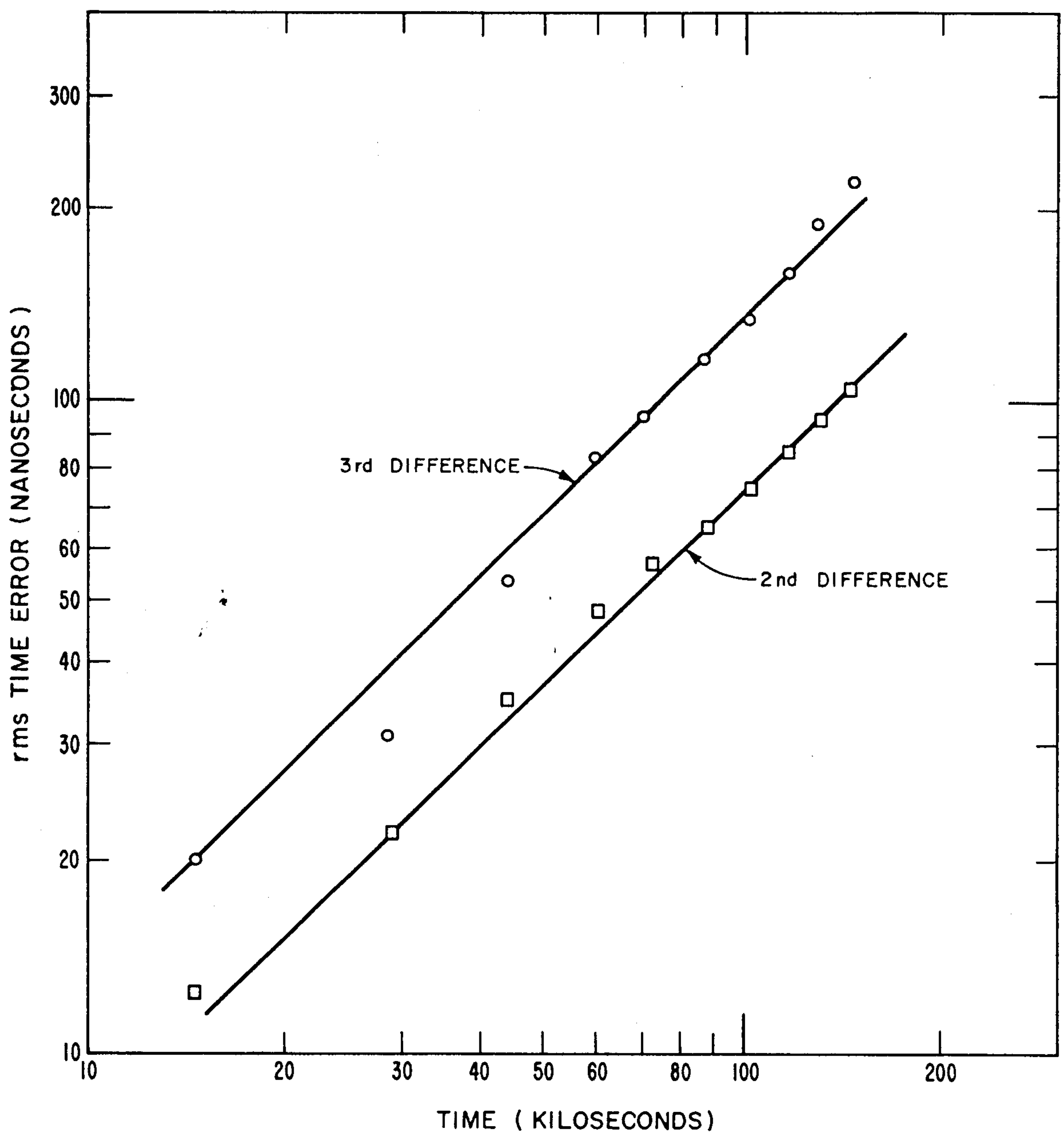

FIG. 3 VARIANCE OF SECOND AND THIRD DIFFERENCE AS A FUNCTION OF τ

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electrical-to-mechanical phase and rounding errors of the digital encoder can be combined into one stationary, statistical quantity, γ, that can be assumed to have delta-function auto-correlation. Let χ represent the rounding errors of the digital encoder defined by the difference between the encoded number and the actual angular position of the shaft. Then these rounding errors form a rectangular distribution from -0.5 nanoseconds to +0.5 nanoseconds, and thus contribute an amount

$$\frac{\int_{-\frac{1}{2}}^{\frac{1}{2}} \chi^2 d\chi}{\int_{-\frac{1}{2}}^{\frac{1}{2}} d\chi} = 0.08 \quad \text{nanoseconds}^2$$

to $\langle \gamma^2 \rangle$. The non-perfect electrical-to-mechanical phase conversion is sinusoidal in nature and thus contributes an amount equal to

$$\tfrac{1}{2}(0.25)^2 \simeq 0.03 \quad \text{nanoseconds}^2.$$

The final value for $\langle \gamma^2 \rangle$ should then be about 0.11 nanosecond^2.

This was checked by taking two very good oscillators operating at a rather large difference frequency ($\sim 5\text{x}10^{-9}$), and printing the phase difference every 10 seconds for several minutes. Since frequency fluctuations of the oscillators are small compared to $1\text{x}10^{-10}$, the resultant scatter can be attributed to the measuring system. The results of this experiment gave the value 0.19 nanosecond^2 for $\langle \gamma^2 \rangle$.

Since γ is assumed to have a delta-function autocorrelation, reference to Table I shows that $\langle (\Delta^3 \gamma)^2 \rangle = 20 \langle \gamma^2 \rangle$, and thus Equation (18) may be more precisely written,

$$\left.\begin{aligned} \sigma_{12}^2 &= \sigma_1^2 + \sigma_2^2 + 20 \langle \gamma^2 \rangle \\ \sigma_{13} &= \sigma_1^2 + \sigma_3^2 + 20 \langle \gamma^2 \rangle \\ \sigma_{23} &= \sigma_2^2 + \sigma_3^2 + 20 \langle \gamma^2 \rangle \end{aligned}\right\} . \qquad (18')$$

For the best oscillators tested, σ_{ij}^2 became about $2\langle (\Delta^3 \gamma)^2 \rangle$ for $\tau = 10^3$ sec., and therefore measurements were limited on the lower end to 20 min. or 1200 sec. The longest run made lasted a little over a month or about one thousand hours. Thus the largest value of τ which might have reasonable averaging is about one hundred hours or 3.6×10^5 sec. This limits the results to about two orders of magnitude variation on τ.

While it is possible to build appropriate frequency multipliers and mixers to improve the resolution of the phase meter and reduce the lower limit on τ, it was considered that the longer time intervals are of greater interest because T $(=3\tau)$ is normally between one day and one week.

II. THEORETICAL DEVELOPMENT

Introductory Remarks

In the development which follows certain basic assumptions are made. It is assumed that the coefficients Ω_o and α of Equation (1) may be so chosen that the average value of $\varepsilon(t)$ is zero; i.e.,

$$\langle \varepsilon(t) \rangle = 0.$$

It is also assumed that a translation in the time axis, $t_A \rightarrow t_A + \xi$, (stationarity) causes no change in the value of the autocovariance function,

$$\begin{aligned} R_\varepsilon(\tau) &\equiv \langle \varepsilon(t) \cdot \varepsilon(t+\tau) \rangle \\ &\equiv \langle \varepsilon(t+\xi) \cdot \varepsilon(t+\xi+\tau) \rangle. \end{aligned} \tag{19}$$

The justification of this assumption lies completely in the fact that the results of the analysis agree well with experiment and the results of others. In the development which follows, one cannot assume that

$$R_\varepsilon(0) = \langle [\varepsilon(t)]^2 \rangle$$

exists (i.e., is finite) and hence reference (17) cannot guarantee that Equations (19) are valid. While it may be that $R_\varepsilon(\tau)$ does not exist, quantities such as

$$U(\tau) = 2[R_\varepsilon(0) - R_\varepsilon(\tau)] \tag{20}$$

may exist and be meaningful if limits are approached properly. It is thus assumed that relations such as Equation (20) have meaning and may be handled by conventional means.

Development of Experimental Results

All oscillator pairs tested which exhibited a stable drift rate as indicated in Figure 1 and did not have obvious diurnal fluctuations in frequency, showed a definite, very nearly linear dependence on τ for both the standard deviation of the second difference and the root mean square third difference of the phase. It was observed that if an oscillator were disturbed accidentally during a measurement, the plot would have a more nearly $\tau^{3/2}$ dependence. This is probably because the assumption of a negligible quadratic dependence of frequency with time is not valid when the oscillator is disturbed. All oscillators tested, therefore, were shock mounted and all load changes and physical conditions were changed as little as possible.

A least square fit of all reliable data to an equation of the form

$$\sqrt{\langle (\Delta^m \varepsilon_n)^2 \rangle} = k_m \tau^\mu \qquad m = 2, 3, 4 \tag{21}$$

gave a value of 1.09 for the average of the μ's. The values of μ ranged from about 0.90 to 1.15 (and even to 1.5 when the oscillators were disturbed during the measurement). It is of interest to postulate that, for an "ideal," undisturbed quartz crystal oscillator, the value of μ is exactly the integer one, and investigate the consequences of this assumption.

Because certain difficulties arise at the value $\mu = 1$, it is essential to calculate with a general μ and then pass to the limits $\mu \to 1^{(-)}$ for the quantities of interest. This is equivalent to considering a sequence of processes which approach, as a limit, the case of the "ideal" crystal oscillator. Thus equation (21) may be rewritten for m=2 in the form

$$\langle (\Delta^2 \varepsilon_n)^2 \rangle = k_2 |\tau|^{2\mu} . \tag{22}$$

Using Table I, Equation (22) may then be rewritten as

$$6\langle[\varepsilon(t_A)]^2\rangle - 8\langle\varepsilon(t_A)\cdot\varepsilon(t_A+\tau)\rangle + 2\langle\varepsilon(t_A)\cdot\varepsilon(t_A+2\tau)\rangle = k_2|\tau|^{2\mu}$$

or equivalently

$$6R_\varepsilon(0)-8R_\varepsilon(\tau)+2R_\varepsilon(2\tau) = k_2|\tau|^{2\mu} . \tag{23}$$

As mentioned above, the function $U(\tau)$ is defined by the relation

$$\begin{aligned} U(\tau) &\equiv \langle[\varepsilon(t_A) - \varepsilon(t_A+\tau)]^2\rangle \\ &= 2[R_\varepsilon(0)-R_\varepsilon(\tau)] \end{aligned} \tag{24}$$

and is assumed to exist. Equations (23) and (24) may be combined to give

$$4U(\tau)-U(2\tau) = k_2'|\tau|^{2\mu} . \tag{25}$$

If a trial solution of the form

$$U(\tau) = A(\beta)\cdot|\tau|^\beta$$

is used, one obtains

$$A(\beta)\cdot|\tau|^\beta[4-2^\beta] = k_2|\tau|^{2\mu}$$

from which one concludes that

$$\beta = 2\mu \leq 2,$$

$$(4-2^\beta)A(\beta) = k_2 ,$$

and

$$U(\tau) = \frac{k_2}{4-2^{2\mu}} |\tau|^{2\mu}, \tag{26}$$

where $\mu \leq 1$ since $U(\tau)$ and k_2 are non-negative. Equation (21) may thus be satisfied if

$$\lim_{\mu \to 1^{(-)}} (4-2^{2\mu})A(2\mu) = k_2, \tag{27}$$

which implies that $A(2)$ is infinite. It is for this reason that the limiting process must be employed.

It is not necessary, however, to assume a particular form for $A(2\mu)$ because Equation (27) is sufficient for the purposes of this development.

It is now possible to determine the mean square third difference of ε_n; i.e., from Table I,

$$\langle(\Delta^3\varepsilon_n)^2\rangle = 20\langle[\varepsilon(t_A)]^2\rangle - 30\langle[\varepsilon(t_A)\cdot\varepsilon(t_A+\tau)]\rangle \tag{28}$$

$$+12\langle[\varepsilon(t_A)\cdot\varepsilon(t_A+2\tau)]\rangle - 2\langle[\varepsilon(t_A)\cdot\varepsilon(t_A+3\tau)]\rangle,$$

where use has again been made of Equations (19). Equation (28) may equivalently be written in the form

$$\langle(\Delta^3\varepsilon_n)^2\rangle = 15U(\tau)-6U(2\tau)+U(3\tau), \tag{29}$$

which may be combined with Equation (26) to yield

$$\langle(\Delta^3\varepsilon_n)^2\rangle = \frac{k_2|\tau|^{2\mu}}{(4-2^{2\mu})}[15-6(2^{2\mu})+3^{2\mu}]. \tag{30}$$

If one now passes to the limit $\mu \to 1^{(-)}$, Equation (30) becomes

$$\langle (\Delta^3 \varepsilon_n)^2 \rangle = \frac{k_2 \tau^2 \ (24 \ln 2 - 9 \ln 3)}{4 \ln 2} \quad . \tag{31}$$

Thus a quadratic dependence of the variance of the second difference of the phase implies a quadratic dependence of the mean square third difference, and the ratio

$$\sqrt{\frac{\langle (\Delta^3 \varepsilon_n)^2 \rangle}{\langle (\Delta^2 \varepsilon_n)^2 \rangle}} = 1.5601\ldots \tag{32}$$

is independent of τ. The average ratio of the points plotted in Figure 3 is 1.65. Values ranging from 1.4 to 1.7 were observed for various runs on different oscillator pairs. The average value for all reliable data taken is 1.52.

Generalization of the Time Error Problem

The average frequency of an oscillator over an interval of time is just the total elapsed phase in the interval divided by the time interval. Since errors of the frequency standard are not presently being considered, the calibration interval of page 7 gives rise to an error time, δt, which could be expressed as a sum

$$\delta t = \frac{1}{\Omega_o} \sum_{n=o}^{m} a_n \phi \ (t_A + b_n \tau) , \tag{33}$$

where m+l is the total number of terms and the set $\{a_n, b_n\}$ are chosen to fit the particular calibration procedure. Indeed, any calibration procedure must give rise to an error time which is expressible in the form of Equation (33).

There are, however, certain restrictions on the $\{a_n, b_n\}$ which are of importance. First, it is a matter of convenience to require that $b_\ell > b_n$ for $\ell > n$. Also, if the oscillator were absolutely perfect, and $\varepsilon(t_A)$ were identically zero, one should logically require that the error time, δt, be identically zero, independent of t_A, the drift rate α, and the basic time interval, τ. That is, from Equation (13)

$$\delta t = \sum_{n=o}^{m} a_n \cdot (t_A + b_n \tau) + \frac{\alpha}{2} \sum_{n=o}^{m} a_n \cdot (t_A + b_n \tau)^2 \equiv 0 \tag{34}$$

for all t_A, α, and τ. One is thus led to the three conditions:

$$\sum_{n=o}^{m} a_n = 0 \,, \tag{35}$$

$$\sum_{n=o}^{m} a_n b_n = 0 \,, \tag{36}$$

$$\sum_{n=o}^{m} a_n b_n^2 = 0 \,. \tag{37}$$

It is of interest to form the quantity

$$\left(\sum_{n=o}^{m} a_n b_n^2\right)\left(\sum_{n=o}^{m} a_\ell\right) = \sum_{n=o}^{m} a_n^2 b_n^2 + \sum_{n<\ell}^{m} a_n a_\ell b_n^2 + \sum_{\ell<n}^{m} a_n a_\ell b_n^2 \,, \tag{38}$$

and it is now possible to interchange the subscripts n and ℓ in the last term of Equation (38) and write the equation in the form

$$\sum_{n=o}^{m} a_n^2 b_n^2 + \sum_{n<\ell}^{m} a_n a_\ell (b_n^2 + b_\ell^2) = 0 \,. \tag{39}$$

Subtracting the square of Equation (36),

$$\left(\sum_{n=o}^{m} a_n b_n\right)\left(\sum_{\ell=o}^{m} a_\ell b_\ell\right) = \sum_{n=o}^{m} a_n^2 b_n^2 + 2\sum_{n<\ell}^{m} a_n a_\ell b_n b_\ell ,$$

one obtains

$$\sum_{n<\ell}^{m} a_n a_\ell (b_\ell - b_n)^2 = 0 \qquad (40)$$

as another condition on the $\{a_n, b_n\}$. Equation (40) is, of course, not independent of Equations (35), (36), and (37).

For the acutal situation where $\varepsilon(t_A)$ is not identically zero, one is reduced, as before, [see Equation (7)] to using conditions (35), (36), and (37) since $\varepsilon(t_A)$ is not knowable. The time error then becomes

$$\delta t = \sum_{n=o}^{m} a_n \cdot \varepsilon(t_A + b_n \tau) \qquad (41)$$

where use has been made of Equations (13), (33), and the restrictions (35), (36), and (37). Note that Equation (41) is the generalization of Equation (10). The square of Equation (41) can be written in the form

$$(\delta t)^2 = \sum_{n=o}^{m} a_n^2 \, \varepsilon^2 (t_A + b_n \tau) + 2\sum_{n<\ell}^{m} a_n a_\ell \left[\varepsilon(t_A + b_n \tau) \cdot \varepsilon(t_A + b_\ell \tau)\right] \qquad (42)$$

which may be averaged over t_A to yield

$$\langle (\delta t)^2 \rangle = \sum_{n=o}^{m} a_n^2 \langle (\varepsilon(t_A))^2 \rangle + 2\sum_{n<\ell}^{m} a_n a_\ell \langle [\varepsilon(t_A) \cdot \varepsilon(t_A + (b_\ell - b_n)\tau)] \rangle \qquad (43)$$

where use has again been made of Equation (19).

The square of Equation (35) may be written as

$$\sum_{n=o}^{m} a_n^2 + 2\sum_{n<\ell}^{m} a_n a_\ell = 0,$$

and Equation (43) then becomes

$$\langle(\delta t)^2\rangle = -\sum_{n<\ell}^{m} a_n a_\ell \{2\langle\varepsilon^2(t_A)\rangle - 2\langle[\varepsilon(t_A)\cdot\varepsilon(t_A + (b_\ell - b_n)\tau)]\rangle\}. \tag{44}$$

Combining Equation (44) with Equation (21) one obtains

$$\langle(\delta t)^2\rangle = -\sum_{n<\ell} a_n a_\ell U((b_\ell - b_n)\tau). \tag{45}$$

Substitution of Equation (26) in (45) yeilds

$$\langle(\delta t)^2\rangle = -k_2|\tau|^{2\mu} \frac{\sum_{n<\ell}^{m} a_n a_\ell (b_\ell - b_n)^{2\mu}}{4-2^{2\mu}}, \tag{46}$$

which is indeterminate of the form $(0/0)$ for $\mu = 1$ because of Equation (40). Making use of L'Hospital's rule, the limit of this expression as $\mu \to 1^{(-)}$ is

$$\langle(\delta t)^2\rangle = \frac{k_2\tau^2}{4\ln 2} \sum_{n<\ell}^{m} a_n a_\ell (b_\ell - b_n)^2 \ln(b_\ell - b_n). \tag{47}$$

Equation (47) is thus the generalization of Equation (31). Since many terms appear in the summation in Equation (47), a computer program was written for evaluation with particular sets of $\{a_n, b_n\}$.

Comparison With Others

If one were to assume that Equation (20) could be solved for $R_\varepsilon(\tau)$, one would obtain

$$R_\varepsilon(\tau) = R_\varepsilon(0) - \frac{1}{2} U(\tau) \tag{48}$$

which expresses the autocovariance function in terms of $U(\tau)$. If one assumes still further that the Wiener-Khinchin theorem applies to Equation (48), the power spectral density of $\varepsilon(t)$, is then the Fourier transform of $R_\varepsilon(\tau)$; i. e.,

$$S_\varepsilon(\omega) = \text{F. T.} \left[R_\varepsilon(0) - \frac{k_2 |\tau|^{2\mu}}{2(4-2^{2\mu})} \right].$$

Fortunately Fourier transforms of functions like $|\tau|^{2\mu}$ have been worked out in refeience (3), page 43. The result is

$$S_\varepsilon(\omega) = \frac{R_\varepsilon(0)}{2\pi} \delta(\omega) - \frac{k_2 \left[\cos \frac{\pi(2\mu+1)}{2}\right] \left[(2\mu)!\right] |\omega|^{-2\mu-1}}{2\pi\,(4-2^{2\mu})}. \tag{49}$$

The factor of $(\frac{1}{2\pi})$ occurs because $S_\varepsilon(\omega)$ is assumed to be a density relative to an angular frequency, ω, rather than a cycle frequency $(f = \frac{\omega}{2\pi})$ as is used in reference (3). The first term on the right of Equation (49) indicates an infinite density of power at zero frequency, i. e., a non-zero average. The zero frequency components are not measurable experimentally and hence this term will be dropped as not being significant to these discussions. The second term on the right of Equation (49) is indeterminate for $\mu = 1$. As before, the limit as $\mu \to 1^{(-)}$ may be obtained, yielding

$$S_\varepsilon(\omega) = \frac{k_2}{8 \ln 2} |\omega|^{-3} \tag{50}$$

for the final result.

