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Frequency Measurement Errors of Passive Resonators Caused by Frequency-Modulated Exciting Signals

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Abstract—The condition of resonance for a signal with FM is defined in this paper as the condition of maximum power transfer by the resonant device. It is shown that if the width of the signal spectrum is small compared to the resonator's linewidth, then the frequency error is proportional to the third moment of the instantaneous signal frequency about its mean.

One expects that this treatment should, at least, give the leading term for a precise treatment of atomic resonances. Experimental results with a cesium beam frequency standard confirm this expectation and add caution to the idea that higher Q atomic resonances make better absolute frequency standards.

I. Introduction

THERE are many situations where one excites a resonator with an RF signal generator to measure frequency. It is also common to measure frequency with a frequency counter and it has been pointed out [1]–[3] that these two techniques may not agree for impure signals. The present paper is confined to instabilities of a frequency modulation nature only and specifically does not consider amplitude modulation. This is justified by the fact that most good signal sources seem to suffer more from FM instabilities than from AM instabilities.

The author's initial interests here are errors in passive atomic frequency standards (e.g., the cesium beam) caused by spectral impurities. It is interesting to note that, to a first order, the frequency errors discussed by Shirley [2] and Harrach [3] can be deduced [4] from considerations of conventional LCR resonators. This suggests that the treatment in this paper should at least give first-order effects of spectral impurities on frequency shifts in passive atomic resonators. The advantages of this treatment are a relative simplicity, an extension to continuous spectra rather than the bright-line spectra assumed in the rigorous treatments, and a relatively convenient means of measurement of the important parameters in the time domain.

The model that is used for the present paper is that of Fig. 1. It is assumed that the voltage output of the variable-frequency oscillator (VFO) has a power spectral density $S_v(f)$ and that there are negligible amplitude fluctuations to the signal. Thus we may write

$$V(t) = V_0 \cos [\varphi(t)]. \tag{1}$$

The condition of resonance is here defined as the maximum power transfer by the resonator, i.e., that frequency that gives maximum indication on the power meter of Fig. 1.

II. Resonant Frequency Errors

In general, we will be interested in an ideal frequency ν_0 that is quite large compared to any errors. Thus it is of value to expand everything about this frequency. In

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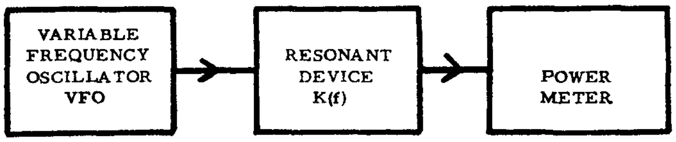

Fig. 1. Block diagram of model.

particular, we will assume that (see Appendix I)

$$\nu_0 \equiv \lim_{T\to\infty} \frac{\varphi(t_0+T)-\varphi(t_0)}{2\pi T} = \frac{2}{V_0^2}\int_0^\infty S_\nu(f) f\, df \tag{2}$$

for a pure FM signal.

It is of value to present a set of four postulates.

1) Tuning the VFO causes a displacement of the spectrum without distorting its shape. That is, if the VFO is tuned to be in error by a constant frequency f_0 (with respect to the frequency ν_0), then the spectral density of the output of the VFO becomes $S_\nu(f-f_0)$.

2) The widths of the spectrum $S_\nu(f)$ and the transfer function $|K(f)|^2$ are small compared to ν_0.

3) If the voltage transfer function of the resonant device is $K(f)$, then the resonant condition requires that the power transfer,

$$P(f_0) = \int_0^\infty |K(f)|^2\, S_\nu(f-f_0)\, df, \tag{3}$$

be a maximum when the VFO is tuned to resonance.

4) The power transfer function $|K(f)|^2$ is symmetric about the ideal frequency ν_0. Thus, we assume that $|K(f)|^2$ may be expanded in a Taylor series about ν_0 and only even powers of $(f-\nu_0)$ will be present, i.e.,

$$|K(f)|^2 = A + B(f-\nu_0)^2 + C(f-\nu_0)^4 + \cdots. \tag{4}$$

The frequency error f_0 results from the assumed spectral shape $S_\nu(f-f_0)$. That is, f_0 can be found by maximizing $P(f_0)$ given in (3):

$$\frac{d}{df_0}P(f_0) = 0 = \int_0^\infty |K(f)|^2 \frac{\partial S_\nu(f-f_0)}{\partial f_0}\, df. \tag{5}$$

It is obvious that

$$\frac{\partial S_\nu(f-f_0)}{\partial f_0} = -\frac{\partial S_\nu(f-f_0)}{\partial f},$$

and then, integration by parts yields

$$\int_0^\infty S_\nu(f-f_0)\frac{\partial\, |K(f)|^2}{\partial f}\, df = 0, \tag{6}$$

since it is assumed that $S_\nu(f)$ (and $|K(f)|^2$) is significant only in the vicinity of $f=\nu_0$, which is far removed from zero frequency.

