Correspondence

ThêoH and Allan Deviation as Power-Law Noise Estimators

Jennifer A. McGee and David A. Howe, *Member, IEEE*

***Abstract*—A spectral interpretation using the frequency sensitivity of the Allan variance (Avar) and Thêo-Hybrid (ThêoH) is used to determine f^{α} noise, or "power-law noise." ThêoH has narrower chi-square confidence than Avar; consequently, ThêoH provides significantly better determination of f^{α} noise types at long term. Furthermore, ThêoH has even narrower confidence than chi-square. Because the algorithms used to calculate these confidence intervals are computationally intensive, we have constructed an empirical formula that approximates confidence intervals as the percent error for ThêoH.**

I. Introduction

"Theoretical variance #1", or "Thêo1", is a new, high-confidence frequency stability statistic that works up to an averaging time τ that is 3/4 of the total time T of a data run given by a sequence of time-error samples $\{x_n : n = 1, \ldots, N_x\}$ with a sampling period between adjacent observations given by τ_0. Thêo1 is given by:

$$\text{Thêo1}(m, \tau_0, N_x) = \frac{1}{0.75(N_x - m)(m\tau_0)^2} \sum_{i=1}^{N_x - m} \sum_{\delta=0}^{\frac{m}{2}-1} \frac{1}{(\frac{m}{2} - \delta)} \times \left[\left(x_{i+m} - x_{i+\delta+\frac{m}{2}}\right) - \left(x_{i-\delta+\frac{m}{2}} - x_i\right)\right]^2 \quad (1)$$

with m even, called the "averaging factor," and $10 \leq m \leq N_x - 1$ [1]–[3].

Thêo1 is biased with respect to the Allan variance, which is computed using the unbiased maximum overlap estimator Avar. One can remove a computed bias between Thêo1 and Avar at large m by using ThêoBR (which stands for "Thêo with bias removed"):

$$\text{ThêoBR}(m, \tau_0, N_x) = \left[\frac{1}{n+1} \sum_{i=0}^{n} \frac{\text{Avar}(m = 9 + 3i, \tau_0, N_x)}{\text{Thêo1}(m = 12 + 4i, \tau_0, N_x)}\right] \times \text{Thêo1}(m, \tau_0, N_x), \quad (2)$$

where $n = \lfloor (0.1N_x/3) - 3 \rfloor$ ($\lfloor\cdot\rfloor$ denotes the floor function) [2], [4]. Then ThêoH (which stands for Thêo-Hybrid and/or Thêo-high confidence), is a frequency stability estimator that has excellent confidence over the largest range of m:

$$\text{ThêoH}(m, \tau_0, N_x) = \begin{cases} \text{Avar}(m, \tau_0, N_x), & \text{for } 1 \leq m < \frac{k}{\tau_0} \\ \text{Thêo1}(m, \tau_0, N_x), & \text{for } \frac{k}{0.75\tau_0} \leq m \leq N_x - 1\ , \\ & \text{m even} \end{cases} \quad (3)$$

where k is the largest $\tau \leq 10\%\ T$ where Avar (m, τ_0, N_x) has sufficient confidence [3]. Note that Avar and Thêo1 have different dependence on τ; for Avar, the averaging time is $\tau = m\tau_0$, and the averaging time associated with Thêo1 is $\tau = 0.75\ m\tau_0$ [2]–[4]. The deviations of Thêo1 and ThêoH are found simply from the square root of the values given by (1) and (3), respectively. In this paper, the term ThêoH-dev refers to the deviation of ThêoH.

II. Chi-Square Distribution and Confidence for Avar and ThêoH

Compared to the Allan variance, ThêoH reports higher confidence for long-term, frequency stability calculations [2], [5]. Avar's constant-Q frequency response is widely used to determine particular power-law noises by noting straight-line slopes on a log-log plot. ThêoH has an even smoother, more ideal frequency response for noise determination [1], [2].