The assumptions needed to arrive at Equation (50) are not wholely satisfying and it is of value to show that Equation (50) implies that $\langle (\Delta^2 \varepsilon_n)^2 \rangle$ is indeed given by Equation (21) for m = 2. Only one additional assumption is needed - - the Wiener-Khinchin theorem. Since $R_\varepsilon(\tau)$ and $S_\varepsilon(\omega)$ are real quantities, this theorem may be written in the form

$$R_\varepsilon(\tau) = 2 \int_0^\infty S_\varepsilon(\omega) \cos \omega\tau \, d\omega . \tag{51}$$

From Table I, one may obtain (after squaring and averaging) the expression

$$\langle (\Delta^2 \varepsilon_n)^2 \rangle = 6R_\varepsilon(0) - 8R_\varepsilon(\tau) + 2R_\varepsilon(2\tau) . \tag{52}$$

Substitution of Equation (51) in (52) yields

$$\langle (\Delta^2 \varepsilon_n)^2 \rangle = 4 \int_0^\infty S_\varepsilon(\omega)[3 - 4\cos(\omega\tau) + \cos(2\omega\tau)] \, d\omega . \tag{53}$$

Using Equation (50) for $S_\varepsilon(\omega)$, it is a lengthy but straightforward process to evaluate the integral in Equation (53). The result is, in fact,

$$\langle (\Delta^2 \varepsilon_n)^2 \rangle = k_2 \tau^2 ,$$

in agreement with Equation (21).

If $g_\varepsilon(\omega)$ is the Fourier transform of $\varepsilon(t)$, it is shown, for example, in reference (3) page 20, that

$$g_{\dot{\varepsilon}}(\omega) = i\omega g_\varepsilon(\omega)$$

is the Fourier transform of $\dot{\varepsilon}(t)$. Thus the power spectral density of $\dot{\varepsilon}(t)$ is related to the power spectral density of $\varepsilon(t)$ by the familiar relation

$$S_{\dot{\varepsilon}}(\omega) = \omega^2 S_{\varepsilon}(\omega) \, .$$

Thus the power spectral density of the frequency fluctuations of an "ideal" oscillator may be given as

$$S_{\dot{\varepsilon}}(\omega) = \frac{k_2}{8 \ln 2} |\omega|^{-1} ,$$

that is, flicker noise frequency modulation. The existence of this type of noise modulating the frequency of good quartz crystal oscillators has been reported by several others [4-8]

Comments on the "Ideal" Oscillator

It has thus been shown that the assumptions of stationarity and "ideal" behavior form a basis for a mathematical model of a quartz crystal oscillator which is in quite good agreement with several experiments and experimenters. On the basis of this model it is now possible to predict the behavior of systems employing "nearly ideal" oscillators with the hope of committing no great errors. There are compelling reasons to believe that $U(\tau)$ actually exists (see Chapter IV) for real oscillators in spite of Equation (26). Such conditions require that the $|\omega|^{-1}$ behavior for the power spectral density of the frequency fluctuations, cut off at some small, non-zero frequency. From some experiments [5] conducted, however, this cut-off frequency is probably much smaller than one cycle per year. Such small differences from zero frequency are of essentially academic interest to the manufacturer and user of oscillators. Quantities which may be expressed in the form of Equation (41), however, where the coefficients satisfy conditions (35), (36), and (37) have finite averages even in the limit as flicker noise behavior approaches zero modulation frequency. Such quantities are called *cut-off independent* in contrast to quantities like $U(\tau)$ which will exist only if the flicker noise cuts off at some non-zero frequency.

III. ATOMIC FREQUENCY STANDARDS

Passive Devices

There are in use today two general types of atomic frequency standards: (1) the active device such as a maser whose atoms actually generate a coherent signal whose frequency is the standard, and (2) the passive type such as a cesium beam or rubidium gas cell. In the passive type, a microwave signal irradiates the atoms and some means is employed to detect any change in the atom's energy state. This paper is restricted to the passive type of frequency standard. Some experiments are in progress, however, to determine the statistical behavior of a maser type oscillator.

It is first of value to discuss in what way a "standard" can have fluctuations or errors. Consider the cesium beam. Ideally the standard would be the exact frequency of the photons emitted or absorbed at zero magnetic field in the $(F=4,\ m_f=0) \longleftrightarrow (F=3,\ m_f=0)$ transition of cesium^{133} in the ground electronic state for an infinite interaction time. This is, of course, impossible. This means that the standard is at least less than ideal and one is thus lead to speak, in some sense of the word, about "errors" or even "fluctuations" of the standard.

Figure (4) shows a block diagram of a typical standard of the passive type. An equivalent diagram of this frequency-lock servo is shown in Figure (5),where $V_1(\omega)$ is the Fourier transform of the noise generated in the detectors, associated demodulating circuitry, and the frequency multipliers of Figure (4). $V_2(\omega)$ is an equivalent noise voltage driving the reactance tube in the oscillator to produce the $\dot{\varepsilon}(t_A)$ term in the unlocked oscillator. The power spectrum of

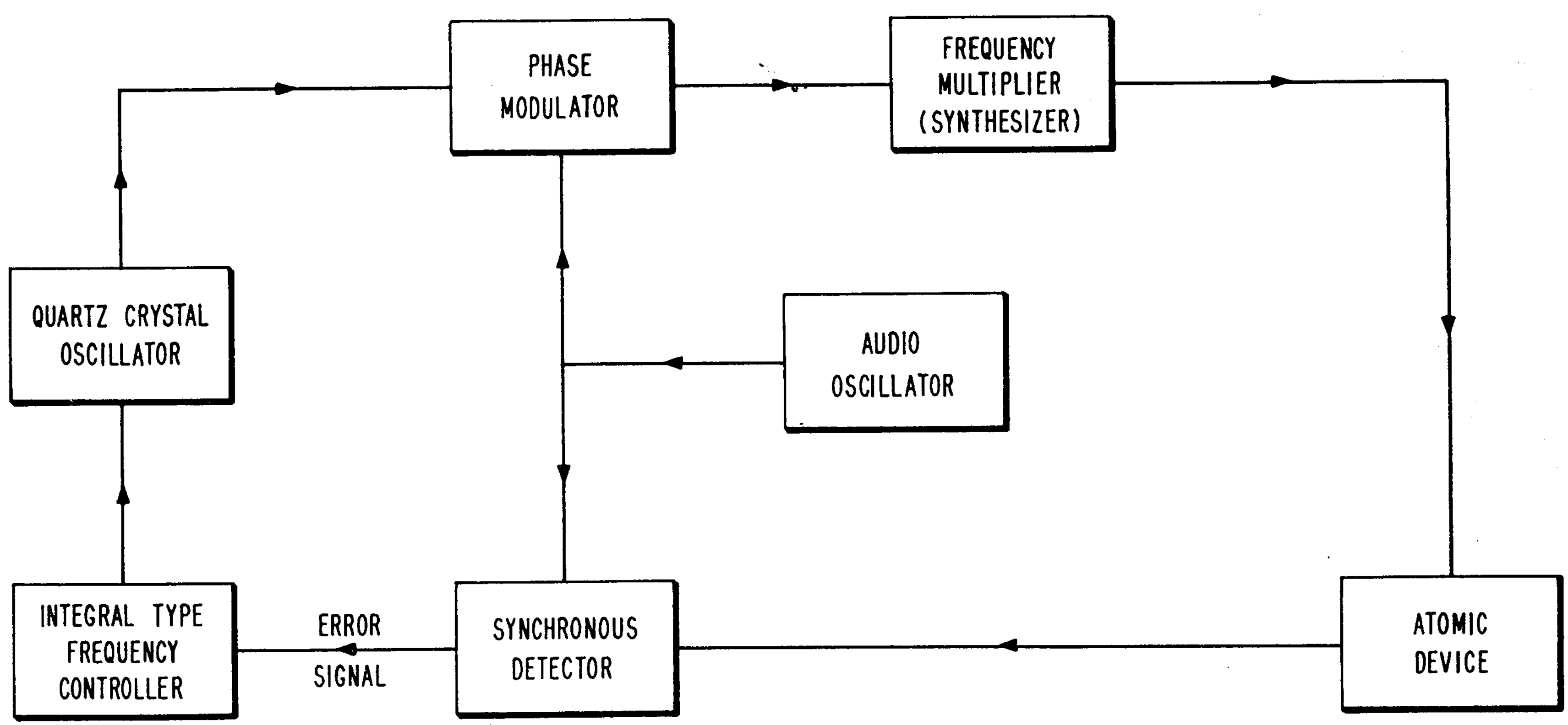
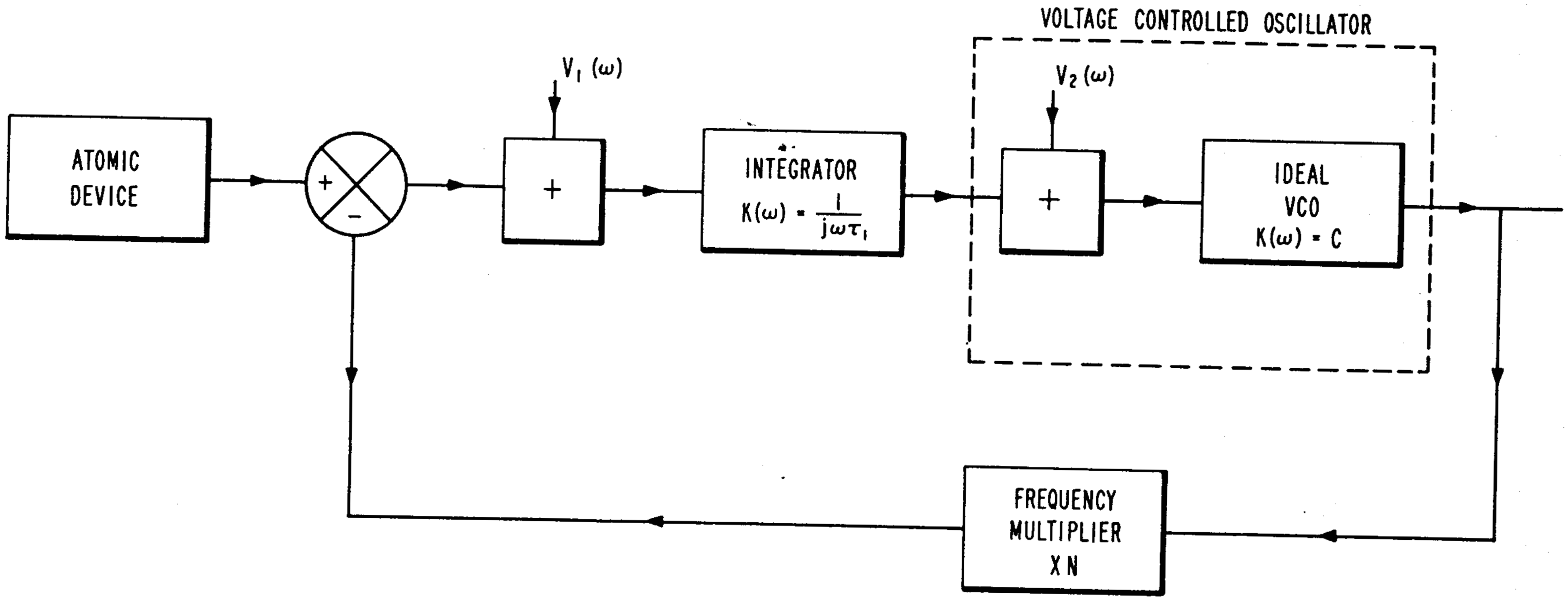


FIG. 4 TYPICAL CONTROL LOOP
FOR AN ATOMICALLY CONTROLLED OSCILLATOR



$V_1(\omega)$ NOISE GENERATED IN DETECTOR
$V_2(\omega)$ EQUIVALENT NOISE TO GENERATE FLICKER NOISE IN VCO

FIG. 5 EQUIVALENT SERVO DIAGRAM OF PASSIVE TYPE FREQUENCY STANDARD

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$V_2(\omega)$, then, is given by

$$C^2 S_{V_2} = \frac{h_s}{|\omega|} \tag{54}$$

where $\frac{h_s}{|\omega|}$ is the power spectral density of the <u>frequency</u> fluctuations of the unlocked oscillator.

It is easiest to treat the servo equations by the use of the variable $\dot{\gamma}$, defined to be the difference between the output frequency of the multiplier and the "ideal" frequency of the atomic transition (the output of the "atomic device" as shown in Figure 5 is then assumed to be the constant zero). In order to preserve the dimensions of voltage for the addition networks, it is convenient to assume that the output of the subtraction network is $-\beta\dot{\gamma}$ where β has the dimensions of volt-seconds. Thus the equation governing the operation of the servo can be expressed, in the frequency domain, as

$$\left[V_2(\omega) + \frac{V_1(\omega) - \beta\dot{\gamma}(\omega)}{j\omega\tau_1}\right] CN = \dot{\gamma}(\omega) \ . \tag{55}$$

This leads to a power spectral density for $\dot{\gamma}$ given by

$$S_{\dot{\gamma}}(\omega) = \frac{(NC)^2 S_{V_1}(\omega) + N^2 \tau_1^2 h_s |\omega|}{\omega^2 \tau_1^2 + (\beta NC)^2} \tag{56}$$

where use has been made of Equation (54).

Normally, $\omega\tau_1$, becomes of the order of (βNC) for ω of the order of 10 sec.$^{-1}$. Thus for small ω, Equation (56) becomes

$$S_{\dot{\gamma}}(\omega) \approx \frac{1}{\beta^2} S_{V_1}(\omega) \tag{57}$$

for $|\omega| < 1$ sec.$^{-1}$.

By applying the techniques of Chapter I to the difference phase of two cesium beams, the curves shown on Figure 6 were obtained. Through least square fits to the data and comparing the ratios of the variances of differences (see Appendix), the result was obtained that the data fit curves of the form

$$\langle (\Delta^m \gamma_n)^2 \rangle = B_m |\tau|^\eta$$

for $\eta = 1.34$, with an uncertainty (standard deviation) of about ± 0.04. Again using reference (3) as before, this leads to the result that

$$S_{\dot{\gamma}}(\omega) = \frac{g}{|\omega|^\mu}, \quad g \approx B_1 \frac{\sqrt{3}}{4\pi} \Gamma\left(\frac{7}{3}\right) \tag{58}$$

where $\mu = 0.34 \pm 0.04$. One is thus led to the very strange conclusion that the spectral distribution of the detector and multiplier noise varies as $\sim |\omega|^{-1/3}$. It is not clear physically where such a noise source might find its origin, although similar spectra have recently been found in other equipment.[9]

Ideally, for measurements over times large compared to the servo time constant, the error accumulated during one measurement interval should be independent of errors accumulated during non-overlapping intervals, i.e., mathematically analogous to Brownian motion.[10] This implies that $S_{\dot{\gamma}}(\omega)$ should be constant for $|\omega| < 1\ \mathrm{sec}^{-1}$ which does not seem to be the case in actuality.[15, 16]

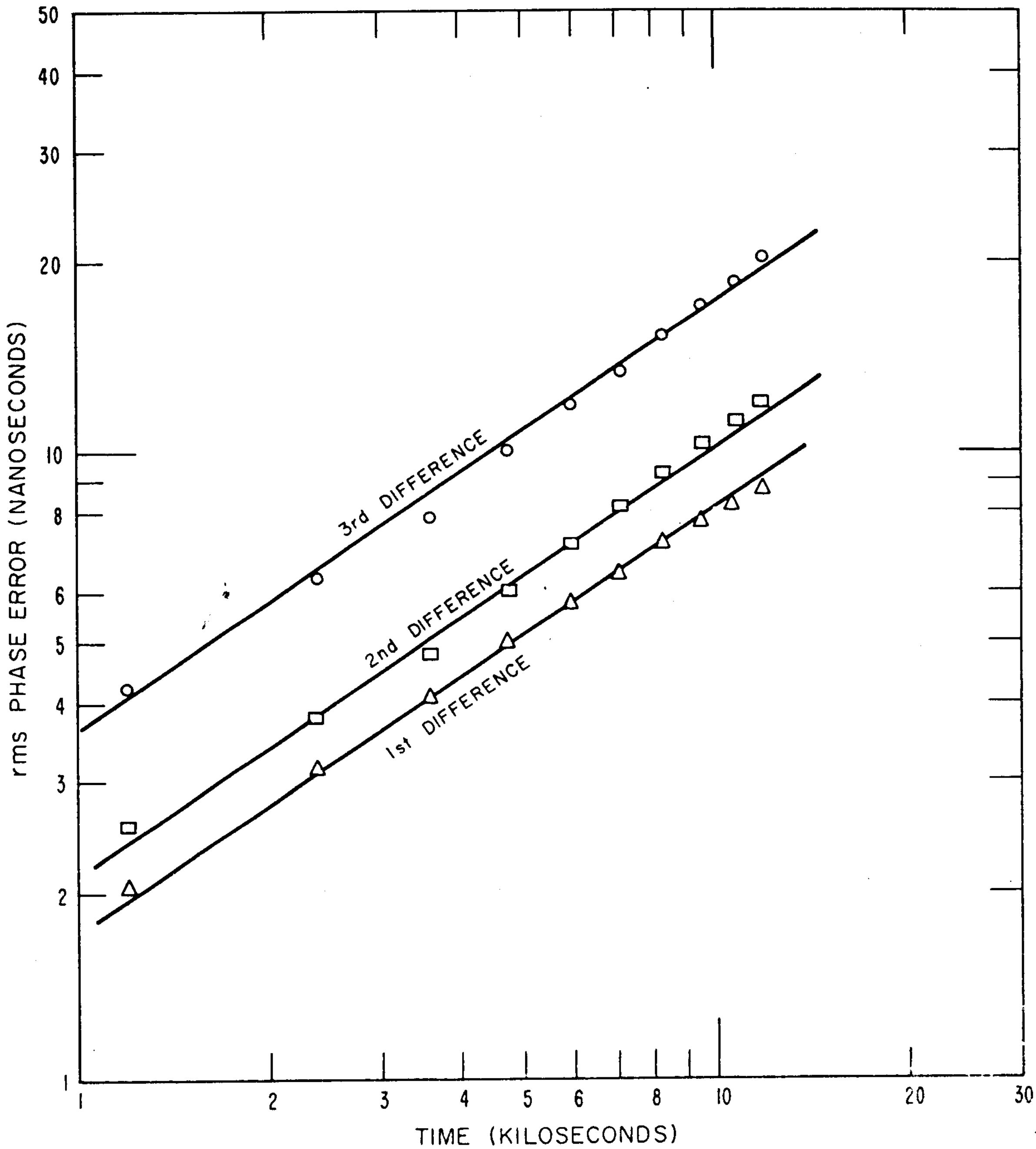

FIG. 6 VARIANCE OF DIFFERENCES OF THE RELATIVE PHASE DIFFERENCE BETWEEN TWO ATOMIC STANDARDS

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The Composite Clock System

If an oscillator's frequency is measured by an atomic standard, the error in measurement of the frequency is given by

$$\delta f = \frac{1}{2\pi\tau}\left[\gamma\left(t_A + \tfrac{1}{2}(T+\tau)\right) - \gamma\left(t_A + \tfrac{1}{2}(T-\tau)\right) \right] \tag{59}$$

for a calibration interval as given on page 7. Also, if f_s is the defined output frequency of the frequency standard, the total error time for the entire calibration interval, T, due to the Standard, is given by $\frac{T\delta f}{f_s}$, and if the clocks contribute an additional error (uncorrelated to the standard's error), of δt, the total error, δT, is given by

$$\delta T = \delta t + \frac{T\delta f}{f_s} . \tag{60}$$

It is of value to explore the dependence of $\langle(\delta t)^2\rangle$ on the ratio of T/τ. In particular, let $T = (2n+1)\tau$ for Equation (10). This can be put in the form of Equation (41) with the following assignments:

n	a_n	b_n
0	-1	0
1	2n + 1	n
2	-2n - 1	n + 1
3	1	2n + 1

It is easy to show that these coefficients satisfy conditions (35), (36), and (37).