Making use of the first two terms in the expansion (4) we obtain

$$2B\int_0^\infty S_\nu(f-f_0)(f-\nu_0)\, df = 0,$$

or

$$2B\int_0^\infty S_\nu(f')(f'+f_0-\nu_0)\, df' \approx 0$$

since $S_\nu(f)$ does not have appreciable value for $0 \le f \le f_0$. Finally, with the aid of Appendix I we obtain

$$f_0 = 0. \tag{7}$$

Thus, if the resonance were a pure quadratic, no frequency error would occur as a result of FM distortion on the signal.

Similarly, if we keep the first three terms of (4), then (6) becomes

$$Bf_0 + \frac{4C}{BV_0^2}\int_0^\infty S_\nu(f-f_0)(f-\nu_0)^3\, df \approx 0$$

or

$$f_0 \approx -\frac{4C}{BV_0^2}\int_0^\infty S_\nu(f')(f'+f_0-\nu_0)^3\, df'.$$

Neglecting terms of order f_0^3, we obtain

$$f_0\left(1+\frac{6C}{B}f_2^2\right) \approx -\frac{2C}{B}f_3^3, \tag{8}$$

where we have written (see Appendix I)

$$f_2^2 \equiv \frac{2}{V_0^2}\int_0^\infty S_\nu(f)(f-\nu_0)^2\, df,$$

and

$$f_3^3 \equiv \frac{2}{V_0^2}\int_0^\infty S_\nu(f)(f-\nu_0)^3\, df.$$

Thus, the first nonvanishing term in the frequency error becomes

$$f_0 \approx -\frac{(2C/B)f_3^3}{1+(6C/B)f_2^2}. \tag{9}$$

It is important to note that the frequency of the VFO as determined by a counter is just the first moment of the spectrum for FM signals (see Appendix I). That is,

$$\nu_0 + f_0 = \frac{2}{V_0^2}\int_0^\infty S_\nu(f-f_0) f\, df.$$

Similarly, the second and third moments (f_2^2 and f_3^3) of the spectrum about its center of gravity may be obtained from the second and third moments of the instantaneous frequency $[(1/2\pi)(d\varphi(t)/dt)]$ about its average. Thus, one can use frequency discriminators and analog squaring and cubing circuits to evaluate f_2 and f_3 for use in (9).

III. Experiment

Two cesium beam frequency standards were used. One was unperturbed and acted as the reference standard for the experiment. For the other instrument, a voltage-variable phase-shift was accomplished by shunting the 5-MHz line from the oscillator to the multiplier with a

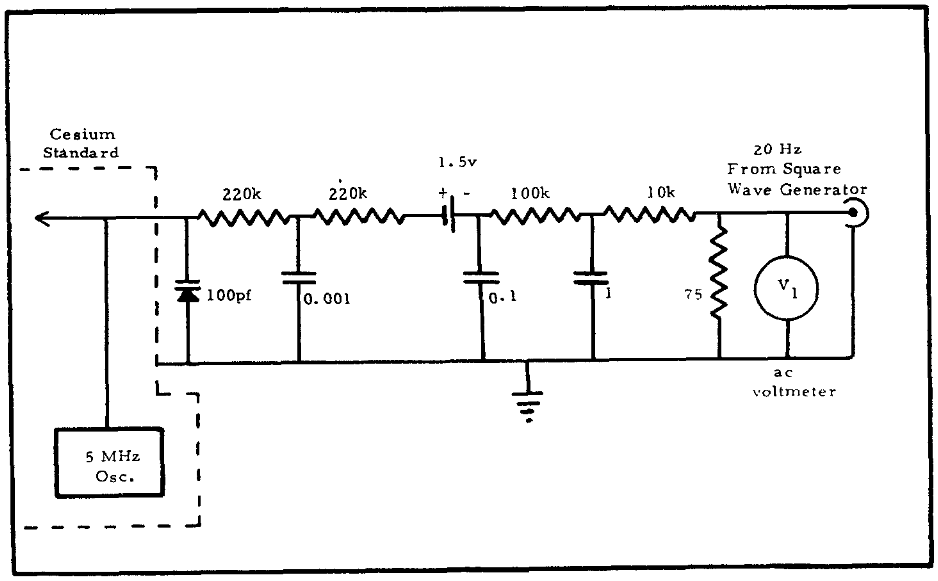

Fig. 2. Phase modulation circuit.