It has been found that the calculated values of Avar are distributed as a chi-square distribution [6]. Using the extensively studied properties of this distribution [7],[1], one can find confidence intervals of each averaging factor m for a data run of N_x points as those for chi-square [6]–[8]:

$$\left[\frac{\eta \cdot \text{Avar}}{\chi^2_{\eta,1-p}}, \frac{\eta \cdot \text{Avar}}{\chi^2_{\eta,p}}\right], \quad (4)$$

where η is the number of chi-square equivalent degrees of freedom (edf), computed for each averaging factor m from standard closed-form formulas [6], [8], and p determines the desired quantile for each specific N_x and m. For example, for a 68.3% confidence interval:

$$68.3\% = (1 - 2p) \times 100\%. \quad (5)$$

To find confidence intervals for the Allan deviation (Adev), one need only take the square root of each factor in (4) [8].

Manuscript received January 11, 2006; accepted September 20, 2006.

The authors are with the National Institute for Standards and Technology, Time and Frequency Division, Boulder, CO (e-mail: mcgeeja@boulder.nist.gov).

Digital Object Identifier 10.1109/TUFFC.2007.260

[1] NIST/SEMATECH e-Handbook of Statistical Methods, http://www.itl.nist.gov/div898/handbook/, 2006.

Confidence factors for ThêoH also can be calculated by use of the chi-square distribution. ThêoH has substantially higher values of edf than Avar, and these edf's can be computed from the following empirical formulas based on simulation studies [1]:

$$\underbrace{\text{edf}}_{\text{WHFM}} = \left[\frac{5.5N_x + 1.07}{m} - \frac{3.1N_x + 6.5}{N_x}\right]\left(\frac{m^{3/2}}{m^{3/2} + 8}\right),$$

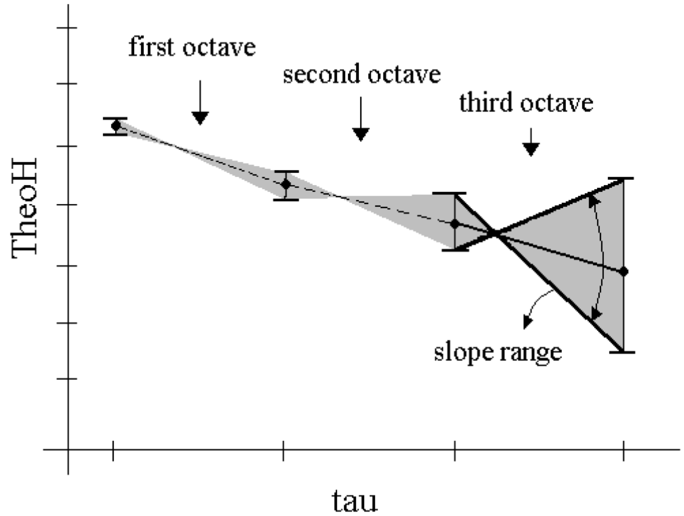
$$\underbrace{\text{edf}}_{\text{FLFM}} = \left(\frac{2.7N_x^2 - 1.3N_x m - 3.5m}{N_x m}\right)\left(\frac{m^3}{m^3 + 5.45}\right),$$

$$\underbrace{\text{edf}}_{\text{RWFM}} = \left(\frac{4.4N_x - 2}{2.175m}\right) \times \left(\frac{(4.4N_x - 1)^2 - 6.45m(4.4N_x - 1) + 6.413m^2}{(4.4N_x - 3)^2}\right),$$

$$\underbrace{\text{edf}}_{\text{WHPM}} = \left(\frac{0.86\,(N_x + 1)\,(N_x - m)}{N_x - 0.75m}\right)\left(\frac{m}{m + 1.52}\right),$$

$$\underbrace{\text{edf}}_{\text{FLPM}} = \left(\frac{5.54N_x^2 - 5.52N_x m + 10.727m}{(m + 48.8)^{1/2}(N_x - 0.75m)}\right)\left(\frac{m}{m + 0.4}\right).$$

Similarly, using these edf's, one can use (4) and (5) above to find ThêoH confidence factors.



(c)

Fig. 1. For the above sample plot of ThêoH with symmetric confidence intervals, the slope range is illustrated as the darkened region between the consecutive τ values of ThêoH, where $m\tau_0 = \tau$, $m = 1, 2, 4, 8 \ldots$. For each octave, the darkened region indicates all possible power-law noise types consistent with the given confidence intervals. The narrower the confidence interval, the smaller the region and the better one can distinguish one noise type from others.