Substitution of these coefficients into Equation (47) yields

$$\langle(\delta t)^2\rangle = 2h\tau^2\{-2n^2(2n+1)\ln(n) + 2(n+1)^2(2n+1)\ln(n+1) - (2n+1)^2\ln(2n+1)\}, \tag{61}$$

where $h = \dfrac{k_2}{8\ln 2}$.

Since τ is normally small compared to T, it is reasonable to approximate Equation (61) for large n. That is, the relative time error is approximately given by

$$\frac{\sqrt{\langle(\delta t)^2\rangle}}{T} \approx \sqrt{2h\ln(n)}\quad, \tag{62}$$

where the approximations,

$$n \gg 1\ ,\ \ln(n+1) \approx \ln(n) + \frac{1}{n}\ ,$$

have been used. It is apparent from Equation (62) that for a given interval, T, the errors accumulated by the clock are not critically dependent on the measuring time, τ, since $\sqrt{\ln(n)}$ is a very slowly changing quantity.

The mean square error associated with the standard, however, can be written in the form

$$T^2\langle(\delta f)^2\rangle = \left(\frac{T}{\tau}\right)^2\langle(\Delta\gamma_n)^2\rangle \tag{63}$$

since Equation (59) may be written as the first difference of γ_n. From Equation (57) it is apparent that the relative time error is then given by

$$\frac{T^2\langle(\delta f)^2\rangle}{T^2 f_s^{\,2}} = \frac{B_1}{(2\pi\tau^{1/3} f_s)^2}\,\tau^{\eta} \tag{64}$$

for η=1.34 ±0.04. Or, combining Equations (59), (62), and (64),

$$\frac{\sqrt{\langle(\delta T)^2\rangle}}{T} \approx \sqrt{2h\ln(n) + \frac{B_1}{(2\pi\tau^{1/3} f_s)^2}} \; . \tag{65}$$

As one should expect, for a given T, the errors get less for larger τ but not very rapidly. In the limit of τ=T [not using the approximate Equation (65)], the errors are those of the standard alone, i.e., $\frac{1}{2\pi}\sqrt{B_1 f_s^{-2}\tau^{-2/3}}$. Also, as $\tau\to 0$ the clock errors are unbounded.

Compounding Time Errors

Equation (65) represents a reasonable approximation to the time errors of a clock system after one calibration interval. The next question is: how do the errors of many calibrations compound to give a total error, δ(NT), after N calibrations? Returning to Equation (59), one may write

$$\delta(NT) = \sum_{n=1}^{N} \delta T_n \quad , \tag{66}$$

where δT_n is the time error associated with the n-th calibration interval. There have been papers published[11] that assume that the errors of one calibration interval are not correlated to the errors of any other calibration interval. It is now possible to investigate this assumption more precisely.

In particular, the clock errors (not including the frequency standard errors), δt_n, compound to give δ(Nt), given by

$$\delta(Nt) = \sum_{n=1}^{N} \delta t_n , \tag{67}$$

which can obviously be put in the form of Equation (41). For N equal to any number larger than one, the algebra becomes much too lengthy for actual calculation by hand and it is desirable to make use of a digital computer. Table III shows the results of this calculation for compounding several third-difference-type calibrations. It is interesting to note that, in fact, the total mean square error after N calibrations is very nearly equal to N-times the mean square error after one calibration. This is in quite good agreement with reference (11).

The errors associated with the frequency standard, unfortunately, are not so easily assessed due to the uncertainty of the origin of the $|\omega|^{-1/3}$ type of noise. If this noise arises from non-fundamental processes effecting such things as phase shifts in the cavity[12] of a cesium beam using Ramsey type excitation, the auto covariance function of this noise must vanish for times of the order of the time between phase alignments of the cavity. On the other hand, if it is, in fact, a fundamental property of the frequency standard itself, one must calculate the errors entirely on the basis of the $|\omega|^{-1/3}$ type of noise.

In the absence of experiment or theory relating the $|\omega|^{-1/3}$ type of noise with fundamental processes in the standard, it does not seem unreasonable to assume the noise to be of a <u>non</u>-fundamental nature and approximate the total mean square error as N-times the mean square error of one calibration. Within one calibration interval, one must certainly calculate on the basis of $|\omega|^{-1/3}$ type noise. Thus the RMS relative error may be approximated by the relation

TABLE III

THEORETICAL DETERMINATION OF TIME ERROR PROPAGATION

N	$\dfrac{\left\langle \left(\sum_{n=1}^{N} \delta t_n \right)^2 \right\rangle}{\left\langle \left(\delta t_n \right)^2 \right\rangle}$
2	2.054
3	3.109
4	4.165
5	5.220
6	6.276
7	7.332

$$\sqrt{\frac{\langle(\delta(NT))^2\rangle}{NT}} \approx \sqrt{\frac{2h\,\ell n(n) + B_1\tau^{-2/3}f_s^{-2}}{N}}\,, \tag{68}$$

where h, n, B_1, f_s, and τ have the same meanings as in Equation (65). It is worth noting that if the $|\omega|^{-1/3}$ noise is of a more fundamental nature than was assumed for Equation (68), the total RMS error of Equation (68) would have a term decreasing as $N^{-1/3}$ instead of $N^{-1/2}$.

The conclusions which can now be drawn are that: (1) the total RMS time error of this clock system from an "ideal" atomic clock is unbounded as time increases, and (2) the relative RMS error time to total elapsed time [Equation (68)] approaches zero about as fast as $N^{-1/2}$. It should be mentioned here that systematic errors in the atomic standard have not been considered. While this is a very important problem, it has been treated rather thoroughly elsewhere[12, 13, 14].

IV. MEASURES OF FREQUENCY STABILITY

General Restrictions

It is of value to consider the problem of establishing a stability measure in a very general sense. Consider some functional of the phase,

$$\chi = \chi\ (\phi(t))\ ,$$

from which the stability measure, Ψ, is obtained according to the relation

$$\Psi^2 = \lim_{T\to\infty} \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} |\chi|^2\, dt\ ,$$

provided this limit exists.

In practice it is not possible to pass to the limit $T \to \infty$, and thus one measures for some fixed time, T; i. e.,

$$\Psi_T^2 = \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} |\chi|^2\, dt\ .$$

Under favorable conditions, Ψ_T may be a reasonable approximation to Ψ. Unfortunately, this may not always be the case.

The frequency emitted by any physically realizable device must be bounded by some upper bound, say B. The following inequalities must, then, be valid:

$$|\Omega(t)| \leq B, \text{ for some } B > 0\,;$$

$$\lim_{T\to\infty} \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} (\Omega(t))^2\, dt \leq B^2\ .$$

With $S_{\dot{\phi}}(\omega)$ being the power spectral density of $\Omega(t)$, it follows from the definition of power spectra that

$$\lim_{T\to\infty} \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} (\Omega(t))^2 \, dt = 2\int_o^\infty S_{\dot{\phi}}(\omega)\, d\omega$$

for real $\Omega(t)$, and thus

$$2\int_o^\infty S_{\dot{\phi}}(\omega)\, d\omega \le B^2 \quad . \tag{69}$$

If $S_{\phi}(\omega)$ has a flicker noise spectrum for small ω, it is apparent that this $\frac{1}{|\omega|}$ type of noise cannot persist to absolute zero frequency or the inequality (69) would be violated. It is thus reasonable to postulate the existence of a lower cut-off frequency, ω_L, for the flicker noise modulation.

It is apparent from the above considerations that there may exist stability measures for which Ψ_T begins to approach Ψ only after T is several times larger than $\frac{1}{\omega_L}$. From some of the experiments on crystal oscillators,[5] this may require T to exceed several years in duration. This is quite inconvenient from a manufacturing or experimental standpoint. The logical conclusion is to consider only those stability measures, Ψ, which are "cut-off independent," that is, those measures of stability which would be valid even in the limit $\omega_L \to 0^+$.

Finite Differences

It was shown in Chapter II that an expression of the form

$$\delta t = \frac{1}{\Omega_o} \sum_{n=o}^{m} a_n \phi(t+b_n\tau) \tag{70}$$

will have a finite variance if the $\{a_n, b_n\}$ satisfy conditions (35), (36), and (37). It is easy to show that the first difference of the phase (i.e., frequency) cannot be put in the form of Equation (70) with the coefficients satisfying conditions (35), (36), and (37). Indeed, the limit

$$\lim_{\mu \to 1^{(-)}} U(\tau) = \lim_{\mu \to 1^{(-)}} \frac{k_2 |\tau|^{2\mu}}{4-2^{2\mu}}$$

does not exist, and hence $U(\tau)$ is not a good measure of stability.

The variances of the second and third finite differences, however, are convergent. It is of interest to note that the first line of Table III may be expressed in the form

$$\frac{\left\langle \left| \Delta^3 \phi_{n+3} + \Delta^3 \phi_n \right| \right\rangle}{\left\langle \left(\Delta^3 \phi_n \right)^2 \right\rangle} = 2.054 \ ,$$

which may be simplified to the form

$$\frac{\left\langle \left(\Delta^3 \phi_{n+3} \right) \left(\Delta^3 \phi_n \right) \right\rangle}{\left\langle \left(\Delta^3 \phi_n \right)^2 \right\rangle} = 0.027 \ .$$

This equation expresses the fact that a third difference has a very small correlation ($\sim 3\%$) to an adjacent, non-overlapping third difference. By extending this procedure with the other values given in Table III, it is found that the correlation of one third difference with a non-overlapping third difference becomes small very rapidly as the interval between these differences becomes large. This is sufficient to insure that the variance of a finite sample of third differences will approach the "true" variance (infinite average) in a well behaved and reasonable fashion as the sample gets larger.

Variance of Frequency Fluctuations for Finite Averaging Times

Even though the variance of the first difference of the phase does not satisfy the condition of being cut-off independent, it is possible (by specifying both the sample time and the total averaging time) to construct a cut-off independent measure of the frequency fluctuations. Instead of Ψ_T, the variance of N adjacent samples of the frequency will be denoted by $\sigma^2(\tau, N)$ where τ is the sample time for each of the N measurements of frequency. The variance is given by the conventional formula

$$\sigma^2(\tau, N) = \frac{1}{N-1} \sum_{i=1}^{N} \left[\frac{\Delta\phi_i}{\tau} - \frac{\phi_{N+1} - \phi_1}{N\tau}\right]^2 .$$

If one neglects the drift rate, α, which is essentially equivalent to obtaining the standard deviation around a linear drift, one obtains

$$\left\langle \sigma^2(\tau, N) \right\rangle = \frac{1}{(N-1)\tau^2} \left\{ N \left\langle \left[\varepsilon(t+\tau) - \varepsilon(t)\right]^2 \right\rangle - \frac{1}{N} \left\langle \left[\varepsilon(t+N\tau) - \varepsilon(t)\right]^2 \right\rangle \right\} . \quad (71)$$

For the case of an "ideal" crystal oscillator, Equations (19), (24), and (26) allow Equation (71) to be simplified (after passing to the limit $\mu \to 1^{(-)}$) to the form

$$\left\langle \sigma^2(\tau, N) \right\rangle = \frac{2h\, N \ln(N)}{N-1} .$$

Thus as N increases, the expected value of $\sigma^2(\tau, N)$ increases without bound (at least until $N\tau \sim \frac{1}{\omega_L}$). It is interesting to note that for a given oscillator, $\left\langle \sigma^2(\tau, N) \right\rangle$ has a minimum value for N = 2. Obviously one would have to average several experimental determinations of $\sigma^2(\tau, 2)$ in order to have a reasonable approximation to $\left\langle \sigma^2(\tau, 2) \right\rangle$.

While $\sigma(\tau, N)$ is a cut-off independent measure of frequency stability, it has the significant disadvantage of being a function of two variables. Indeed, in order to compare the stability of two oscillators, both the τ's and the N's should have nearly corresponding values.

Delayed Frequency Comparison

In radar work, often the frequency of a signal is compared to the frequency of the same source after it has been delayed in traversing some distance--often a very great distance. One might thus be interested in defining a stability measure in an analogous fashion:

$$\Psi^2(\tau, T) = \left\langle \left[\frac{\phi(t+T+\tau) - \phi(t+T)}{\tau} - \frac{\phi(t+\tau) - \phi(t)}{\tau} \right]^2 \right\rangle. \quad (72)$$

Again neglecting the drift rate, α, of the oscillator, Equations (24) and (72) combine to yield

$$\Psi^2(\tau, T) = \frac{1}{\tau^2} \left[2U(\tau) - U(T-\tau) + 2U(T) - U(T+\tau) \right] . \quad (73)$$

After substitution of Equations (26) into Equation (73) the equation can be rearranged to give (again passing to the limit, $\mu \to 1^{(-)}$)

$$\Psi^2(\tau, T) = -2h \left[\rho^2 \ln \rho - (1+\rho)^2 \ln (1+\rho) - (\rho-1)^2 \ln (\rho-1) \right] ,$$

where $\rho \equiv \frac{T}{\tau}$. Although this is a rather complicated expression, it may be simplified with the approximation $\rho \equiv \frac{T}{\tau} >> 1$. The result is

$$\Psi^2 (\tau, T) = 4h \left(2 + \ln \frac{T}{\tau}\right), \text{ for } \frac{T}{\tau} >> 1 \quad (74)$$

for an "ideal" oscillator. It is interesting to note here that even when considering only $\frac{1}{|\omega|}$ type of noise, one cannot pass to the limit $\tau = 0$ for this problem. In the limit as $\tau \to 0^+$, the expression

$$\lim_{\tau \to 0^+} \left[\frac{\phi(t+\tau) - \phi(t)}{\tau} \right] \equiv \dot{\phi}(t) \equiv \Omega(t)$$

and thus

$$\lim_{\tau \to 0^+} \Psi^2(\tau, T) = \left\langle \left[\Omega(t+T) - \Omega(t) \right]^2 \right\rangle \to \infty$$

from Equation (74), even though $T << \frac{1}{\omega_L}$.

Again $\Psi(\tau, T)$ is a function of two variables with all of the associated annoyances. It may, however, be useful in certain applications.

V. THE NBS - A TIME SCALE

The Clock System

For reliability of operation it is necessary to have several independent oscillator-counter (clock) systems. In order to determine a malfunction of a particular unit with some degree of reliability, three independent systems are an absolute minimum. For a more certain determination of such a malfunction even in the event that one system is down for maintainance purposes (e.g. its frequency is being reset to maintain agreement with nominal value ± one point in 10^8), five independent clocks is a more desirable number.

The clock system maintaining the NBS-A time scale is comprised of five independent oscillator-counter (clock) systems. Each clock has its own, independent power supply with battery back-up to maintain uninterrupted operations in the event of failure of the 60 Hz commercial power. These batteries will maintain clock operation in excess of 10 hours. Emergency power from a diesel generating system is available for more prolonged outages. The mean "down time" for a clock in the system is less than one day per year of operation and hence the probability of simultaneous failure of all five clocks is less than the probability of some catastrophic event (e.g., a flood) destroying the time kept by the whole system. During the past three years no interruption of operation or complete loss of time has, in fact, occurred.

The frequencies of the oscillators are measured against the United States Frequency Standard (USFS) by "beating" the nominal 5 MHz signal from each oscillator in turn with the 5 MHz signal from the USFS. The period of the beat note is counted on a frequency counter and about a 15-minute average of each oscillator's frequency is obtained. The date, time-of-day, period, and coding information are automatically transferred to punched cards for computer analysis. Also the five clocks are intercompared by using the frequency counter in a time-interval mode. With the counter set to count a standard 10 MHz signal, the gate is opened by a one-second "tick" from one clock and closed by a "tick" from another clock. Thus time differences between clocks are readable to ± 0.1 μsec, and again all data are automatically transferred to punched cards.

<u>Operation of the Clock System</u>

In Chapter IV it was shown that the expected value of the variance of the frequency fluctuations, $\langle \sigma^2(\tau, N) \rangle$, is given by

$$\langle \sigma^2(\tau, N) \rangle = \frac{2h\, N \ln N}{N-1}$$

where $h = \dfrac{k_2}{8 \ln 2}$. Also, in Chapter II it was shown that for a given calibration procedure [Equation (47)] the mean square error time, $\langle (\delta t)^2 \rangle$, is proportional to k_2. For the situation where there exists M non-identical clocks (M=5 for the NBS-A system), the errors of the i-th clock may be determined from the constant $k_2^{(i)}$, or equivalently, the merit of one clock relative to the other M-1 clocks may be expressed in terms of $\langle \sigma_i^2(\tau, N) \rangle$ since all clocks are calibrated with the same τ, and N. That is,

$$\langle (\delta t_i)^2 \rangle = K \langle \sigma_i^2(\tau, N) \rangle, \; i = 1, 2, \cdots M, \tag{75}$$

where K is the same for all oscillators.

The final form of the time error problem may now be stated as follows: After N calibrations of frequency have been performed on M (=5) oscillators (which should not be assumed identical), what is the least biased estimate of the true elapsed time? That is, if we take W_i

to be the weighting factor for the i-th clock and define the elapsed time, t, to be given by

$$t = \sum_{i=1}^{M} W_i \, t_i \tag{76}$$

where the weighting factors satisfy the equation

$$\sum_{i=1}^{M} W_i = 1 , \tag{77}$$

for what set of weighting factors, W_i, will t have the least probable error?

While the solution to this type of problem can be found in many statistics books, it is reasonable to go through this derivation again, here. This problem may be stated as a problem in Legrange undetermined multipliers: Find the minimum value of the variance of t [obtained from Equation (76)] subject to the constraint of Equation (77). Letting λ be the undetermined multiplier, the solution to the problem may be found by the simultaneous solution of the M equations

$$\frac{\partial}{\partial W_j} \left[\sum_{i=1}^{M} W_i^2 \langle (\delta t_i)^2 \rangle - \lambda \left(\sum_{i=1}^{M} W_i - 1 \right) \right] = 0,$$

$$j = 1, 2, \cdots M. \tag{78}$$

and also Equation (77). In order to obtain Equation (78) it is necessary to assume that

$$\langle (\delta t_i \, \delta t_j) \rangle = \langle (\delta t_i)^2 \rangle \, \delta_{ij}$$

where δ_{ij} is the Kronecker δ-function; that is, the oscillators errors are not correlated to each other and thus

$$\text{Variance } (t) = \sum_{i=1}^{M} W_i^2 \langle (\delta t_i)^2 \rangle .$$

Solving the M Equations (78) for W_j, one obtains

$$W_j = \frac{\lambda}{2\langle (\delta t_j)^2 \rangle} \qquad j = 1, 2, \cdots M \tag{79}$$

which may be combined with Equation (77) to yield the solution

$$W_j = \frac{1}{\langle (\delta t_j)^2 \rangle} \cdot \left[\sum_{i=1}^{M} \frac{1}{\langle (\delta t_i)^2 \rangle} \right]^{-1} . \tag{80}$$

Some trial values can quickly satisfy one that Equation (80) does indeed represent a minimum and not a maximum. By virtue of Equation (75) this may be rewritten in the form

$$W_j = \frac{1}{\langle \sigma_j^2 (\tau, N) \rangle} \cdot \left[\sum_{i=1}^{M} \frac{1}{\langle \sigma_i^2 (\tau, N) \rangle} \right]^{-1} . \tag{81}$$

Equation (81) expresses the fact that the least biased estimate of the elapsed time is a weighted average of the M(=5) clocks where each clock is weighted inversely proportional to the variance of the frequency fluctuations around a linear drift (see Chapter IV). The NBS-A system is normally operated by accumulating data on a working-day basis for about two weeks. On the basis of 10 to 12 frequency measurements for each oscillator a least square fit of a straight line frequency drift is obtained for each oscillator and the mean square deviation from this line is also obtained. The weighting factor for each oscillator is then calculated according to Equation (81), all calculations being performed automatically on a digital computer.