voltage-variable capacitor (see Fig. 2). This variable capacitance was biased at 1.5 volts and was capable of producing a phase-shift of a few milliradians at 5 MHz with a few tens of millivolts change in bias. The modulation signal was a square wave (whose relative duty cycle could be adjusted) derived from a 75-ohm source. The actual square wave used is shown in Fig. 3. The purpose of using this signal source was to allow one to adjust the level of distortion but maintain the ratios f_3/f_1 and f_2/f_1 constant, where f_1 is the peak-to-peak instantaneous frequency excursion. It is thus relatively easy to estimate f_2 and f_3 from readings of the voltmeter V_1.

By comparing the modulated signal with the reference signal in a phase-error multiplier, one can obtain estimates of the amplitude, and second and third moments of the instantaneous frequency. From a photograph of the frequency analog output of the comparator, it was possible to calculate (graphically) the ratio of f_3 and f_2 to the peak-to-peak frequency fluctuation f_1 (see Fig. 4).

The results are

$$f_3 = -(0.26)f_1$$

and

$$f_2 = (0.36)f_1.$$

The peak-to-peak amplitude was measured relative to the signal-generator output voltage by observing the frequency change in the oscillator needed to displace the frequency analog signal on the oscilloscope by its peak-to-peak value. The results of these experiments are summarized by the equation

$$\frac{f_1}{V_1} = (1769 \pm 60) \text{ Hz/V}.$$

The resonance of the cesium beam can be approximated

Fig. 3. Square wave, 5 ms/div.

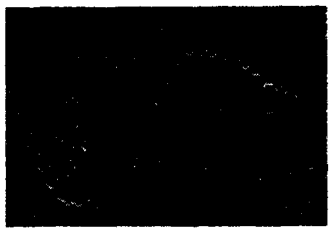

Fig. 4. Analog of instantaneous frequency, 5 ms/div.

[5] by making approximations for low-power and small frequency deviations from resonance.

$$|K(f)|^2 \approx A' + B' \cos \pi\tau(f - \nu_0)$$
$$\approx (A' + B') - \frac{B'(\pi\tau)^2}{2!}(f - \nu_0)^2$$
$$+ \frac{B'(\pi\tau)^4}{4!}(f - \nu_0)^4 + \cdots$$

where τ is the time of flight through the Ramsey cavity and with a linewidth of 529 Hz for the unit used. This approximation should be reasonably valid in the range

$$(\nu_0 - 529) \text{ Hz} \leq f \leq (\nu_0 + 529) \text{ Hz}.$$

We may write

$$\pi\tau(529) = \pi$$

or

$$\tau = 1.89 \times 10^{-3} \text{ second}$$

and

$$\frac{2C}{B} = -\frac{(\pi\tau)^2}{6} = -5.88 \times 10^{-6}(\text{second})^2. \tag{10}$$

From the graphical analysis of the modulation signal, we obtain

$$\frac{f_2}{V_1} = (644 \text{ Hz})/\text{V}$$

and

$$\frac{f_3}{V_1} = (-455 \text{ Hz})/\text{V}.$$

Also,

$$1 + \frac{6C}{B} f_2^2 = 1 - 7.32V_1^2,$$

and

$$\frac{2C}{B\nu_0} f_3^3 = +(6.0 \times 10^{-8})V_1^3,$$

or, finally,

$$\frac{f_0}{\nu_0} = -\frac{(6 \times 10^{-8})V_1^3}{1 - 7.32V_1^2}.$$

The results of the experiment and this equation are plotted in Fig. 5.

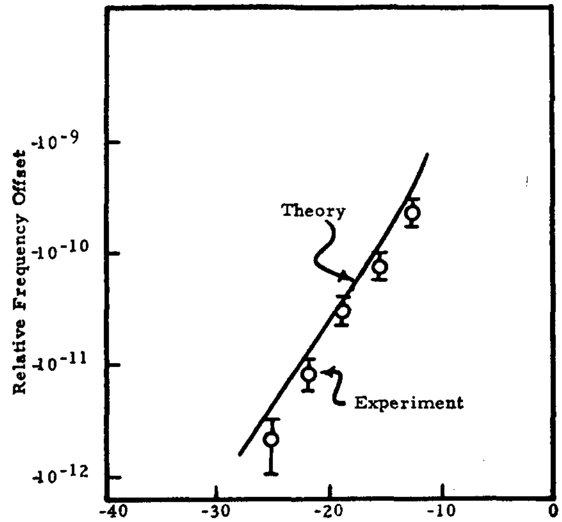

Fig. 5. Modulation voltage V_1, dBm (600 ohms).