III. Defining Slope Range in Power-Law Noise Estimation

Power-law noise processes, models of oscillator noise types, produce a particular slope on a spectral density plot [6], [8]. For spectral density of frequency fluctuations of the form $S_y(f) = h_\alpha f^\alpha$, the power-law noise process is completely specified by α, a number modeling the most appropriate type of power law for the data for a given range of f, and the corresponding level, h_α. Thus, on a log-log plot of frequency stability versus τ, α becomes the slope and f^α is the straight line that relates $S_y(f)$ to f. The range of α for which TheoH and Theo1 converge is the same as for Avar [8]. In general, a plot of $S_y(f)$ represents a linear combination of these integer power-law processes [9], [10]. The five common noise types, random walk FM (RWFM), flicker FM (FLFM), white FM (WHFM), flicker PM (FLPM), and white PM (WHPM), have slopes described by $\alpha = -2, -1, 0, 1, 2$, respectively [6]. Avar, and similarly ThêoH, can be related to the power spectral density; therefore, noise types also can be found by the slope, given by μ, on a plot of variance versus τ of these statistics [11], [12].

The noise type of an oscillator noise process is not necessarily known up front. But, because of the power-law nature of noise processes and the narrow confidence interval of ThêoH, we introduce a method that allows us to make some determination of the noise type by analyzing allowed slopes within the confidence intervals.

If we calculate the uncertainty of ThêoH-dev with a chi-square distribution, the confidence intervals are substantially narrower than those of Adev, especially at longer averaging times for which edf's are lower [5]. Fig. 1 illustrates a sample plot of ThêoH-dev in which confidence intervals are calculated at each octave of $\tau = m\tau_0$, where $m = 1, 2, 4, 8 \ldots$. Stability computations that occur in power-of-2, or "octave", increments of m are used for determining power-law noise types because these points are sufficiently independent [3]. Due to the power-law nature of the noise, ThêoH-dev, as Adev, also has a linear representation in a log-log scale with a slope of $\mu/2$ as each is the square root of a variance.

There are two lines, each having a certain slope, associated with the consecutive confidence intervals, as shown in Fig. 1. One connects the upper confidence factor of the first octave, Upper1, to the lower confidence factor at the second octave, Lower2. The other connects the lower confidence factor of the first octave, Lower1, to the upper confidence factor at the second octave, Upper2. The possible slopes that ThêoH could take on between these two lines constitute the slope range of that octave. For a narrower confidence interval, we have a narrower calculated range. There are fewer possible slopes that ThêoH can take on; thus, we can determine the type of noise. However, if the slope range crosses a transition from one noise type to another, we cannot confidently distinguish the noise type.

Fig. 2 demonstrates the divergence of the slope range of both ThêoH-dev and Adev for RWFM, FLFM, and WHFM calculated for $N_x = 1025$ at 90% confidence. For all three noise types, ThêoH has a consistently smaller slope range than Adev out to longer averaging times, indicating its ability to better estimate the noise type from the slope.