Assuming a linear drift for the frequency between each pair of calibration points for an oscillator, it is thus possible to estimate the increment in the time difference between indicated time and true atomic time for each clock. The difference between center-interval calibration (Chapter I) and linear interpolation between end-point calibrations can easily be shown to be insignificant for exactly periodic calibrations as the number of calibrations increases without bound. However, linear interpolation between end-point calibrations allows an extension of the treatment to where calibrations are not exactly periodic and is therefore a more practical approach.

Assume that $t_A^{(i)}$ is the indicated time of the i-th clock and that Q_i is the total accumulated correction for the i-th clock at some instant according to some calibration process. Thus there exists M (in this case M=5) equations

$$t_i = t_A^{(i)} + Q_i \quad . \qquad i = 1, 2, \cdots M \tag{82}$$

Data are also taken to give the time difference between pairs of clocks. It is a matter of convenience to check all clocks against the last or M-th clock and thus their exists M-1 independent, measurable quantities, Θ_{Mi} given by

$$\Theta_{Mi} = t_A^{(M)} - t_A^{(i)} \qquad i = 1, 2, \cdots M-1 \tag{83}$$

Equation (83) is a set of M-1 linearly independent equations in the M unknowns $t_A^{(i)}$ and the known quantities Q_M, Q_i, and Θ_{Mi}.

Equations (83) and (82) may be combined to form

$$t_i = t_M + Q_i - Q_M - \Theta_{Mi} \qquad i = 1, 2, \cdots M. \tag{84}$$

It is now possible to substitute the M Equations (84) [for i = M, Equation (84) reduced to the trivial equation $t_M = t_M$] into Equation (76) and obtain

$$t - t_M = \sum_{i=1}^{M-1} W_i \, (Q_i - Q_M - \Theta_{Mi}), \tag{85}$$

which may be combined with Equation (82) to give

$$t - t_A^{(M)} = Q_M + \sum_{i=1}^{M-1} W_i \, (Q_i - Q_M - \Theta_{Mi}). \tag{86}$$

Thus the M-th clock's indicated time, $t_A^{(M)}$, differs from the least biased estimate of true time, t, by the amount given in Equation (86) and is expressed entirely in terms of known quantities. Since the difference time between the M-th clock and all others is known (Θ_{Mi}), it is an easy matter to obtain the total best correction for each clock in the system. These total best corrections are incremented by the next calibrations of the oscillators to form the new Q_i's for the next calculation.

Evaluation of the System

In the present operation of the NBS-A time system, clock No. 5 is located in a separate room from the other four oscillators with independent ambient temperature control. If clocks within the system are influenced by diurnal variations in ambient temperature, one would expect a systematic accumulation of time difference between clock No. 5 and some other clock when each clock is treated independently and not corrected to the mean. Figure 8 shows the accumulation of this difference for clocks 5 and 4. It is apparent that systematic errors are certainly less than one part in 10^{12}; however, there is an indication of a variation with about a one-year period which will require a longer investigation to verify.

An analysis of the data for two-months periods indicated that the data fit well into random walk behavior. The variance of the n-th finite difference ($n \geq 1$) is nearly linear with delay time and the ratio of variances of finite differences agree well with the values given in the Appendix for a random walk ($\eta = 1$).

Table IV shows the relative RMS time errors expected from the five clocks of the NBS-A time system as determined from an analysis of finite differences. The values listed are computed from Equation (47) for a calibration interval of one day with a 15-minute comparison to the USFS. Also indicated on the table are the weighting factors one would expect on this basis. The weighted average should have a standard deviation of 1.8×10^{-12} or a three-sigma value of 5.4×10^{-12}. Thus the present clock system contributes an error after one day of about the same magnitude as the USFS which has been estimated[13] at 5×10^{-12} for a three-sigma limit also. After N days the clock error should be reduced by a factor of $N^{-\frac{1}{2}}$ and thus for periods longer than a few days the limitation of the time keeping is any systematic errors of the USFS itself which are believed to be less than the 5×10^{-12} figure.

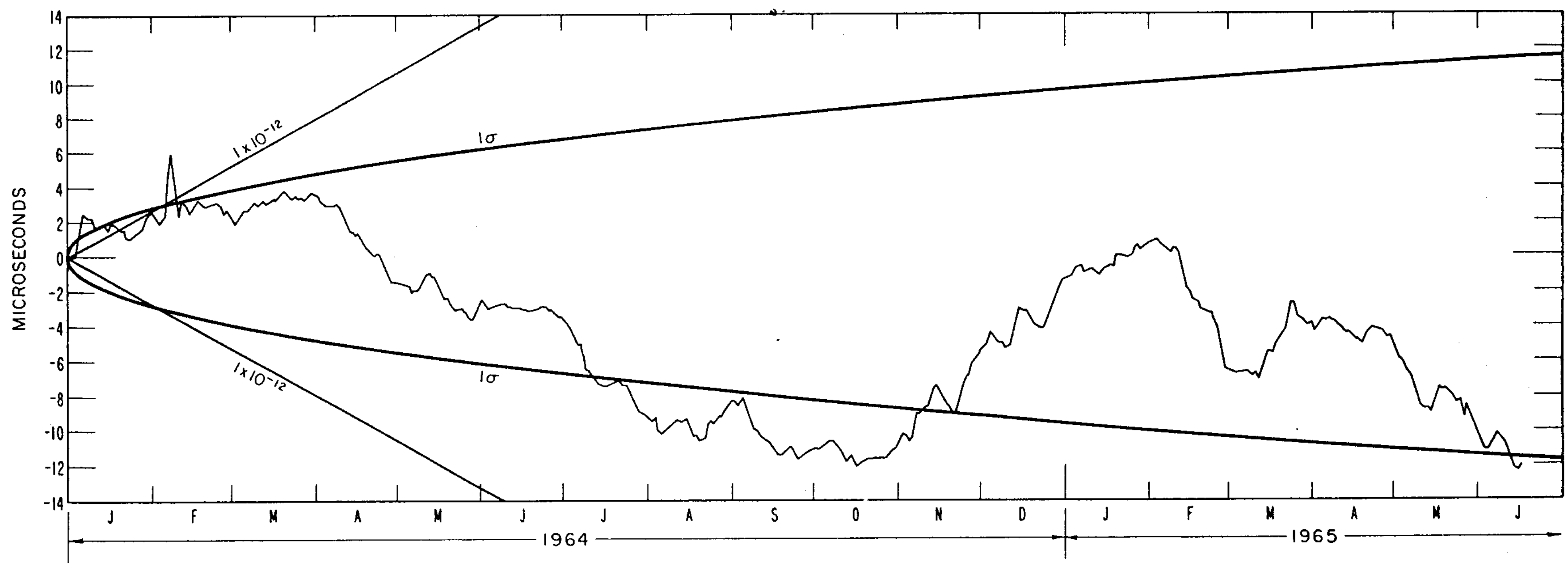

FIGURE 8. ACCUMULATED, UNACCOUNTABLE DIFFERENCES BETWEEN CLOCKS 4 AND 5 OF THE NBS-A TIME SYSTEM

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TABLE IV

QUALITY OF THE OSCILLATORS IN THE NBS-A SYSTEM

Clock No.	$\frac{\sqrt{\langle (\delta t)^2 \rangle}}{T}$	Weighting Factor
1	10×10^{-12}	3.4%
2	22×10^{-12}	0.7%
3	4×10^{-12}	21.1%
4	3×10^{-12}	37.4%
5	3×10^{-12}	37.4%

Conclusion

The assumptions of stationarity and "ideal" behavior for a quartz crystal oscillator lead to a statistical model which agrees well with many different experiments. One finds, however, that certain quantities are unbounded as averaging times are extended and it is important to consider only those quantities which have reasonable hope of converging toward a good value in reasonable time. Thus the concepts of cut-off dependent and cut-off independent measures of frequency stability form a natural classification for all possible frequency stability measures.

On the basis of "ideal" behavior, it has been shown that the errors of a clock run from a quartz crystal oscillator and periodically referenced to an atomic frequency standard accumulates error at a probable rate proportional to the square root of the number of calibrations. That is, the errors of one calibration interval are essentially uncorrelated to errors of non-overlapping intervals in spite of the fact that "ideal" behavior is highly correlated for long periods of time.

It has also been shown that the method of finite differences can be a useful method of determining spectral distributions of noise as well as being a possible measure of frequency stability. By using higher order finite differences, phase fluctuations with even a high order pole at zero modulation frequency can similarily be treated. The need for higher than second or third differences, however, has not yet been demonstrated.

It should also be noted that the existence of higher frequency modulation noise of different origin also has significant affect on stability measures. In general, the factors which limit the system to a finite bandpass are sufficient to insure convergence of the stability measures as $\omega \to \infty$. If it is primarily the measuring system which limits the system bandpass, however, the results may be significantly altered by the measuring system itself.

The techniques of the first four chapters have been applied to an actual clock system (NBS-A) and an evaluation of its precision obtained. The clock system should contribute an uncertainty of less than 0.5μsec. per day (~6 parts in 10^{12}) in a random uncorrelated fashion -- i.e., a random walk process. The precision (not accuracy) of the USFS being only a few parts in 10^{13} for a 15 minute comparison(13), it is primarily the precision of the clocks which determine the overall precision. In accuracy, however, it is the USFS which ultimately limits all measurements to an accuracy (not precision) of 5 parts in 10^{12}.

Acknowledgments

The author is sincerely indebted to Drs. Paul Wacker and Edwin Crow of the National Bureau of Standards for some very valuable suggestions and references. The author also wishes to acknowledge the assistance of Mr. D. Allan for his aid in data acquisition and analysis, Mr. Roger E. Beehler for making the cesium beams of the NBS available, and to Mr. R. L. Fey for some helpful criticism.

APPENDIX

Ratio of Variances

Let $\varepsilon(t)$ be a real generalized function such that $\langle \varepsilon(t) \rangle = 0$ and define the discrete variable ε_m by the relation

$$\varepsilon_m \equiv \varepsilon\,(t + m\tau)\ . \tag{A-1}$$

Also let the autocovariance function of $\varepsilon(t)$ be, as before, independent of a simple time transition. One may now write (see Table I)

$$\langle (\Delta\varepsilon_m)^2 \rangle = 2\,[\langle (\varepsilon_n)^2 \rangle - \langle [\varepsilon(t + \tau)\ \varepsilon\,(t)] \rangle] \tag{A-2}$$

and assume that

$$\langle (\Delta\varepsilon_m)^2 \rangle = k_1 \tau^\eta\ , \tag{A-3}$$

where k_1 is a constant for a given η. It is also possible to obtain the variance of the second difference:

$$\langle (\Delta^2\varepsilon_m)^2 \rangle = 6\,\langle (\varepsilon_m)^2 \rangle - 8\,\langle [\varepsilon(t+\tau) \cdot \varepsilon(t)] \rangle + 2\,\langle [\varepsilon(t+2\tau)\ \varepsilon\,(t)] \rangle\ . \tag{A-4}$$

Using Equations (A-3) and (A-4), one may obtain

$$\frac{\langle (\Delta^2\varepsilon_m)^2 \rangle}{\langle (\Delta\varepsilon_m)^2 \rangle} = \frac{4k_1\tau^\eta - k_1(2\tau)^\eta}{k_1\tau^\eta} \tag{A-5}$$

$$= 4 - (2)^\eta\ .$$

Since Equation (A-5) must be non-negative (ε is real), the exponent is restricted to the range $\eta \leq 2$.

Similarly one may obtain the variance of the third difference

$$\langle (\Delta^3 \varepsilon_m)^2 \rangle = 15 k_1 \tau^\eta - 6 k_1 (2\tau)^\eta + k_1 (3\tau)^\eta , \tag{A-6}$$

and hence the ratio

$$\frac{\langle (\Delta^3 \varepsilon_m)^2 \rangle}{\langle (\Delta^2 \varepsilon_m)^2 \rangle} = \frac{15 - 6 \cdot (2)^\eta + 3^\eta}{4 - (2)^\eta} . \tag{A-7}$$

Similarly,

$$\frac{\langle (\Delta^4 \varepsilon_m)^2 \rangle}{\langle (\Delta^3 \varepsilon_m)^2 \rangle} = \frac{56 - 28 \cdot (2)^\eta + 8 \cdot (3)^\eta - (4)^\eta}{15 - 6 \cdot (2)^\eta + (3)^\eta} . \tag{A-8}$$

Equations (A-5), (A-7), and (A-8) are plotted in figure 7 as a function of the exponent, η.

For $\eta = 4/3$, as in figure 6, the theoretical ratios,

$$\frac{\langle (\Delta^{n+1} \varepsilon_m)^2 \rangle}{\langle (\Delta^n \varepsilon_m)^2 \rangle}$$

for n = 1 and 2 are 1.48 and 2.84, respectively. The straight lines drawn in figure 6 were made to have these ratios and slope 2/3 (the square root of the variances).

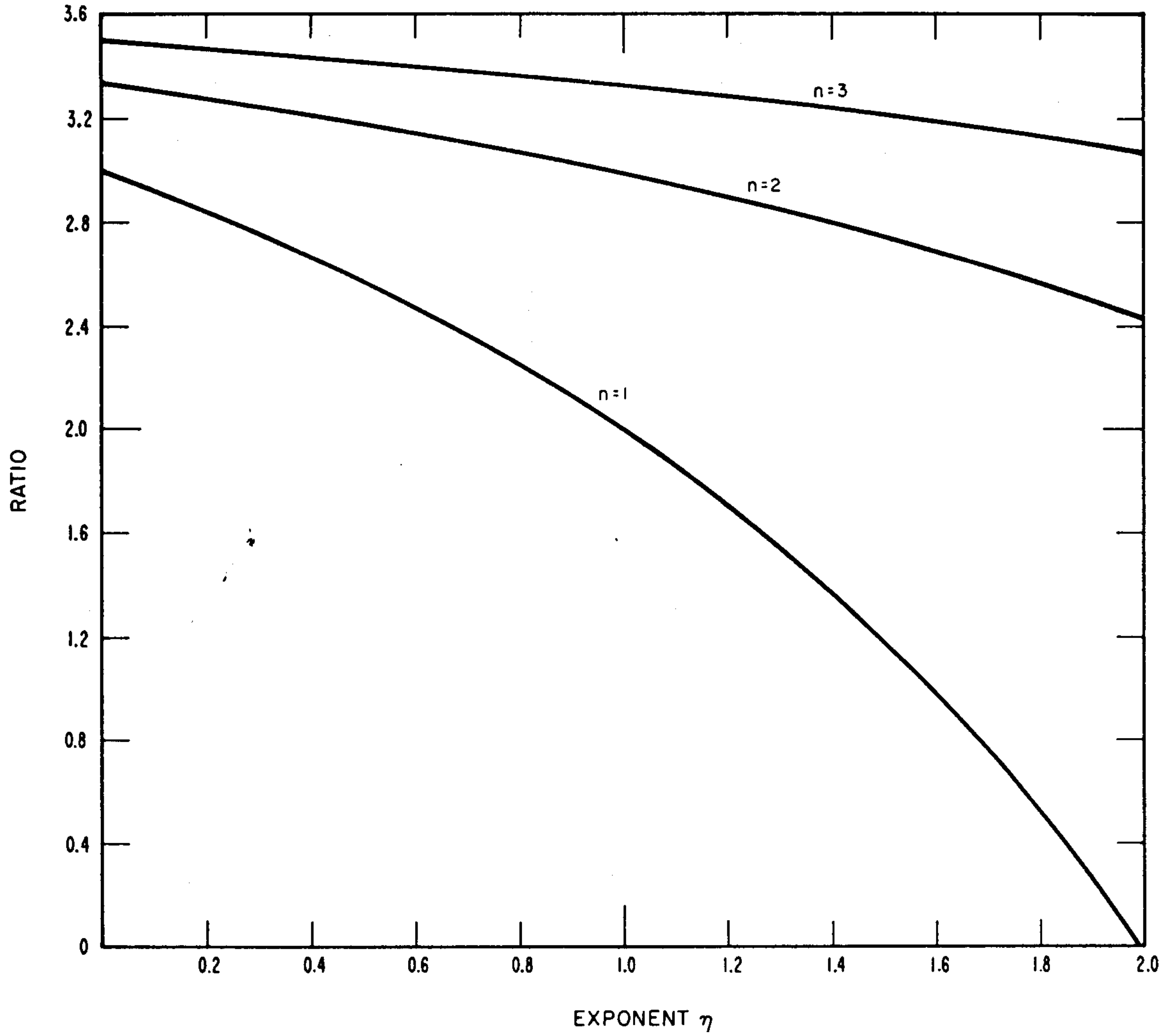

FIGURE 7: RATIO OF VARIANCES, $\frac{\langle(\Delta^{n+1}\epsilon_m)^2\rangle}{\langle(\Delta^n\epsilon_m)^2\rangle}$, AS A FUNCTION OF THE EXPONENT η

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Atomic Timekeeping and the Statistics of

Precision Signal Generators

by

James A. Barnes

ERRATA

Equation (6), page 209 should read:

$$\Delta t \simeq \Delta t_A - \int_{t_{A1}}^{t_{A2}} [\alpha t_A + \dot{\epsilon}(t_A)] dt_A. \tag{6}$$

Equation (8), page 209 should read:

$$\delta t = \int_{t_A}^{t_A + T} [\alpha t_A + \dot{\epsilon}] dt_A - T \left\langle \frac{\delta\Omega}{\Omega_0} \right\rangle_T, \tag{8}$$

In Table II on page 210 under "Quantity", the last line should read:

$$\Big\} \left\langle \left(\Delta^3 \phi_n \right)^2 \right\rangle$$

Equation (18), page 211 should read

$$\sigma_{12}^2 = \sigma_1^2 + \sigma_2^2 + 20 \langle \gamma^2 \rangle$$

$$\sigma_{13}^2 = \sigma_1^2 + \sigma_3^2 + 20 \langle \gamma^2 \rangle \tag{18}$$

$$\sigma_{23}^2 = \sigma_2^2 + \sigma_3^2 + 20 \langle \gamma^2 \rangle$$

On page 211 under II. Theoretical Development, the sixth line of text should read: $t_A \rightarrow t_A + \xi, \ldots.$

Equation (44) on page 213 should read:

$$\langle(\delta t)^2\rangle = -\sum_{n<\ell}^{m} a_n a_\ell \left\{2\langle\epsilon^2(t_A)\rangle - 2\langle[\epsilon(t_A)\cdot\epsilon(t_A + (b_\ell - b_n)\tau)]\rangle\right\}. \tag{44}$$

The equation between (48) and (49) should read:

$$S_\epsilon(\omega) = \text{F. T.}\left[R_\epsilon(0) - \frac{k_2|\tau|^{2\mu}}{2(4 - 2^{2\mu})}\right].$$

The quantity in the text just after (65), page 217 should read:

..., the errors are those of the standard alone, i. e.,

$$\frac{1}{2\pi f_s}\sqrt{\frac{B_1}{\tau}}.$$

Equation (68) should read:

$$\frac{\sqrt{\langle(\delta(NT))^2\rangle}}{NT} = \sqrt{\frac{2h\ \ln(n) + B_1\tau^{-1}(2\pi f_s)^{-2}}{N}} \tag{68}$$

Table III, the column opposite column "N" should read:

$$\frac{\left\langle\left(\sum_{n=1}^{N}\delta t_n\right)^2\right\rangle}{\langle(\delta t_n)^2\rangle}$$

The second equation after (70), page 218 should read:

$$\frac{\langle|\Delta^3\phi_{n+3} + \Delta^3\phi_n|^2\rangle}{\langle(\Delta^3\phi_n)^2\rangle} = 2.054,$$

The equation between (73) and (74) on page 219 should read:

$$\Psi^2(\tau, T) = -2h[2\rho^2 \ln\rho - (1+\rho)^2 \ln(1+\rho) - (\rho-1)^2 \ln(\rho-1)],$$

On page 219, second column, second paragraph, the first line of text should read:

Again, $\Psi(\tau, T)$ is a function of two variables with all ....