IV. Conclusions

Errors in the measured resonant frequency of a passive resonator can result from an impure exciting spectrum. These errors result from a nonzero third moment of the instantaneous frequency fluctuations about the average frequency. Measurement of the second and third moments allow one to predict the resulting frequency error by (9).

The experiments have shown that these results are reasonably applicable to a quantum device (cesium beam) where effects of transit time and saturation have been neglected. Appendix II shows that this technique also gives reasonable quantitative agreement with rigorous treatments of frequency errors caused by second-harmonic distortion in intentionally modulated exciting spectra used for servo control in atomic standards. Combining (9) and (10) and assuming

$$1 + \frac{6C}{B} f_2^2 \approx 1,$$

one may show that

$$f_0/\nu_0 \approx \frac{\pi^2}{6} \frac{\tau^2}{\nu_0} f_3^3. \tag{11}$$

It has been shown [6] that ideal frequency multiplication effectively multiplies phase fluctuations. Thus, one expects that the ratio f_3/ν_0 is independent of the order of multiplication (i.e., independent of ν_0) if the multipliers are good. Equation (11) may be rewritten in the form

$$f_0/\nu_0 \approx \frac{\pi^2}{6} Q^2(f_3/\nu_0)^3, \tag{12}$$

where the "Q" is given by $Q = \tau\nu_0$. Equation (12) shows that (for spectral impurities within the resonance linewidth) fractional frequency errors caused by spectral "pulling" increase as the square of the Q of the device for a given signal source. This could prove to be very important for atomic frequency standards.

Appendix I

There are various means of defining power spectral densities. For convenience assume $\psi_T(t)$ is a function of time with

$$\psi_T(t) = \int_{-\infty}^{\infty} a_T(f)e^{i\omega t}\, df,$$

$$a_T(f) = \int_{-T/2}^{T/2} \psi_T(t)e^{-i\omega t}\, dt,$$

and

$$\omega = 2\pi f.$$

Then

$$\frac{1}{T}\int_{-T/2}^{T/2} \psi_T^*(t)\psi_T(t)\, dt$$
$$= \int_{-\infty}^{\infty} df \left\{\int_{-\infty}^{\infty} a_T^*(f')a_T(f) \frac{1}{T}\int_{-T/2}^{T/2} e^{i(\omega-\omega')t}\, dt\, df'\right\}.$$

One may define $S_T(f)$ by the ensemble average of the quantity in the curly brackets, i.e.,

$$S_T(f) = E\left[\int_{-\infty}^{\infty} a_T^*(f')a_T(f) \frac{1}{T}\int_{-T/2}^{T/2} e^{i(\omega-\omega')t}\, dt\, df'\right].$$

In the limit as $T \to \infty$, $S_T(f)$ will approach $S(f)$, the power spectral density of $\psi(t)$, both in probability and in mean square.

Consider the real function $\varphi(t)$ and define $\psi(t)$ by the relation

$$\psi(t) = e^{i\varphi(t)}.$$

Then,

$$\psi^*(t)\frac{d}{dt}\psi(t) = i\dot{\varphi}(t)$$

and, taking ensemble and time averages as before,

$$\int_{-\infty}^{\infty} i\omega S(f)\,df = i2\pi\nu_0$$

or

$$\nu_0 = \int_{-\infty}^{\infty} fS(f)\,df,$$

where

$$\nu_0 \equiv \lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{T/2}\frac{1}{2\pi}\dot{\varphi}\,dt.$$

Similarly,

$$\left(\frac{d}{dt}\psi^*\right)\left(\frac{d}{dt}\psi\right) = +\dot{\varphi}^2,$$

and, thus,

$$\overline{(\dot{\varphi})^2} = \int_{-\infty}^{\infty}\omega^2 S(f)\,df,$$

or (using the first moment and zeroth moment)

$$\overline{(\nu(t)-\nu_0)^2} = \int_{-\infty}^{\infty}(f-\nu_0)^2 S(f)\,df,$$

where the bar over a quantity implies infinite time average. Again, in a similar fashion,

$$Im\left[\left(\frac{d^2}{dt^2}\psi^*\right)\left(\frac{d}{dt}\psi\right)\right] = -i\dot{\varphi}^3.$$