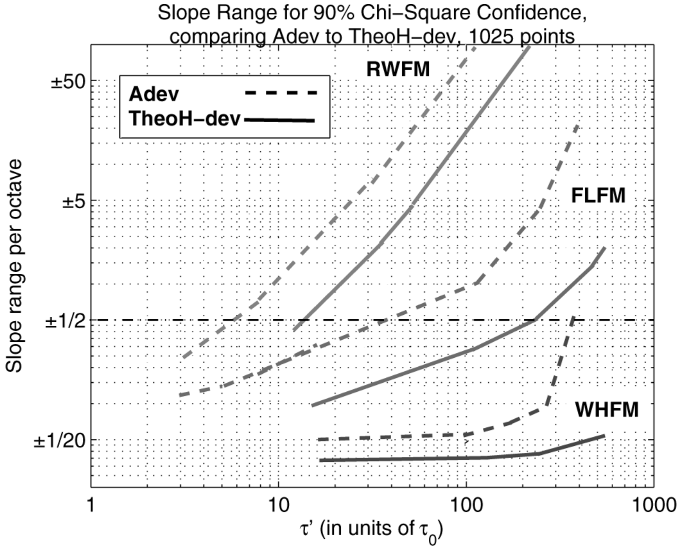


Fig. 2. Slope range per octave for simulated frequency noise consisting of 1025 data points. ThêoH, in the presence of WHFM, FLFM, and RWFM, has a dependence on τ that goes as $\tau^{-1/2}, \tau^{0}, \tau^{+1/2}$, respectively. $\tau' = 1/2(\tau_{\text{upper}} + \tau_{\text{lower}})$ is the midpoint between the consecutive calculations of ThêoH or Adev, τ_{upper} and τ_{lower} separated by an octave. In the plots above, a spread greater than $\pm 1/2$, indicated by the region above the horizontal dashed line, means that prevailing power-law noise type is indistinguishable from its neighboring type.

IV. Calculating Slope Range

The possible ranges of slope are determined by a weighted, linear, least-squares regression to the time variable τ; this provides a probabilistic interpretation for each octave of the slope range. Reference [13] describes this weighted regression technique in the context of spectral-like statistics such as ThêoH and Adev.

The calculation is based on the typical expression for finding a slope or rise over run. For a calculation of slope range for the chi-square uncertainty of the variance of ThêoH-var or Avar, we want to find the rise and run of both lines connecting the confidence intervals. The rise, or Δy, is given by $\Delta y_1 = |\text{Upper2} - \text{Lower1}|$ and $\Delta y_2 = |\text{Upper1} - \text{Lower2}|$. Then we divide by the run, Δx, for each octave, we denote as τ', the midpoint between the consecutive calculations of ThêoH-var or Avar:

$$\Delta x = \tau' = \frac{\tau_{\text{upper}} - \tau_{\text{lower}}}{2}. \tag{6}$$

Then the two slopes are given by:

$$\frac{\Delta y_1}{\Delta x} = \frac{|\text{Upper2} - \text{Lower1}|}{\tau'}, \tag{7}$$

and

$$\frac{\Delta y_2}{\Delta x} = \frac{|\text{Upper1} - \text{Lower2}|}{\tau'}. \tag{8}$$

Subtracting these two slopes, we find the range of slopes possible between the two consecutive calculations. The confidence factors of ThêoH-dev and Adev vary by a square root of their associated variance statistics; and, in a log-log representation, their slope ranges will vary by dividing all points down by 2. This already has been done in Fig. 2.

A transition from one noise type to another happens when the slope range spans more than 1, indicated in the plot as $\pm 1/2$. For example, in Fig. 2 we see that near $\tau' = 40\tau_0$, about 4% of the data run, FLFM becomes indistinguishable from RWFM or WHFM as calculated by Adev. However, for ThêoH, FLFM is distinguishable up to $\tau' = 200\tau_0$, about 1/5 of the data run. At the long term, ThêoH has a slope range narrow enough to be able to confidently detect the onset of the nonstationary noise types FLFM or RWFM earlier than Adev.

V. Exact Confidence of ThêoH

Up to this point, we have been assuming a chi-square distribution, but ThêoH's confidence intervals are actually narrower than those determined by a normal chi-square distribution. It has been shown that one can bring the ThêoH statistic into the form of a quadratic distribution of chi-square random variables [5], [14], and numerical techniques for the calculation of quadratic quantiles are available [5], [15], [16]. Once these are calculated, the exact confidence factors for ThêoH are given by:

$$\left[\frac{k \cdot \text{ThêoH-var}}{q_{1-p}}, \frac{k \cdot \text{ThêoH-var}}{q_p}\right], \tag{9}$$

where $k = (N_x - m + 1) \times (m/2)$ for each averaging factor m and number of data points N_x, and q_p and q_{1-p} define the desired quantiles of the new distribution with p and $1 - p$ defined as described for chi-square in (5). Factors for ThêoH-dev are the square root of the upper and lower limits [5].