Equation (78) on page 220 should read:

$$\langle(\Delta^2\epsilon_m)^2\rangle = 6\langle(\epsilon_m)^2\rangle - 8\langle[\epsilon(t+\tau)\cdot\epsilon(t)]\rangle + 2\ \langle[\epsilon(t+2\tau)\epsilon(t)]\rangle. \tag{78}$$

ERRATA SHEET

STATISTICS OF ATOMIC FREQUENCY STANDARDS

by

David W. Allan

Equation (12) p.223 should have summation from n = 1 to N - 1 rather than from n = 0

$$\left(\sum_{n=1}^{N-1} (N-n) [\quad] \right)$$

Equation (22), page 225 should read

$$\lim_{\tau \to 0^+} \left\langle \left[\frac{\varphi(t+\tau) - \varphi(t)}{\tau} \right] \right\rangle = \langle \Omega(t) \rangle \quad .$$

Equation (24), page 225 should read

$$\langle \sigma^2(N, T, \tau) \rangle = \frac{k}{N(N-1)} \sum_{n=1}^{N-1} (N-n) \left[-2(nr)^2 \ln(nr) \right.$$

$$\left. + (nr+1)^2 \ln(nr+1) + (nr-1)^2 \ln(nr-1) \right] \quad .$$

The next to last line in paragraph 1 of the Appendix should read:

"zero; and if $|\tau \omega_B|^{|\alpha+1|} >> 1$, then the equation for $U(\tau)$. . ." .

USCOMM-BL

PAPER II

STATISTICS OF ATOMIC FREQUENCY STANDARDS

by

David W. Allan

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INDEX OF TABLES

STATISTICS OF ATOMIC FREQUENCY STANDARDS

David W. Allan

ABSTRACT

A study has been made of the types of noise on the signals of various atomic and molecular frequency standards made available in the Atomic Frequency and Time Standards Section of the National Bureau of Standards, Boulder Laboratories. These include masers (both H and $N^{15}H_3$), the cesium beam frequency standard employed as the United States Frequency Standard, and rubidium gas cell frequency standards.

The theoretical development results in a practical and straightforward method for determining the power spectral density of the frequency fluctuations from the variance of the freqency fluctuations, the sampling time, and the number of samples taken. Additional insight is also given into some of the problems that arise from the presence of "flicker noise" (power spectrum proportional to $|\omega|^{-1}$) modulation of the signal of an oscillator.

Due to the inherent dimensional instabilities in maser cavities one might have expected "flicker noise" frequency modulation to be present on the signal of a maser; however, in a comparison between the NBS hydrogen maser and the NBS III cesium beam, uncorrelated random noise was observed on the frequency fluctuations for sampling times extending to 4 hours; the fractional standard deviations of the frequency fluctuations were as low as 5 parts in 10^{14}. The absence of "flicker noise" frequency modulation was observed in a comparison between two ammonia masers with sampling times ranging from 0.1 sec. to 1000 sec. and with fractional standard deviations of the frequency fluctuations as low as 1 part in 10^{12}. "Flicker noise" frequency modulation was observed in rubidium gas cells and in quartz crystal oscillators used in the comparisons. An unusual power spectral density (proportional to $|\omega|^{-1/3}$)

of the frequency fluctuations was observed in a comparison between NBS II and NBS III cesium beams. This has been observed by others and its source is, at present, unexplained.

Key Words: Atomic Frequency Standards, Power Spectral Density, Short and Long Term Frequency Stability.

1. INTRODUCTION

As atomic timekeeping has come of age, it has become increasingly important to identify quality in an atomic frequency standard. Some of the most important quality factors are directly related to the inherent noise of a quantum device and its associated electronics. For example, a proper measurement and statistical classification[1] of this inherent noise makes it possible to determine the probable rate of time divergence of two independent atomic time systems, as well as giving insight concerning the precision and accuracy obtainable from an atomic frequency standard.

In the realm of precise frequency measurements, the properties of noise again play an important role. The relative precision obtainable with atomic frequency standards is unsurpassed in any field, and the precision limitations in this field are largely due to inherent noise in the atomic device and the associated electronic equipment. The standard deviation of the frequency fluctuations can be shown to be directly dependent on the type of noise in the system, the number of samples taken, and the dead time between samples. An understanding of this dependence becomes very interesting in light of an unusual type of noise that has been observed[1] on the output frequency of cesium beam atomic frequency standards-- namely, a power spectral density proportional to $|\omega|^{-1/3}$, where ω is 2π times the spectral frequency.

A very common and convenient way of making measurements of the noise components on a signal from a frequency standard is to compare two such standards by measuring the period of the beat frequency between the two standards. It is again the intent of the author to show a practical and easy way of classifying the statistics, i.e., of determining the power spectral density of the frequency fluctuations using this type of measuring system.

An analysis has already been made of the noise present in passive atomic frequency standards,[1] such as cesium beams, but a classification of the types of noise exhibited by the maser type of quantum-mechanical oscillator has not been made in the long term area, i.e., for low frequency fluctuations. Though this paper is far from exhaustive, the intent is to give additional information on the noise characteristics of masers. Because a maser's output frequency is more critically parameter dependent than a passive atomic device, it has been suggested[2] that the output frequency might appear to be "flicker noise" modulated where "flicker noise" is defined as a type of power spectral density which is inversely proportional to the spectral frequency, $\omega/2\pi$. It has been shown that if "flicker noise" frequency modulation is present on a signal from a standard some significant problems arise, such as the logarithmic divergence of the standard deviation of the frequency fluctuations as the number of samples taken increases, and also the inability to define precisely the time average frequency. It thus becomes of special interest to determine whether "flicker noise" is or is not present on the signal from a maser so that one might better evaluate its quality as a frequency standard.

Throughout the paper the paramount mathematical concern is the functional form of the equations with the hope of maintaining simplicity and of providing better understanding of the material to be covered.

2. METHODS EMPLOYED TO MEASURE NOISE

A. Power Spectrum and Variance Relationship

The average angular frequency, $\Omega_\tau(t)$, of an oscillator (to distinguish it from spectral frequency ω) over a time interval, τ, can be written

$$\Omega_\tau(t) = \frac{1}{\tau}\left[\phi(t+\tau) - \phi(t)\right], \tag{1}$$

where ϕ is the phase angle in radians. Now the variance of the frequency deviations is the square of the standard deviation, σ. Define the time average of a function as

$$\langle f(t)\rangle = \lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} f(t)\,dt \ . \tag{2}$$

One may, therefore, write the square of the standard deviation as follows:

$$\sigma^2 = \langle \Omega_\tau(t)^2\rangle - \langle \Omega_\tau(t)\rangle^2 \ . \tag{3}$$

One may assume with no loss of generality that the second term in Eq. (3) can be set equal to zero by a proper translation since it is the square of the time average frequency. $\Omega_\tau(t)$ is, therefore, now the frequency deviation from the average value, and $\phi(t)$ the integrally related phase deviation.

Substituting Eq. (1) into Eq. (3) gives

$$\sigma^2 = \frac{1}{\tau^2}\left[\langle \phi(t+\tau)^2\rangle - 2\langle \phi(t+\tau)\cdot\phi(t)\rangle + \langle \phi^2(t)\rangle\right] \ . \tag{4}$$

The time average of $\phi(t+\tau)\cdot\phi(t)$ is the auto-covariance function of the phase--denoted $R_\phi(\tau)$. One is justified in assuming that a time translation has no affect on the auto-covariance function,[1] therefore,[4]

$$\sigma^2 = \frac{2}{\tau^2} \left[R_\phi(o) - R_\phi(\tau) \right] \quad . \tag{5}$$

It is now possible to relate the variance of the squared frequency deviations to the power spectral density by use of the Wiener-Khinchin Theorem, which states that the auto-covariance function of the phase is equal to the Fourier transform, (F. T.), of the power spectral density of the phase, $S_\phi(\omega)$. The power spectral density of the frequency is related to this by the useful equation

$$S_{\dot\phi}(\omega) = \omega^2 S_\phi(\omega) \ .$$

Most of the discussion that follows is based on the restriction

$$S_\phi(\omega) = h|\omega|^\alpha \ . \tag{6}$$

That a singular type of power spectrum predominates over a reasonable range of ω has been verified experimentally. The region of interest for α is $-3 \leq \alpha \leq -1$, and $\alpha = 0$. This covers white noise phase modulation ($S_\phi(\omega) = h$), "flicker noise" frequency modulation ($S_{\dot\phi}(\omega) = h|\omega|^{-1}$), and includes, of course, white noise frequency modulation ($S_{\dot\phi}(\omega) = h$). Fortunately there have been tabulated the Fourier transforms of functions of the above form,[3] and the following transforms can be established:

$$\begin{aligned}
&\text{F.T. } |\omega|^\alpha = a'(\alpha) \cdot |\tau|^{-\alpha-1} \text{ for } \alpha \neq 0 \text{ or not an integer} \\
&\text{F.T. } |\omega|^{-0} = \delta(\tau) \\
&\text{F.T. } |\omega|^{-1} = a'(-1)\ (\ln|\tau| + C) \\
&\text{F.T. } |\omega|^{-2} = a'(-2) \cdot |\tau| \\
&\text{F.T. } |\omega|^{-3} = a'(-3) \cdot |\tau|^2\ (\ln|\tau| + C) \ ,
\end{aligned} \tag{7}$$

where C is an arbitrary constant and a is an α dependent coefficient. A useful substitution is the following:[1]

$$U(\tau) = 2\,(R_\phi(o) - R_\phi(\tau))\,. \tag{8}$$

Now $R_\phi(o)$, though finite and non-zero in general, has no importance to the problem at hand, and is physically non-observable because of its dependence on zero frequency components. (See Appendix A.) It, therefore, can be neglected for cases of physical interest, and thus $U(\tau)$ has the same functional form as $R_\phi(\tau)$, and may be written as follows:

$$U(\tau) = a(\alpha)\,|\tau|^{-\alpha-1};\ -3 \le \alpha \le -1$$

and (9)

$$U(\tau) = a(0);\ \alpha = 0,\ \tau \ne 0.$$

The standard deviation squared may, therefore, be written:

$$\sigma^2 = a(\mu) \cdot |\tau|^\mu$$

where

$$\mu = -\alpha - 3 \text{ if } -3<\alpha<-1 \tag{10}$$

and

$$\mu = -2 \text{ if } \alpha = 0.$$

The cases with $\alpha = -1$ or -3 will be discussed in Section D of Chapter 2.

Using Eqs. (6) and (10) one may, therefore, deduce the power spectral density from the dependence of the standard deviation of the frequency fluctuations on the sampling time.

B. Adjacent Sampling of Data

In actual practice, of course, the number of frequency or phase samples must be finite. The case to be considered now is one for which the phase or the frequency is monitored on a continuous basis. Two of the techniques used by the author to accomplish this were as follows: A device, described elsewhere,[1] was used to monitor the phase of the beat frequency between two oscillators at prescribed time intervals; the second technique was to measure the period of the beat frequency between two oscillators with two counters so that the dead-time of one counter corresponded to the counting time of the other.

A very powerful and meaningful method of analysis of data taken by a phase monitoring technique has been developed by James A. Barnes[1] at the National Bureau of Standards, Boulder, Colorado. The method employs the use of finite differences and is especially useful in analyzing "flicker noise" and long term frequency fluctuations in general. On the other hand, if one uses the period counting technique for data acquisition, the following form of analysis is useful. It also may be cast in a form where one can use the finite difference technique.

Data are often obtained with one counter measuring the frequency or the period of the beat note between two oscillators with a dead-time between counts. The concern of this section is with the dead-time being zero, but it is convenient to develop the general case for which the dead-time is non-zero for use in the next section, and specialize this to the continuous sampling case for which the dead-time is zero, which is the case of interest in this section.

Let T be the period of sampling, τ the sample time, and N the number of samples. The standard deviation, $\sigma(N, T, \tau)$,* of the frequency fluctuations may, therefore, be written as:

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*Note that this $\sigma(N, T, \tau)$ is not the same as the σ in Eqs. (3), (4), (5) and (10). $\sigma(N, T, \tau)$ is over a finite number of data samples, N, and to avoid confusion the variable N will always be used with σ in the finite sampling case as in Eq. (11).

$$\sigma^2(N, T, \tau) = \frac{1}{N-1}\left\{\sum_{n=0}^{N-1}\left[\frac{\phi(nT+\tau) - \phi(nT)}{\tau}\right]^2 - \frac{1}{N}\left[\sum_{n=0}^{N-1}\frac{\phi(nT+\tau) - \phi(nT)}{\tau}\right]^2\right\}. \tag{11}$$

Taking the time average of $\sigma^2(N, T, \tau)$ and making the substitution given in Eq. (8) yields

$$\langle\sigma^2(N, T, \tau)\rangle = \frac{1}{\tau^2}\left\{U(\tau) + \frac{1}{N(N-1)}\sum_{n=0}^{N-1}(N-n)\left[2U(nT) - U(nT+\tau) - U(nT-\tau)\right]\right\}. \tag{12}$$

If the dead-time were zero, then $T = \tau$, and Eq. (12) becomes[1]

$$\langle\sigma^2(N, \tau)\rangle = \frac{1}{(N-1)\tau^2}\left[N\,U(\tau) - \frac{1}{N}U(N\tau)\right]. \tag{13}$$

Remembering that $\mu = -\alpha-3$ and substituting Eq. (9) into (13) gives

$$\langle\sigma^2(N, \tau)\rangle = \frac{a(\mu)\,N\,|\tau|^{\mu}}{N-1}\left[1 - N^{\mu}\right];\ -3<\mu<0* \tag{14}$$

which establishes the interesting result of the dependence of the expectation value of the standard deviation of the frequency fluctuations on the numbers of samples, the sample time, and the power spectral density. It will be noted that

$$\langle\sigma^2(\infty, \tau)\rangle = a(\mu)\,|\tau|^{\mu};\ -3<\mu<0 \tag{15}$$

in agreement with Eq. (10).

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* See Appendix A

By keeping N constant and assuming μ to be constant over several different values of τ, it may be seen that the value of μ is the slope on a log-log plot of $\langle\sigma^2(N,\tau)\rangle$ versus τ. This provides a means of determining the power spectral density simply by varying the sample time over the region of interest.[4]

It is informative to look at the family of curves obtained from a plot of the dependence of $\langle\sigma^2(N,\tau)\rangle$ as a function of N for various pertinent values of μ to see how it approaches $\langle\sigma^2(\infty,\tau)\rangle$. The family of curves is shown in Fig. 1. One may notice that the convergence is much faster in the region between white noise frequency modulation and white noise phase modulation than between white noise frequency modulation and "flicker noise" frequency modulation.* In fact as $\mu \rightarrow 0$, the ratio approaches zero, and one would conjecture that the

$$\lim_{N\rightarrow\infty} \langle\sigma^2(N,\tau)\rangle$$

is infinite in the presence of "flicker noise" frequency modulation -- a result proven by J. A. Barnes.[1]

The data points plotted in Fig. 1 were extracted from a cesium beam-cesium beam comparison analyzed elsewhere,[1] and exhibit in this new formulation a type of frequency modulation proportional to $|\omega|^{-1/3}$, giving confirmation to this strange type of power spectral density.

It is possible to utilize the dependence of the standard deviation on the number of samples to determine a value of μ by considering a function which takes into account the extreme values of N obtainable from a finite set of data, namely,

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*To see that $\mu = 0$ corresponds to "flicker noise" see Chapter 2, Section D.

$$\chi(N, \mu) = \frac{\langle \sigma^2(N,\tau)\rangle}{\langle \sigma^2(2,\tau)\rangle}^{*} . \quad (16)$$

Tabulated values of this function are given in Table I and a plot of $\chi(N, \mu)$ as a function of μ for various values of N is given in Fig. 2. It may be noted that the function is most sensitive in the region between "flicker noise" frequency modulation, $\mu = 0$, and white noise frequency modulation, $\mu = -1$ -- one of basic interest. In practice the table and graph have proven very useful. $\chi(\infty, \mu)$ is plotted for comparison and computational purposes.

Note that $\chi(\infty, 0) = \infty$. In the development thus far no consideration has been given to the experimental fact that there must exist a lower cutoff frequency which keeps the functions considered from going to infinity, corresponding to certain types of noise, such as "flicker noise". The value of the cutoff frequency is not important other than to say that the functions considered are valid for times up to the order of $1/(\omega$ cutoff). This time is apparently more than a year for quartz crystal oscillators.[5] If "flicker noise" is present in some atomic frequency standards, the value of $1/(\omega$ cutoff) is probably shorter than for quartz crystal oscillators for reasons discussed later, and if "flicker noise" is not present, infinities do not occur in the functions considered for most other types of pertinent noise and hence there is no concern.

C. Non-Adjacent Sampling of Data

The next consideration is to determine the effect of counter dead-time on one's ability to deduce the statistics of an oscillator using

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* It will be noted that the τ dependence cancels in the expression for χ and hence it is N and μ dependent only. In the table and graphs, the τ dependence is not shown and is, therefore, suppressed since the μ dependence is the thing emphasized.

the techniques developed in the previous section. This form of data acquisition is one of the most common, and hence merits attention.

It was shown earlier that Eq. (12) is applicable to the present case, and if Eq. (9) is substituted into Eq. (12), with the assignment that $r = T/\tau$ (the ratio of the period of sampling to the sample time), then

$$\langle \sigma^2 (N, T, \tau) \rangle = a(\mu) |\tau|^{\mu} \left\{ 1 + \sum_{n=1}^{N-1} \frac{(N-n)\,(nr)^{\mu+2}}{N(N-1)} \left[2 - \left(1 + \frac{1}{nr}\right)^{\mu+2} - \left(1 - \frac{1}{nr}\right)^{\mu+2} \right] \right\} ; \tag{17}$$

$$-3 < \mu < 0.$$

An important result from Eq. (17) is that if N and r are held constant it is still possible to determine the value of μ by varying τ. Therefore, the relationship between the power spectral density and the standard deviation has the same form as for the continuous sampling case. Additional insight may be obtained by considering some special cases. If $\mu = -1$, $(S_{\dot{\phi}}(\omega) = h)$,[6] the series in Eq. (17) goes to zero for all possible values of r and hence

$$\langle \sigma^2 (N, T, \tau) \rangle \doteq \frac{a(-1)}{|\tau|} \tag{18}$$

for white noise frequency modulation; this is the same result obtained in the continuous data sampling case. One notices that Eq. (18) is independent of N as would be expected since the frequency fluctuations are uncorrelated. If $\mu = -2$, $(S_{\phi}(\omega) = h)$,[6] the series in Eq. (18) again goes to zero for all values of $r > 1$, and the frequency fluctuations appear to be uncorrelated in the presence of white noise phase modulation as long as the measurement dead-time is non-zero. Using Eq. (14) for the case where $r = 1$, gives $\langle \sigma^2 (N, T, \tau) \rangle$ being equal to $[a(-2) \cdot (N+1)/N\tau^2]$.

One therefore has the unusual result that

$$\langle\sigma^2(N, T, \tau)\rangle = \frac{N + \delta(r-1)}{N} \cdot \frac{a(\mu)}{\tau^2}, \tag{19}$$

where

$$\delta(r-1) = \begin{cases} 1, & r=1 \\ 0, & r\neq 1. \end{cases}$$

A slight N dependence then appears only in the continuous sampling case. Experimentally, one can show that the Kronecker δ-function is replaced by $R_\phi(r-1)/R_\phi(0)$ because of the finite band widths involved.

It will be recalled that the curves in Figs. 1 and 2 are for $r=1$ (the dead-time equal zero). The results of Eq. (19) show a character change in the curves for $r>1$ and $-3<\mu<-1$. For now the curve for $\mu = -2$ is coincident with the curve for $\mu = -1$ in Fig. 1 and $\chi(N, -2) = \chi(N, -1) = 1.0$ in Fig. 2. No profound character change occurs for $-1<\mu<0$.

It is possible to show that the series in Eq. (17) approaches zero as N approaches infinity for all values of $r>1$ and $\mu<0$, hence $\langle\sigma^2(N, T, \tau)\rangle$ has the same asymptotic value as for the continuous sampling technique, independent of the counter dead-time.

If a binomial expansion is made of the second two terms in the series expression of Eq. (17), and 4th order terms and higher in $1/nr$ are neglected, the following simplification occurs:

$$\langle\sigma^2(N, T, \tau)\rangle = a(\mu)\,|\tau|^\mu\left[1 - \frac{(\mu+2)(\mu+1)}{N(N-1)}\; r^\mu \sum_{n=1}^{N} (N-n)\, n^\mu\right]. \tag{20}$$

On the first observation of Eq. (20) one may notice that as r becomes large, the standard deviation approaches its asymptotic value. This occurs when one is taking samples much shorter than the capable reset-time of the counter. The dependence on N is, therefore, reduced as r^{μ}. In fact, it has been determined by a computer analysis of Eq. (17) that the net effect of increasing the dead-time is to collapse the curves in Figs. 1 and 2 towards the unit axis.