After all averaging,

$$\overline{[\nu(t)]^3} = \int_{-\infty}^{\infty} f^3 S(f)\,df.$$

Making use of the zeroth, first, and second moments,

$$\overline{[\nu(t)-\nu_0]^3} = \int_{-\infty}^{\infty}(f-\nu_0)^3 S(f)\,df.$$

If $\varphi(t)$ is a monotonically increasing function of t, then $S(f)$ has essentially no power for negative f. Thus a pure FM signal $V(t)$ may be constructed in the form

$$V(t) = \tfrac{1}{2}V_0[\psi(t)+\psi^*(t)].$$

Obviously,

$$\int_{-\infty}^{\infty} S_v'(f)\,df = \tfrac{1}{2}V_0^2,$$

where $S_v'(f)$ is the two-sided spectral density of $V(t)$. Define

$$S_v(f) = \begin{cases}2S_v'(f) & f \geq 0\\ 0 & f < 0\end{cases}$$

where $S_v(f)$ is the one-sided spectral density of $V(t)$. Then,

$$\tfrac{1}{2}V_0^2 = \int_0^{\infty} S_v(f)\,df,$$

$$\nu_0 = \frac{2}{V_0^2}\int_0^{\infty} fS_v(f)\,df,$$

$$\overline{(\nu(t)-\nu_0)^2} = \frac{2}{V_0^2}\int_0^{\infty}(f-\nu_0)^2 S_v(f)\,df,$$

and

$$\overline{(\nu(t)-\nu_0)^3} = \frac{2}{V_0^2}\int_0^{\infty}(f-\nu_0)^3 S_v(f)\,df,$$

provided $|(\nu(t))/(2\pi\nu_0)| \ll 1$ for all time. Note, $\nu(t)$ is the instantaneous frequency of

$$V(t) = V_0\cos\varphi(t).$$

Another treatment can be found in [1].

Appendix II

Effects of Second-Harmonic Distortion of the Modulation for a Cesium Beam

Following Shirley [7], we may take

$$\varphi(t) = 2\pi\nu_0 t + (b_1/\omega_m)\cos(\omega_m t+\delta_1) + (b_2/2\omega_m)\cos(2\omega_m t+\delta_2), \tag{13}$$

where ω_m is the modulation (angular) frequency.

With

$$\nu(t) = \frac{1}{2\pi}\frac{d\varphi}{dt},$$

we obtain

$$\nu(t)-\nu_0 = -\frac{b_1}{2\pi}\sin(\omega_m t+\delta_1) - \frac{b_2}{2\pi}\sin(2\omega_m t+\delta_2).$$

The third moment of $(\nu(t)-\nu_0)$ may be written in the form

$$\overline{(\nu(t)-\nu_0)^3} = -\frac{\omega_m}{2\pi)^4}\int_0^{2\pi/\omega_m}[b_1\sin(\omega_m t+\delta_1) + b_2\sin(2\omega_m t+\delta_2)]^3\,dt. \tag{14}$$

We will make the ass mption that $b_2 \ll b_1$ and keep at most those terms of f_0 linear in b_2. That is,

$$f_3^3 = \overline{(\nu(t)-\nu_0)^3} \approx -\frac{3b_1^2b_2}{4(2\pi)^3}\sin(2\delta_1-\delta_2) \tag{15}$$

and, similarly,

$$f_2^2 \approx \frac{b_1^2}{2(2\pi)^2}. \tag{16}$$

Substituting (15), (16), and (10) into (9), one obtains

$$f_0 \approx \frac{\tau^2 b_1^2 b_2}{4\pi(16 - \tau^2 b_1^2)} \sin(2\delta_1 - \delta_2).$$

Again following Shirley [7], we let $b_1/\omega_m \sim 1$, $(2\delta_1 - \delta_2) \sim \pi/2$, and $b_2/2\omega_m = \delta\varphi_2$. If the total frequency deviation is equal to the linewidth, then $\tau\omega_m \sim \pi/2$. With these substitutions, we obtain

$$f_0/\nu_0 \sim (0.37) \frac{\delta\varphi_2}{4\pi\nu_0\tau}, \tag{17}$$

which agrees with Shirley's order-of-magnitude calculations to within the factor (0.37).

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