Percent error is another way of talking about the confidence of a measurement that relates the size of the confidence interval to the value of the computed statistic [5]. For a set of ThêoH calculations, the percent error for each averaging factor is given by the following formula:

$$\text{\%-error, ThêoH-dev} = \frac{|\text{closest confidence factor} - \text{ThêoH-dev}|}{\text{ThêoH-dev}} \times 100\%. \tag{10}$$

VI. Empirical Formula

The algorithms for exact calculations of ThêoH confidence factors are computationally intense and can take more time to run than ThêoH itself, especially for increasing values of N_x and m. Because of this, computing exact confidence intervals for every data set becomes impractical. We have constructed an empirical formula that conveniently and reliably predicts the percentage upper bound (indicating how much worse the measurement might be)

TABLE I
QUANTILES AND EDF FOR RWFM AT 68.3%.

N_x	m	k	$q_{.159}$	$q_{.841}$	edf
32	2	31	23.22	38.78	29.85
32	4	58	37.39	78.60	13.48
32	8	100	47.28	152.7	5.352
32	16	136	34.53	236.7	1.420
64	2	63	51.85	74.15	62.23
64	4	122	91.37	152.6	29.65
64	8	228	142.9	313.0	13.39
64	16	392	180.8	601.6	5.323
64	32	528	130.0	963.5	1.418

at 68.3% of low values of edf of ThêoH-dev, the low values being the most important here [2].

For general population statistics, the formulation of error is proportional to one over the square root of the number of independent points [7][1]. For our empirical formula, we begin with this general form, in which we have the computed edf's as the number of independent points and produce a result that conservatively approximates our exact calculations. Percentage confidence is given by:

$$\%\text{-error using ThêoH-dev}(\tau) = \frac{100}{\sqrt{2(edf + 6.6)}}. \quad (11)$$

The lower bounds of percent error (indicating how much better the estimate might be) are skewed downward, meaning they are skewed optimistically. These are not particularly as useful in interpreting stability estimates as upper bounds.

Table I lists quantiles for RWFM noise calculated with a numerical algorithm for quadratic distributions [13]. It also lists the ThêoH equivalent degrees of freedom for RWFM given in Section II. For each noise type, edf is given by empirical formulas and depends on the number of data points, N_x, and $\tau = 0.75\ m\tau_0$ [1].

Fig. 3 compares upper-bound percent error versus edf for Adev and ThêoH for RWFM at one standard deviation. The upper confidence factor was calculated with chi-square for Adev and with both the quadratic distribution confidence factors, (9), and the empirical formula, (11), for ThêoH. At edf's near 1, corresponding to long-term τ, both formulations for ThêoH show a percent error at or below 25% compared to Adev's 50%, indicating again that ThêoH reports substantially higher confidence. The bottom plots in Fig. 3 show that the percent error for ThêoH as given by the empirical formula is conservative with respect to the exact formula given in (9).

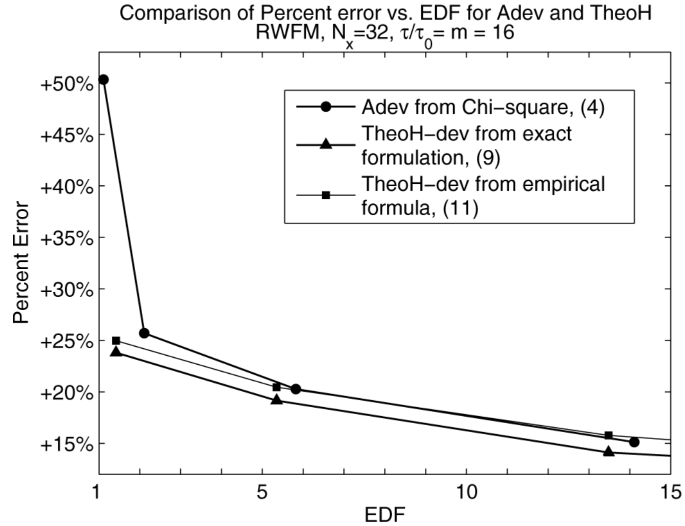


Fig. 3. The upper bound of a 68.3% confidence interval comparing values for Adev calculated from chi-square and ThêoH calculated from both quadratic distribution, (9), and the empirical formula, (11). At lower edf values, hence, in long-term, ThêoH reports substantially lower percent error for both formulations than possible for Adev.