D. The "Flicker Noise" Problem

The existence of "flicker noise" frequency modulation on the signal of quartz crystal oscillators has caused difficulty in handling such quantities as the auto-covariance function of the phase and the standard deviation of the frequency fluctuations. As mentioned previously the development by J. A. Barnes [1] makes it possible to classify "flicker noise" frequency modulation without any divergence difficulties or dependence on the value of the low frequency cutoff.

Some of the other difficulties associated with the presence of "flicker noise" frequency modulation, as Barnes has shown, are illustrated in the following equations:

$$\langle \sigma^2(N, \tau) \rangle = \frac{2h\, N \ln N}{N-1} , \tag{21}$$

and

$$\lim_{\tau \to o^+} \left\langle \left[\frac{\phi(t+\tau) - \phi(t)}{\tau} \right] \right\rangle = \Omega(t) . \tag{22}$$

The fact that the standard deviation diverges with N, as shown in Eq. (21), is an annoyance, aside from the fact that it becomes more difficult to write specifications on the frequency fluctuations of an oscillator. The limit expressed in Eq. (22) does not exist or is dependant on the low frequency cutoff; this, of course, affords difficulty in defining a frequency, and would be an unfortunate property to be present in a frequency standard. As has been stated previously for all values of μ considered thus far, $-3 < \mu < o$, the limit as N approaches infinity of $\langle \sigma^2(N, \tau) \rangle$ exists; and as the sampling time increases, $\langle \sigma^2(N, \tau) \rangle$ converges toward zero or perfect precision.

Data have been analyzed that indicate the presence of "flicker noise" frequency modulation on the signals from rubidium gas cell frequency standards. It is of concern to determine if this type of noise is present on the signals of other atomic frequency standards, and specifically masers. It is not the intent to determine the source or sources of "flicker noise" as this is a ponderous problem in and of itself, but perhaps only infer where such noise might arise.

Consider, now, ways to establish the presence of "flicker noise." If the data were on a continuous basis, the method of finite differences[1] developed by Barnes would be very useful. One may also notice from Eq. (21) that if N is held constant, the value of μ is zero for values of τ in the "flicker noise" frequency modulation region. If data were taken by the common technique of non-adjacent samples, the following considerations are of value:

For "flicker noise" frequency modulation $U(\tau)$ takes on a different form as a result of the divergence of the auto-covariance function of the phase,[1]

$$U(\tau) = \frac{k|\tau|^{\mu+2}}{4-2^{\mu+2}} . \tag{23}$$

The constant k is dependent on the quality of the oscillator. If Eq. (23) is substituted into Eq. (12) letting $\mu \to o$ ($\mu = o$ corresponds to "flicker noise"), an indeterminate form results for $\langle \sigma^2(N,T,\tau) \rangle$. Applying L'Hospital's rule and then passing to the limit gives the following equation:

$$\langle \sigma^2(N,T,\tau) \rangle = \frac{k r^2}{N(N-1)} \sum_{n=1}^{N-1} (N-n) \Big[-2n^2 \ln (n r) + (n+\frac{1}{r})^2 \ln (nr+1) + (n-\frac{1}{r})^2 \ln (nr-1) \Big] . \tag{24}$$

Hence for a fixed ratio $r = T/\tau$, and a determined number of samples, N, the standard deviation is constant, independent of τ. So for both adjacent and non-adjacent data sampling, the value of μ is zero. One may therefore determine if "flicker noise" frequency modulation is present on a signal sampled in a non-adjacent fashion, subject to the above constraints.

A consideration of interest at this point is "flicker noise" phase modulation ($S_\phi(\omega) = h|\omega|^{-1}$). Substituting F. T. $|\omega|^{-1}$ from Eq. (7) into Eq. (13) gives

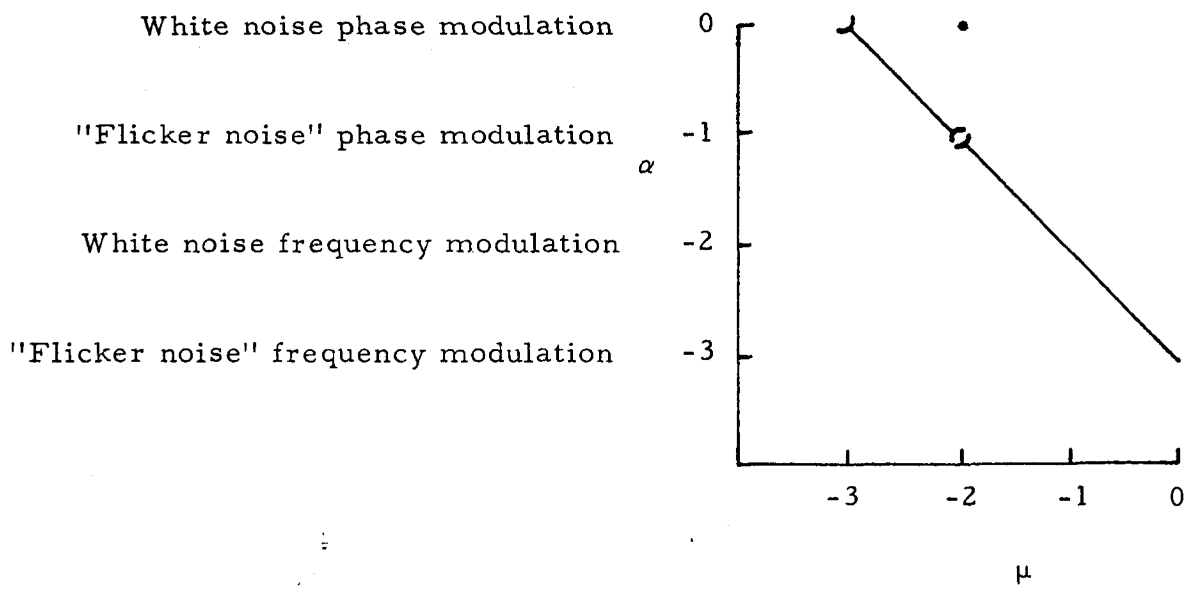
$$\langle\sigma^2(N,T,\tau)\rangle = \frac{N+1}{N\tau^2}(a\ln|\tau| + C) - \frac{a\ln N}{N(N-1)\tau^2}, \qquad (25)$$

where C is an arbitrary constant and a is determined from the Fourier transform. One thus obtains the somewhat unfortunate result that experimentally it would be unlikely that one could distinguish between "flicker noise" phase modulation and white noise frequency modulation, since $\mu \simeq -2$ in both cases. The only difference is an additional logarithmic dependence on τ for the "flicker noise" case. However, one might be able to infer from the experimental setup which of the two types of noise was being observed. If this were not possible, one would be forced to determine the type of noise present by some other technique.

If the number of samples, N, and the ratio of the period of sampling to the sample time, r, be held constant the following general equation can be written for both adjacent and non-adjacent sampling of data:

$$\langle\sigma^2(N,T,\tau)\rangle = K(N,r)\,|\tau|^{\mu}. \qquad (26)$$

Using Eq. (26) and Eq. (6) ($S_\phi(\omega) = h|\omega|^2$) coupled with the results established thus far, one is now able to extract the power spectral density from the dependence of the standard deviation on sample time for values of μ ranging from $-3 < \mu \leq 0$ with some limitation in the region of $\mu = -2$ as discussed above. The following mapping of μ into α is an informative summary.



3. ATOMIC FREQUENCY STANDARDS

A. Passive Atomic Standards

Devices such as atomic beam machines, and rubidium gas cells have been made to generate impressively stable frequencies.[7,8] In October 1964, the appropriately authorized International Committee of Weights and Measures adopted as a provisional definition for the measurement of time the transition between the $F = 4$, $m_F = 0$ and $F = 3$, $m_F = 0$ hyperfine levels of the ground state $^2S_{1/2}$ of the atom of cesium 133, unperturbed by external fields, with the assigned frequency for the transition of 9,192,631,770 Hz. The cesium beam at NBS, Boulder, has also been established as the United States Frequency Standard. The analysis of the theory of operation of these quantum devices has been covered in many publications,[7,8] along with the methods of slave-locking an oscillator to a given transition.[8] An analysis of the noise that should be present in an oscillator servoed to an atomic transition has been made by Kartaschoff,[9] Cutler,[6] and others. The author desires to reiterate at this point some of the results of these analyses: the frequency should appear white noise modulated, therefore, the phase fluctuations will go as the random walk phenomena, and hence the mean square time error in a clock running from one of these frequency standards would be proportional to the running time. Barnes has also shown that[1]

$$\langle (\Delta^3 \phi)^2 \rangle \sim \langle (\delta t)^2 \rangle, \tag{27}$$

where Δ^3 denotes the 3rd finite difference and δt the clock error time for a running time t, and also that

$$\begin{gathered} \langle (\Delta^3 \phi)^2 \rangle \sim |\tau|^{\mu+2} \\ -2 \leq \mu \leq 0 . \end{gathered} \tag{28}$$

The results from combining Eq. (27) and Eq. (28) are obvious, but still very important, i.e., if either the power spectrum is known or the dependence of $\langle \sigma^2 \rangle$ on the sample time, one may then determine the rate of time divergence and conversely, [see Eqs. (7) and (10)],

$$\langle (\delta t)^2 \rangle \sim |\tau|^{\mu+2}; \quad -2 \leq \mu \leq 0. \tag{29}$$

An example of the above is illustrated by the following: if in fact "flicker noise" frequency modulation is present on the signal of rubidium gas cells, and if one assumes the r.m.s. time errors were equal on clocks driven by a rubidium gas cell and a cesium beam of theoretical form at 1/3 of a day -- say for example 0.1 microseconds, then the accumulated r.m.s. time error after 1 year would be of the order of 100 microseconds for the rubidium cell and 10 microseconds for the cesium beam. If, however, the $|\omega|^{-1/3}$ type of frequency modulation that has been observed on cesium beams persisted for frequencies down to one cycle per year, then the r.m.s. time error would be about $100^{2/3}$ or about 20 microseconds instead of its theoretical value of 10 microseconds.

B. Masers

Masers, in contrast to the passive atomic devices discussed in the previous section, may be used as quantum mechanical oscillators.[10] Since the sources of "flicker noise" frequency modulation could arise from many different mechanisms, the suggestion exists that perhaps the active character of masers could be influenced by one of these mechanisms. Most of the basic experimental research associated with this paper and performed by the author has been to determine if the above suggestion is valid or not, and specifically to determine the type or types of noise present on the signal from masers.

One $N^{15}H_3$ maser has been in operation at NBS, Boulder, for several years. It is desirable to compare similar atomic devices in looking at noise; so another $N^{15}H_3$ maser was put into operation for direct comparison purposes. Worth considering at this time is the basic operation of a maser so that arguments made later on will be understandable.

Three basic elements of a maser are the source, the energy state selector, and the resonant cavity. A picture of these three elements in an $N^{15}H_3$ maser is shown in Plate 1. A needle valve and a collimating nozzle are coupled to the source to provide a highly directed beam of $N^{15}H_3$ down the axis of the four-pole focusers. The four-pole focusers are the energy state selectors and consist of four electrodes with alternate high voltages so that on axis (in line with the resonant cavity) the electric field is zero, but slightly off axis the field is large and increases as the distance off axis. Because ammonia has induced electric dipole moment, an interaction occurs as it enters this electric field region, and as it has been shown[10] the low energy states of the inversion levels of ammonia are defocused while the high energy states are focused on axis and into the resonant cavity. Therefore, if a noise component of the proper frequency is present and the ammonia beam flux is sufficient, a regenerative process will take place in which a high energy state molecule will decay and radiate a quantum of energy , "$h\nu$" causing the field to increase in the cavity and hence inducing other molecules to undergo the transition, etc. The radiation may be coupled off with a wave guide as seen on the right of Plate 1, and then with an appropriate detection scheme, as seen in Plate 2, one may observe the maser oscillations at a frequency ν where, neglecting any frequency pulling effects, ν is approximately 22, 789, 421, 700 Hz for $N^{15}H_3$.

There are many parameters that affect the output frequency of an ammonia maser. It is from these parameters that one may expect correlation of frequency fluctuations over long times such as exist for "flicker noise" frequency modulation. To see the mechanism,consider the fundamental pulling equation [10]

$$\nu_o - \nu \doteq \frac{\Delta\nu_1}{\Delta\nu_c} (\nu_o - \nu_c) + B , \tag{30}$$

where ν_o is the unperturbed transition frequency, ν the maser output frequency, $\Delta\nu_l$ the transition line width, $\Delta\nu_c$ the cavity band width, ν_c the cavity's frequency, and B is a term involving the basic parameters such as ammonia beam flux, the voltage on the state selectors, and the magnetic and electric field intensities inside the cavity. Most of the quantities in Eq. (30) are dependent on temperature either directly or through the coupling electronics. Some of the quantities in Eq. (30) are dependent on the maser's alignment affording another mechanism for correlation of the long term frequency fluctuations.

As described in detail elsewhere,[11] it is possible to construct an electronic servo that will tune the resonant frequency of the cavity to that of the ammonia transition frequency ($\nu_o = \nu_c$). The tuning is not prefect because of the noise present. The effect this has, as can be seen from Eq. (30), is profound, greatly reducing the effect of the basic parameters and almost entirely eliminating cavity dimensional instability. The correlation now existing in the long term frequency fluctuations may well be expected to be masked out by other noise in the system.

Since the servo mechanism is of the frequency lock type, and if the noise in the servo is white over some finite band width, then the fluctuations on the output frequency would be white inasmuch as they were caused by the servo. This is the same as for passive atomic devices -- a result not altogether unexpected.

4. EXPERIMENTS PERFORMED

A. $N^{15}H_3$ Maser Comparison

Two $N^{15}H_3$ masers were compared by the following technique. Each maser has coupled onto its output wave guide a balanced crystal detector and onto each detector a 30 MHz I. F. amplifier. Since the frequency of the inversion transition in $N^{15}H_3$ is about 22, 790 MHz, a local oscillator signal at 22,760 MHz is inserted into each remaining leg of the balanced crystal detectors. The 30 MHz beat frequencies resulting from the output of the I. F. amplifiers are then compared in a balanced mixer and its audio output is analyzed. The frequency of the audio signal can be adjusted to a reasonable value by offset tuning the cavity of one of the masers. The period of this audio signal is determined with a counter; the data are punched on paper tape for computer analysis. The computer determines the average value of the fractional standard deviation "σ^2" along with its confidence limit for pertinent discrete values of the sampling time "τ."

In all of the maser data taken it was necessary to subtract out a systematic but well understood linear drift -- due to a change of the ammonia beam flux in the $N^{15}H_3$ maser and due to cavity dimensional drift from changing temperature in the case of the H maser. This was accomplished by the method of least squares.[12] The ability to change the number of samples "N" was also built into the computer program. A typical computer fortran program is given in Appendix B.

A plot of the computer output for some of the data analyzed in the $N^{15}H_3$ maser comparison is shown in Fig. 4. The output frequencies of the masers were not servo-controlled for these data. The slope is very nearly that of white noise frequency modulation. It will be noted that for a time of one second, the fractional standard deviation of the frequency fluctuations is less than 2×10^{-12}.

B. Cesium Beam, Maser, Quartz Oscillator Intercomparisons

Many comparisons were made between all the different types of atomic devices made available in the Atomic Frequency and Time Standards Section of the National Bureau of Standards. Most of the comparisons were made at 5 MHz by measuring the period of the beat frequency with a counter, and the resulting data were processed by a computer (see Appendix B).

A comparison was made between the double beam $N^{15}H_3$ maser with the cavity servo in operation as illustrated in Fig. 3, and the NBS III cesium beam. A 5 MHz quartz crystal oscillator of good spectral purity was phase locked by a double heterodyne technique to the output frequency of the maser, and this oscillator was compared with a high quality quartz crystal oscillator that was frequency locked to NBS III. A plot of the data is shown in Fig. 5. The noise is undoubtedly that of the maser cavity servo system, and though it exhibits very nearly its theoretical value of $\mu = -1$ for white noise frequency modulation the noise level will be seen to be well over an order of magnitude higher than for the free-running maser.

The free-running, single beam, $N^{15}H_3$ maser was compared with NBS III on a longer time basis and indicated the presence of "flicker noise" frequency modulation. There was indicated a fair amount of uncertainty to the data, however.

In Fig. 6 one sees a very interesting plot in the comparison of the NBS III cesium beam with a very high quality quartz crystal oscillator. The comparison was again made at 5 MHz as indicated above. The first part of the curve shows the white noise frequency modulation in the cesium beam servo system, and then the curve changes slope indicating the presence of "flicker noise" frequency modulation as is typically exhibited by quartz crystal oscillators. This shows experimentally the theoretical result discussed previously, that one may have quite a high level of white noise in a system and eventually the system will be better than one with "flicker noise" frequency modulation.

As extended comparison was made (58 hours) between the H maser, NBS III cesium beam, a rubidium gas cell, and a quartz crystal oscillator. Some of the data are still to be analyzed, but some interesting and impressive results have been obtained thus far. Fig. 7 shows the analysis of some of the data obtained from the H maser, cesium beam (NBS III) comparison. White noise frequency modulation is exhibited for sampling times extending to 4 hours with the very impressive standard deviation of $\sim 6 \times 10^{-14}$ for $\tau = 4$ hours. "Flicker noise" frequency modulation was observed on both the rubidium gas cell and on the quartz crystal oscillator in this comparison.

5. CONCLUSIONS

"Flicker noise" frequency modulation was not observed on the frequency fluctuation of the maser type of atomic frequency standard for any of the reliable data obtained -- the time coverage here being from about 0.1 seconds to 4 hours. The presence of "flicker noise" was indicated for free running masers (both H and $N^{15}H_3$) in the long term range, though the data were somewhat unreliable. This type of noise was eliminated in the case of the $N^{15}H_3$ maser by use of a cavity servo mechanism, and one might infer from the similarities in the masers that the same would hold true for the H maser.

It is acknowledged that if "flicker noise" is present on the frequency fluctuations of a free-running maser for longer times, the lower cut-off frequency must be much shorter than for quartz crystal oscillators -- for indeed the maser operator is himself a part of a cavity servo when he retunes the cavity or perhaps recoats the quartz bulb.

The existence of "flicker noise" frequency modulation on the frequency fluctuation of rubidium gas cells is just another indication that they would not at present make a reliable primary frequency standard.

There are many areas where additional data analysis would be informative. It would be very interesting to look at the behavior of the cesium beam, hydrogen maser comparison, shown in Fig. 7, for extended times. Another anlysis of interest would be that of a cavity servoed H maser. Such a servo system is under development for the NBS H maser. To then compare the cavity servoed H maser with a cesium beam and its associated servo system would afford great insight into determining which of these competitive atomic standards should be primary.

6. ACKNOWLEDGMENTS

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APPENDIX A

To avoid certain divergence difficulties the following approach is useful in the region from $-3 < \mu \leq -2$. Assume that

$$U(\tau) = a(\mu)\,\tau^{\mu+2} \quad , \tag{A1}$$

and rewrite Eq. (8) as

$$R_\phi(\tau) = R_\phi(o) - \frac{1}{2}\,U(\tau) \; . \tag{A2}$$

By taking the inverse of the Wiener - Khinchin Theorem:

$$S_\phi(\omega) = F.\,T.^{-1}\,R_\phi(\tau) \quad , \tag{A3}$$

and applying to Eq. (A2) yields

$$S_\phi(\omega) = \frac{R_\phi(o)}{2\pi}\,\delta(\omega) - \frac{1}{2}\,F.\,T.^{-1}U(\tau) . \tag{A4}$$

Since experimentally one cannot look at $\omega = o$, the delta function contributes nothing, and therefore:

$$S_\phi(\omega) = h\,|\omega|^{-\mu-3} \tag{A5}$$

$$\text{for } -3 < \mu < -2 \, ,$$

and

$$S_\phi(\omega) = h$$

$$\text{for } \mu = -2,$$

where

$$\langle \sigma^2 \rangle = a(\mu)\,\tau^{\mu} \; .$$

APPENDIX B

Fortran Computer Program for Determining the Dependence of the Standard Deviation of the Frequency Fluctuations as a Function of the Sampling Time with the Ability of Varying N (The Number of Samples Taken)

```
 1 format (f7.0,i2)
 2 format (f9.2,2x,e8.3,5x,i3)
 3 format (50h;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;)
 4 format (8hsigma;sq,6x,4h,or-,5x,4htime)
   dimension f(256)
 5 pause 0011,ii
26 read 3
   punch 3
   punch 4
   do 27 n=1,256
   read 1 f(n),k
   f(n)=.2e11/f(n)
   if (k-1) 27,28,17
27 continue
28 a=0.
   aa=0.
   ab=0.
   b=0.
   do 29 k=1,n
   a=a+k
   aa=aa+k'k
   ab=ab+k'f(k)
29 b=b+f(k)
   ab+(ab'a-aa'b)/(a'a-n'aa)
   b=(b-n'ab)/a
```

```
      do 30 k=1,n
   30 f(k)=f(k)-ab-k'b
      k=1
    7 m=n/ii
      n=n-ii+1
      ab=0.
      aa=0.
      do 13 j=1,n,ii
      a=0.
      b=0.
      jj=ii+j-1
      do 10 i=j,jj
      a=f(i)+a
   10 b=f(i)'f(i)+b
      a=(b-(a'a)/ii)/(ii-1)
      aa=aa+a
   13 ab=ab+a'a
      aa=aa/m
      ab=(ab-m'aa'aa)/(m'm-m)
      ab=sqrtf(ab)
      punch 2,aa,ab,k
      k=2'k
      n=n+ii-1
      n=n/2
      nn=ii-1
      if (n-nn) 16,16,14
   14 do 15 i=1,n
      nn=2'i-1
```

```
15 f(i)=(f(nn)+f(nn+1))/2.
   go to 7
16 pause 0001,i
   if (i-1) 17,5,26
17 stop 7777
   end
   end
```

A PLOT SHOWING THE DEPENDENCE OF THE STANDARD DEVIATION OF THE FREQUENCY FLUCTUATION ON THE NUMBER OF SAMPLES AND THE TYPE OF NOISE PRESENT

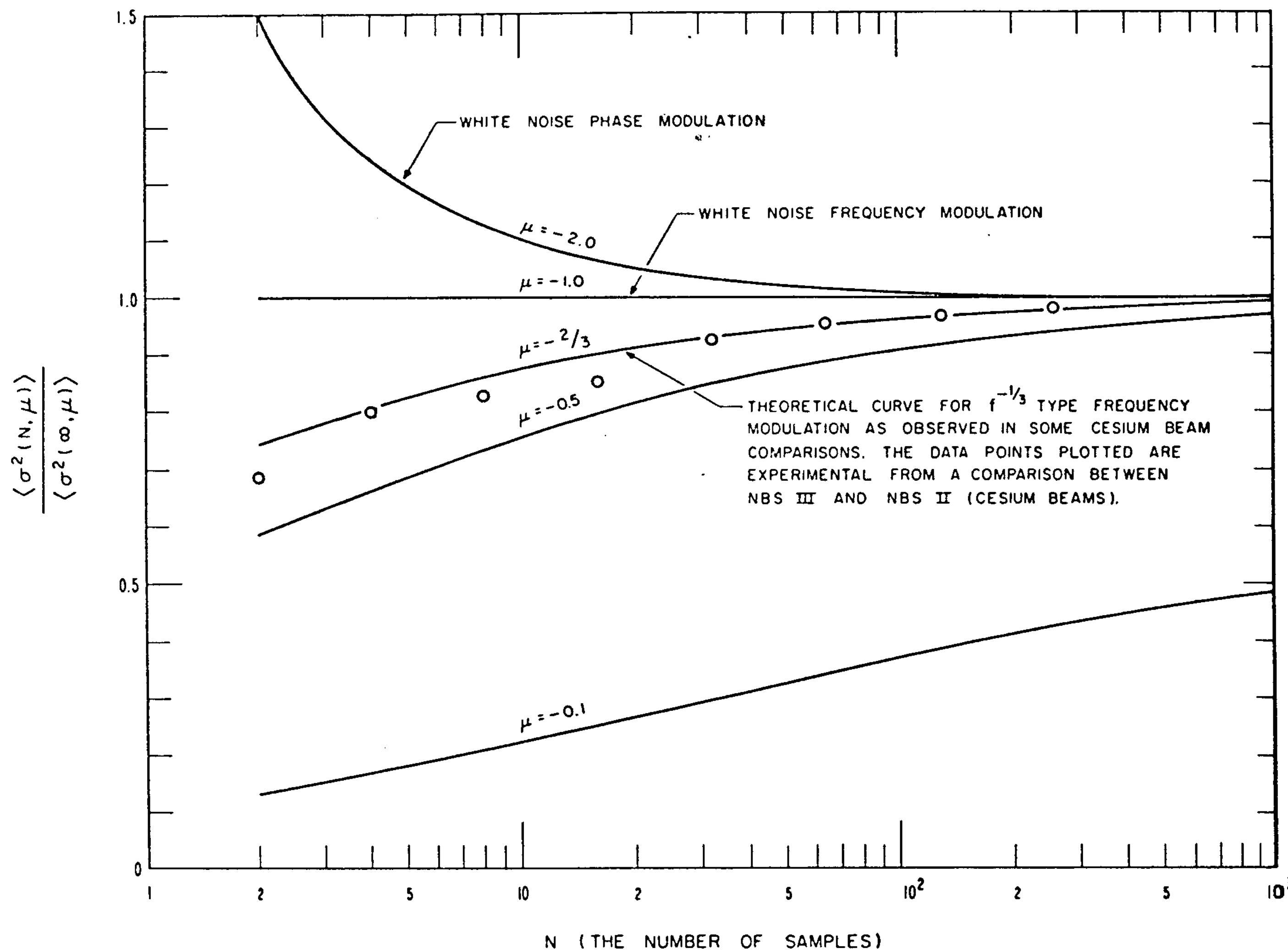


Figure 1

B-52284

A PLOT ENABLING ONE TO EXPERIMENTALLY EXTRACT THE STATISTICS OF AN OSCILLATOR BY KNOWING THE STANDARD DEVIATION FOR N SAMPLES AND FOR 2 SAMPLES USING $G_{\dot{\phi}}(\omega) = h|\omega|^{-\mu-1}$ EXCEPT AT $\mu = -2, -3$.

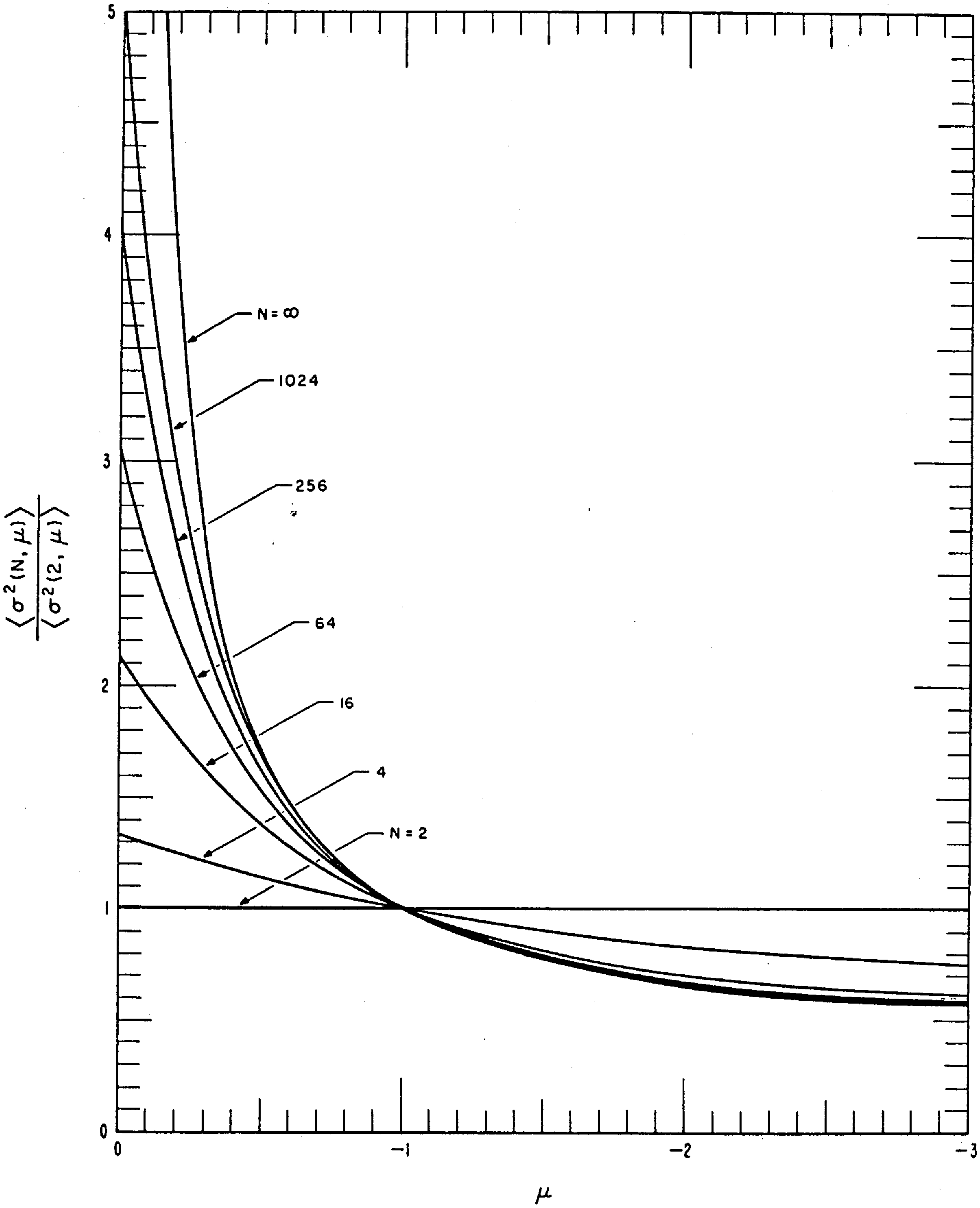


Figure 2

BLOCK DIAGRAM OF MASER CAVITY SERVO SYSTEM

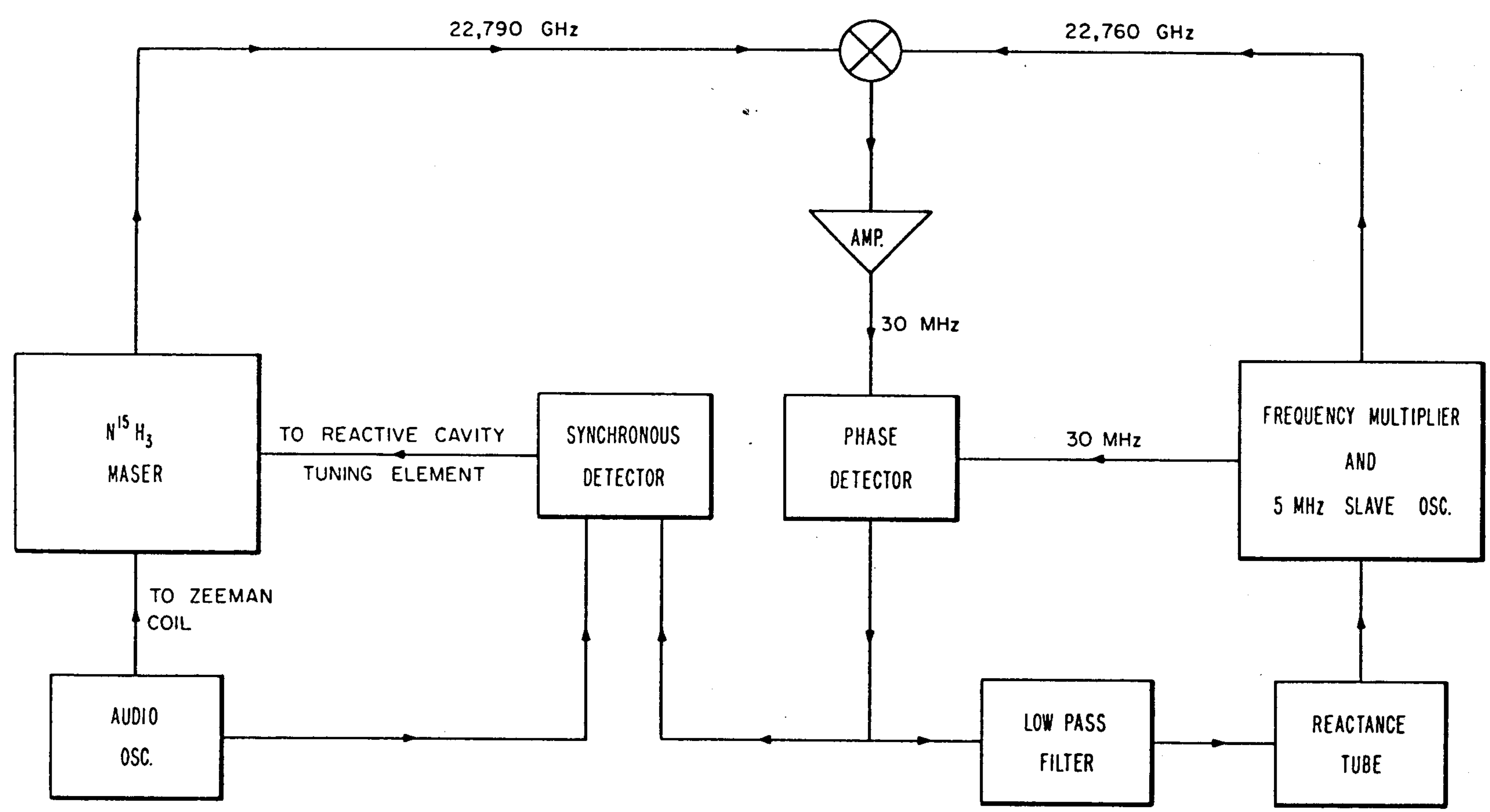


Figure 3

COMPARISON OF TWO $N^{15}H_3$ MASERS; STANDARD DEVIATION OF THE FREQUENCY FLUCTUATIONS AS A FUNCTION OF SAMPLING TIME

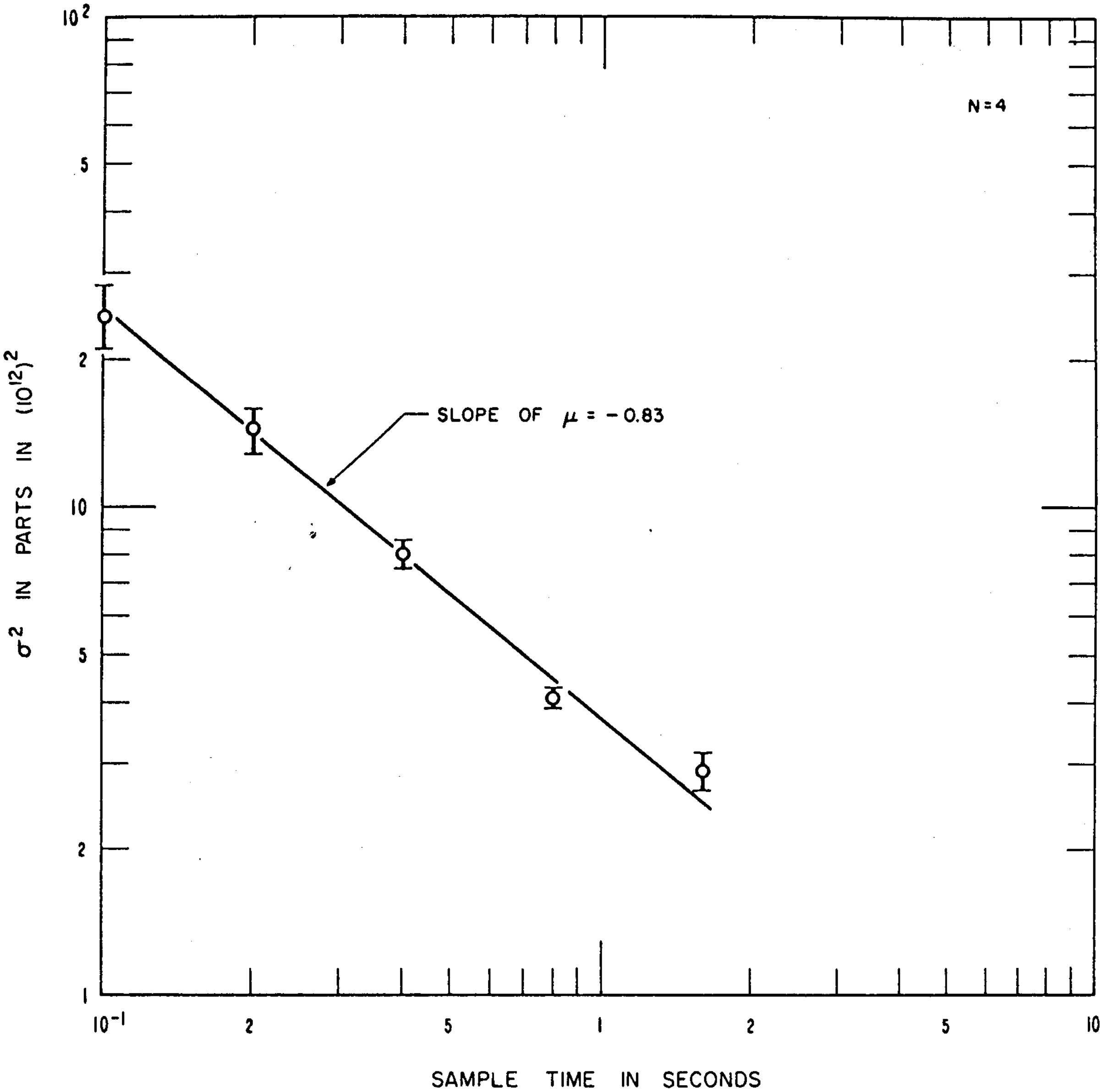


Figure 4

B- 52286

CESIUM BEAM (NBS III),
$N^{15}H_3$ MASER (2) COMPARISON; STANDARD DEVIATION OF
THE FREQUENCY FLUCTUATIONS AS A FUNCTION OF SAMPLING TIME.