VII. CONCLUSIONS

ThêoH reports long-term frequency stability for up to 3/4 of a given data run. By defining a slope range with high confidence between chi-square confidence factors for each averaging time, we can use maximum and minimum slopes in a probabilistic sense to determine noise types. Because of its high confidence, ThêoH is a better estimator of noise type than Adev.

ThêoH more exactly fits a distribution with narrower confidence intervals than chi-square. In order to benefit from these narrow confidence intervals, one must work with the numerical techniques needed for calculating the quantiles of a quadratic distribution. However, these can be difficult and computationally intense to implement. To aid the estimation of slope type, we have introduced a very simple, empirical formula for percent error that can be used to calculate upper confidence factors for ThêoH easily and reliably without extensive numerical calculation.

REFERENCES

[1] D. A. Howe and T. K. Peppler, "Very long-term frequency stability: Estimation using a special-purpose statistic," in *Proc. Joint Mtg. IEEE Int. Freq. Cont. Symp. and EFTF Conf.*, May 2003, pp. 233–238.
[2] D. A. Howe and T. N. Tasset, "Thêo1: Characterization of very long-term frequency stability," in *Proc. 18th Eur. Freq. Time Forum*, Apr. 2004, pp. 581–587.
[3] D. A. Howe, "ThêoH: A new high-confidence, hybrid statistic that improves on the Allan deviation," *Metrologia*, vol. 43, pp. 322–331, Aug. 4, 2006.
[4] D. A. Howe, D. B. Percival, J. A. McGee, and T. N. Tasset, "ThêoH bias-removal method," in *Proc. IEEE Freq. Cont. Symp.*, 2006, pp. 788–792.
[5] T. N. Tasset, D. A. Howe, and D. B. Percival, "$\widehat{\text{Theo1}}$ Confidence intervals," in *Proc. IEEE Freq. Cont. Symp.*, Aug. 2004, pp. 725–728.
[6] D. B. Sullivan, D. W. Allan, D. A. Howe, and F. L. Walls, Eds. "Characterization of clocks and oscillators," Natl. Inst. Stand. Technol. Technical Note 1337, 1990.
[7] O. J. Dunn, V. A. Clark, and R. M. Mickey, *Applied Statistics: Analysis of Variance and Regression*. 3rd ed. New York: Wiley, 2004.

[8] D. A. Howe, D. W. Allan, and J. A. Barnes, "Properties of signal sources and measurement methods," in *Proc. 35th IEEE Freq. Cont. Symp.*, 1981, pp. A1–A47.

[9] T. Walter, "Characterizing frequency stability: A continuous power-law model with discrete sampling," *IEEE Trans. Instrum. Meas.*, vol. 43, no. 1, pp. 69–79, Feb. 1994.

[10] D. B. Percival, "Characterization of frequency stability: Frequency-domain estimation of stability measures," *Proc. IEEE*, vol. 79, pp. 961–972, 1991.

[11] D. W. Allan, M. A. Weiss, and J. L. Jespersen, "A frequency-domain view of time-domain characterizations of clocks and time and frequency distribution systems," in *Proc. 45th IEEE Freq. Cont. Symp.*, 1991, pp. 667–678.

[12] J. A. Barnes, A. R. Chi, L. S. Cutler, D. J. Healy, D. B. Leeson, T. E. McGunigal, J. A. Mullen, Jr., W. L. Smith, R. L. Syndor, R. F. C. Vessot, and G. M. R. Winkler, "Characterization of frequency stability," *IEEE Trans. Instrum. Meas.*, vol. IM-20, pp. 105–120, 1971.

[13] D. B. Percival and A. Walden, *Wavelet Methods for Time Series Analysis*. Cambridge, UK: Cambridge University Press, 2000, Sec 9.5–9.7 pp. 374–386.

[14] D. B. Percival, "The statistics of long memory processes," Ph.D. dissertation, Department of Statistics, University of Washington, Seattle, WA, 1983.

[15] J. Sheil and I. O'Muircheartaigh, "The distribution of nonnegative quadratic forms in normal variables," *Appl. Statistics*, vol. 22, pp. 92–98, 1977.

[16] R. B. Davies, "The distribution of a linear combination of chi-square random variables," *Appl. Statistics*, vol. 29, pp. 323–333, 1980.