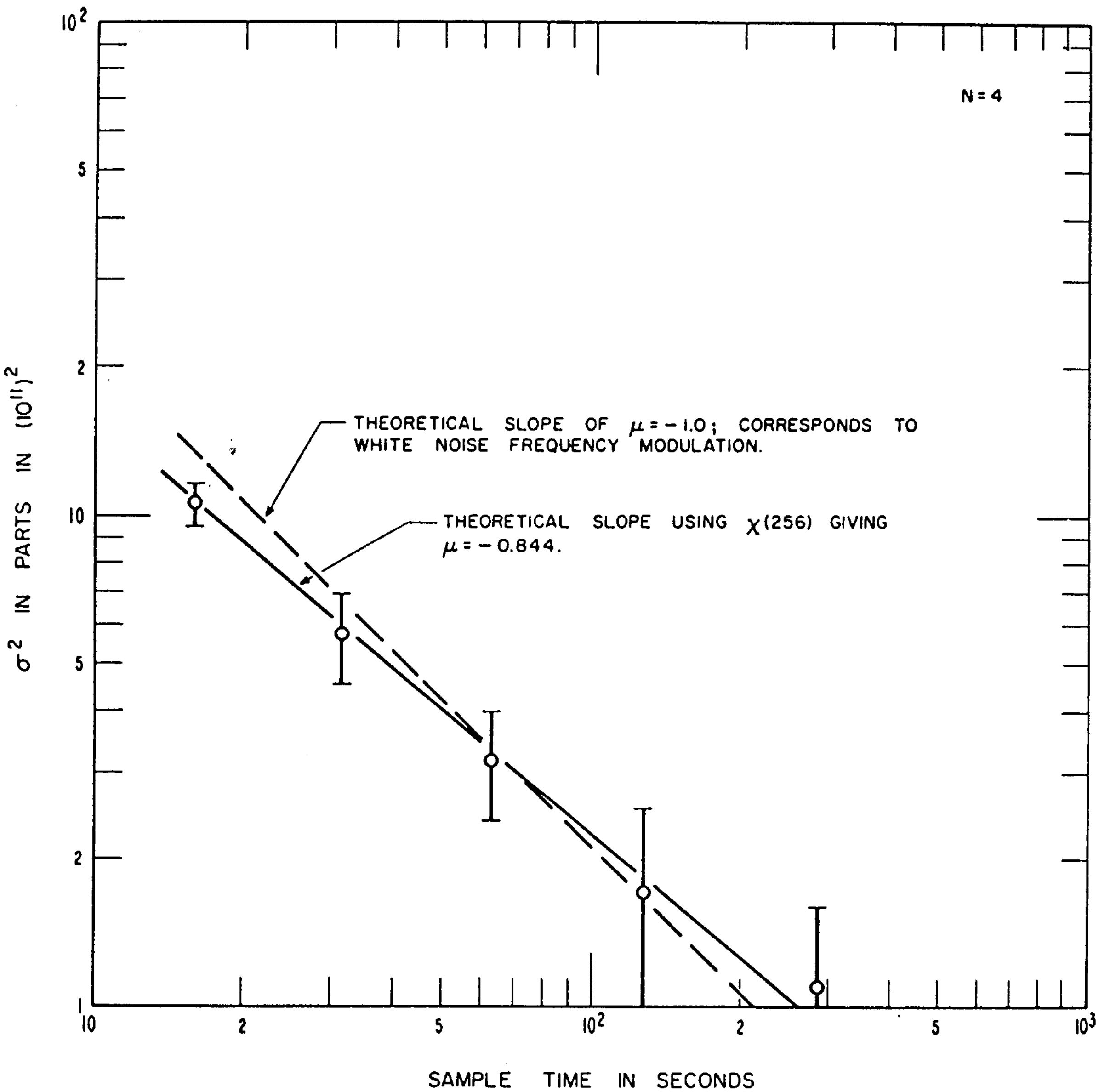


Figure 5

CESIUM BEAM (NBS III),
QUARTZ CRYSTAL OSCILLATOR COMPARISON; STANDARD DEVIATION
OF THE FREQUENCY FLUCTUATIONS AS A FUNCTION OF SAMPLING TIME

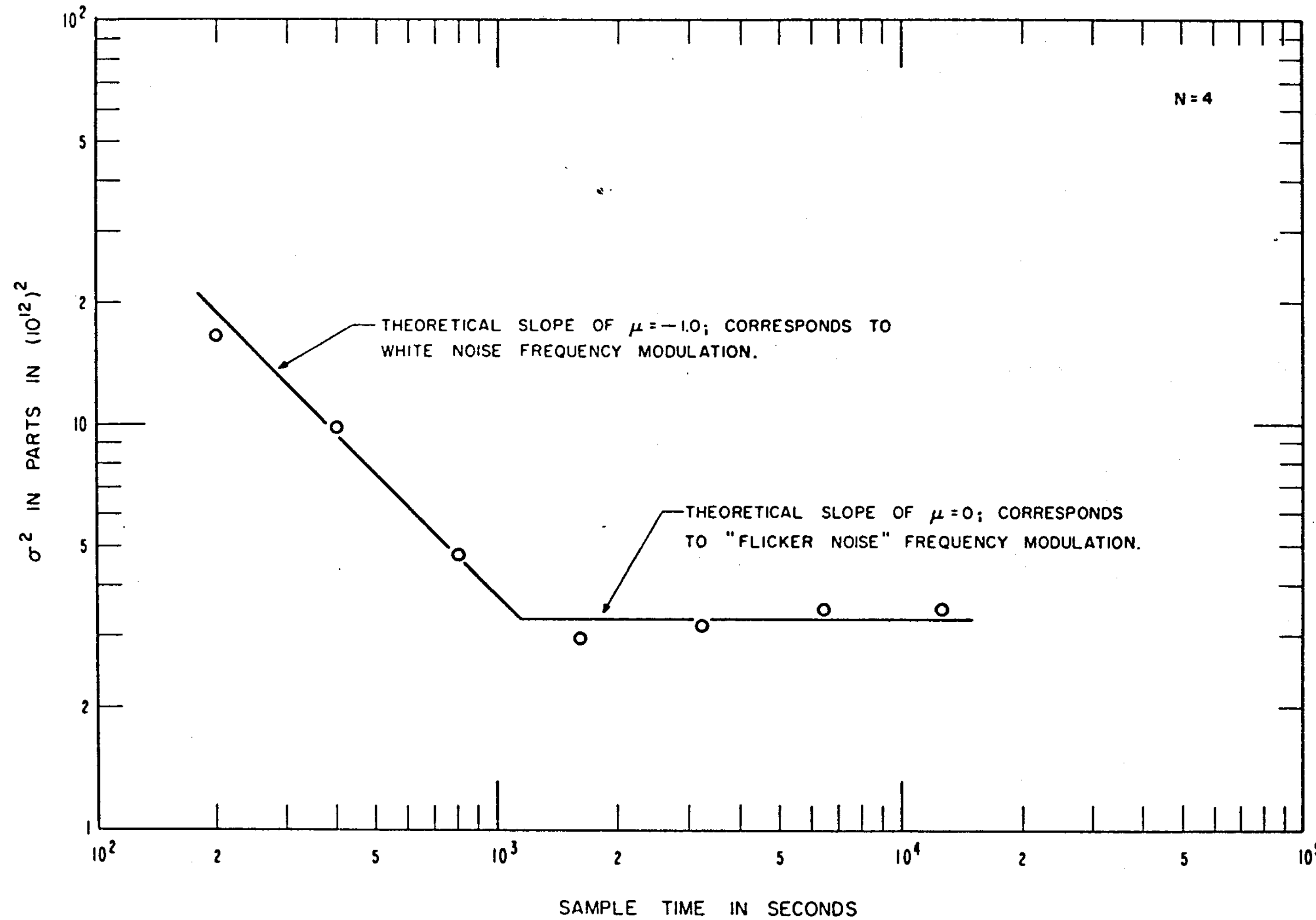


Figure 6

B-52289

CESIUM BEAM (NBS III),
HYDROGEN MASER COMPARISON; STANDARD DEVIATION OF
THE FREQUENCY FLUCTUATIONS AS A FUNCTION OF SAMPLING TIME

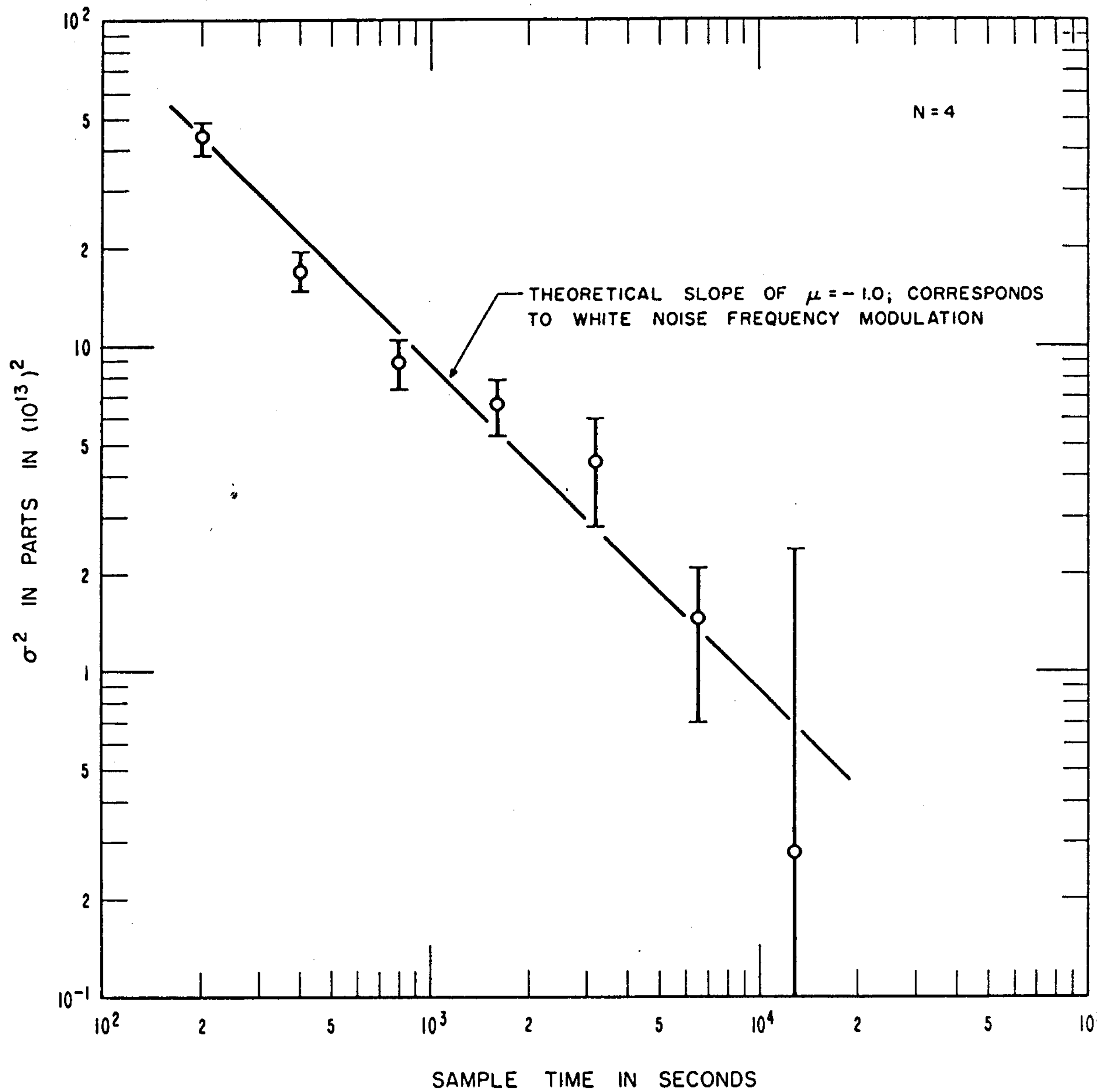
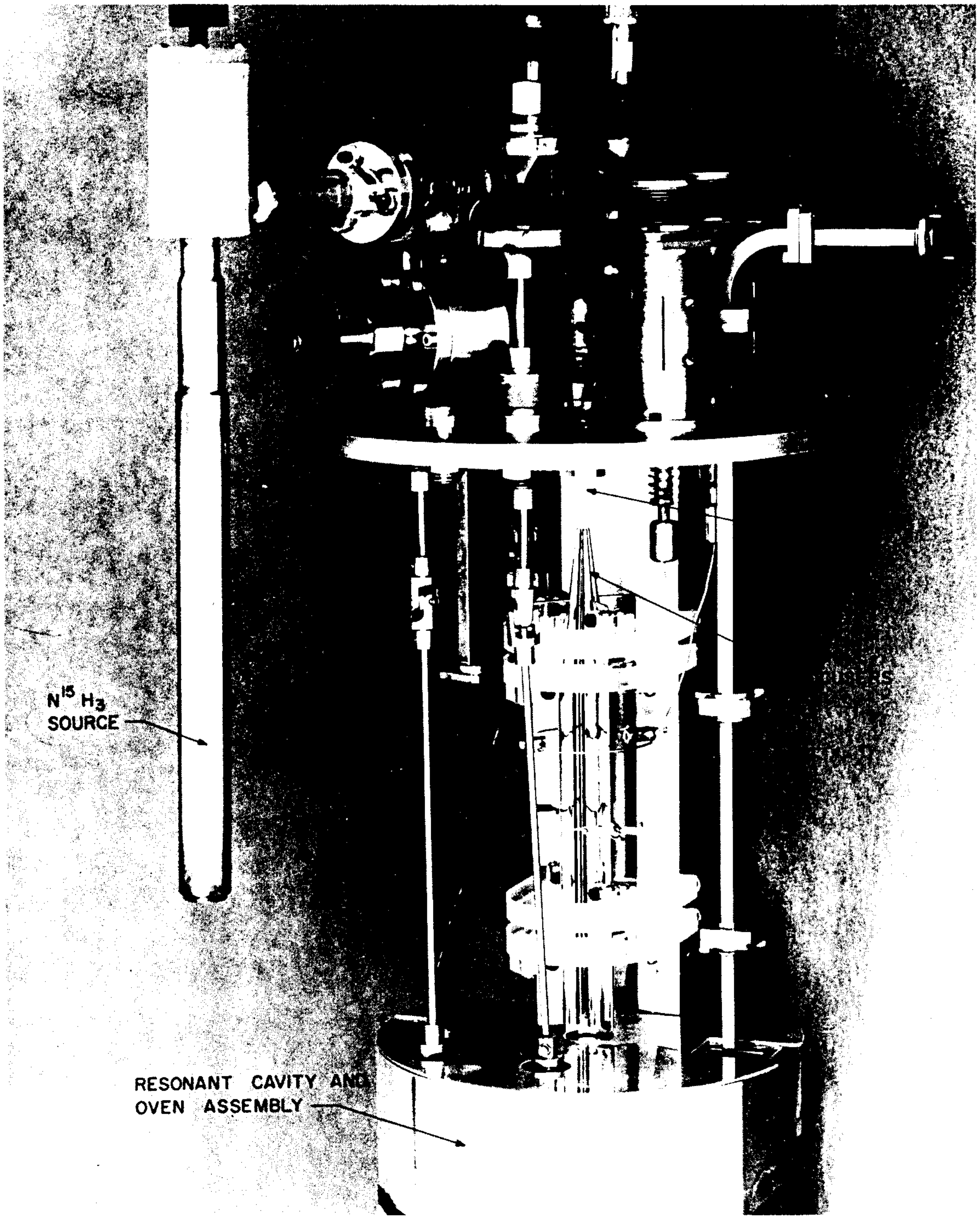


Figure 7

B- 52290

PLATE I

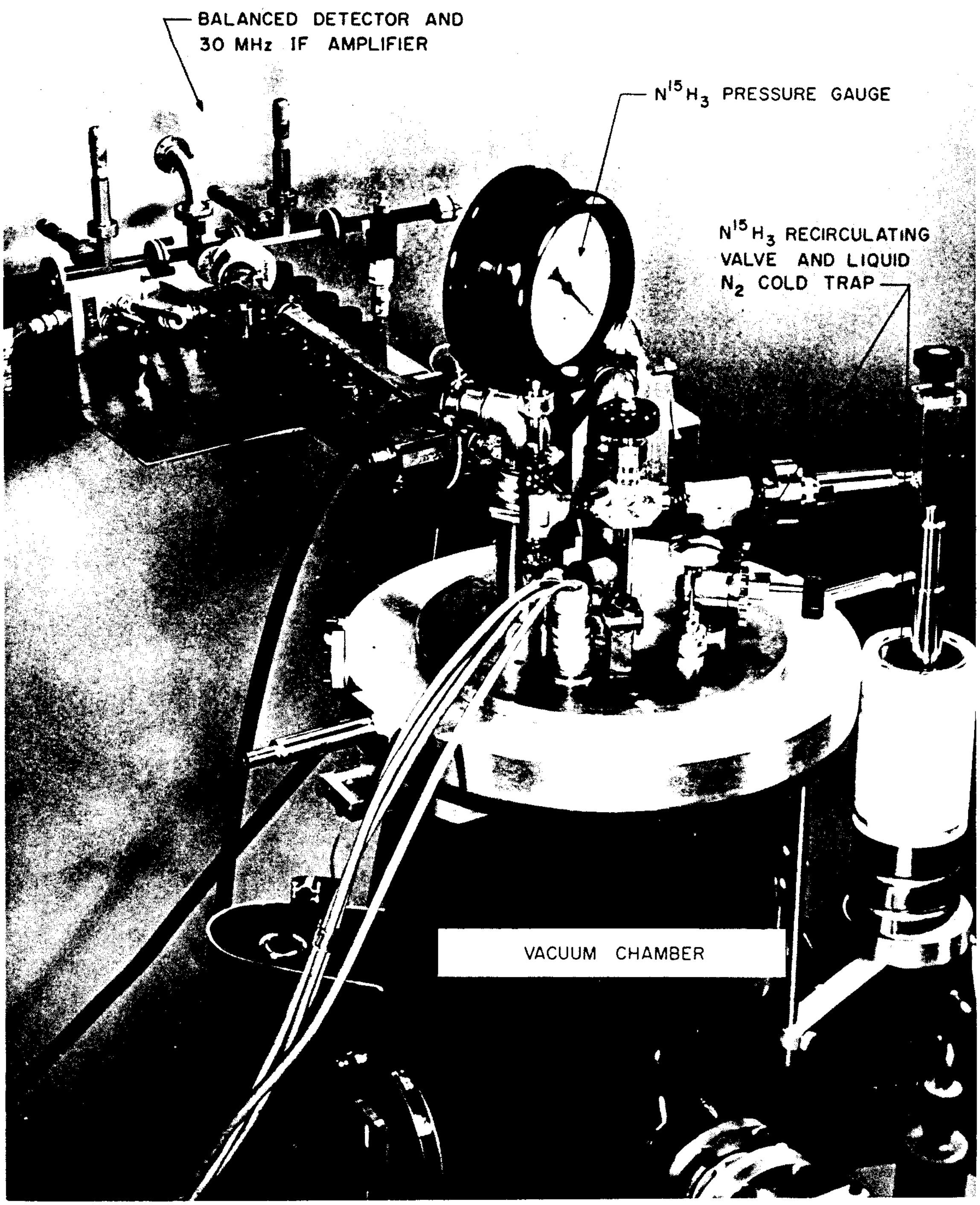
RESONANT CAVITY (IN OVEN),
FOCUSERS, NOZZLE, AND PLUMBING OF AN $N^{15}H_3$ MASER



B-52282

PLATE 2

THE ASSEMBLED $N^{15}H_3$ MASER
WITH THE RESONANT CAVITY IN THE VACUUM CHAMBER



B-52281

TABLE I

FOR A FIXED SAMPLE TIME τ, ONE CAN OBSERVE THE DEPENDENCE OF $\langle \sigma^2 (N, \tau) \rangle$ ON THE NUMBER OF SAMPLES AND THUS DETERMINE A VALUE OF μ AND HENCE THE STATISTICS. THE VALUES LISTED ARE OF

$$\chi(N,\mu) = \frac{\langle \sigma^2 (N, \tau) \rangle}{\langle \sigma^2 (2, \tau) \rangle}$$

μ	N (NUMBER OF SAMPLES)					
	4	16	64	256	1024	∞
0.0	1.337	2.133	3.048	4.016	5.004	∞
-0.1	1.288	1.928	2.580	3.190	3.736	7.464
-0.2	1.247	1.753	2.215	2.598	2.899	3.855
-0.3	1.208	1.604	1.928	2.167	2.332	2.660
-0.4	1.171	1.475	1.700	1.847	1.937	2.062
-0.5	1.138	1.365	1.517	1.606	1.655	1.705
-0.6	1.106	1.270	1.369	1.422	1.447	1.467
-0.7	1.077	1.188	1.249	1.278	1.291	1.299
-0.8	1.049	1.116	1.150	1.165	1.171	1.174
-0.9	1.023	1.054	1.068	1.074	1.076	1.076
-1.0	1.000	1.000	1.000	1.000	1.000	1.000
-1.1	0.977	0.952	0.942	0.938	0.937	0.937
-1.2	0.956	0.910	0.893	0.887	0.886	0.886
-1.3	0.937	0.873	0.851	0.844	0.842	0.842
-1.4	0.919	0.841	0.815	0.807	0.805	0.805
-1.5	0.902	0.812	0.784	0.776	0.774	0.773
-1.6	0.886	0.786	0.756	0.748	0.746	0.745
-1.7	0.871	0.763	0.733	0.725	0.723	0.722
-1.8	0.858	0.743	0.712	0.704	0.702	0.701
-1.9	0.845	0.724	0.693	0.685	0.683	0.682
-2.0	0.833	0.708	0.677	0.669	0.667	0.667
-2.1	0.822	0.693	0.662	0.654	0.652	0.652
-2.2	0.811	0.680	0.649	0.641	0.639	0.638
-2.3	0.802	0.668	0.637	0.629	0.628	0.627
-2.4	0.792	0.657	0.626	0.619	0.617	0.616
-2.5	0.784	0.647	0.616	0.609	0.607	0.607
-2.6	0.776	0.638	0.608	0.601	0.599	0.598
-2.7	0.769	0.629	0.600	0.593	0.591	0.591
-2.8	0.762	0.622	0.593	0.586	0.584	0.583
-2.9	0.755	0.615	0.586	0.579	0.577	0.577
-3.0	0.750	0.609	0.580	0.573	0.571	0.571

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