Bibliographic reference:
Short-Term Frequency Stability, NASA SP-80, pp. 119-123, Proc. of the IEEE-NASA Symposium on the Definition and Measurement of Short-Term Frequency Stability held at Goddard Space Flight Center, Greenbelt, Md., November 23-24, 1964, prepared by Goddard Space Flight Center (Scientific & Technical Information Division, National Aeronautics and Space Administration, Washington, D. C., 1965).

11. EFFECTS OF LONG-TERM STABILITY ON THE DEFINITION AND MEASUREMENT OF SHORT-TERM STABILITY

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Several authors have reported the measurement of a "flicker noise" spectrum for the frequency fluctuations, below a few cycles per second, of good quartz crystal oscillators. Experimental work carried out at the National Bureau of Standards at Boulder is in good agreement with these results.

The influence of this longer term type of noise turns out to be of considerable importance in the definition and measurement of shorter term noise, since averages of this short-term noise normally cover a total averaging time well into the flicker noise region. A mathematical formalism which satisfactorily avoids convergence difficulties has been developed around a set of physically meaningful quantities. Some theoretically reasonable definitions of short- and long-term stability are given.

It has been established by many people that quartz crystal oscillators are frequency-modulated by a "flicker" or $1/\omega$ type of noise which extends to at least as low a frequency as 1 cycle per year and probably even lower. Because the frequency emitted by any physically realizable device is bounded, this flicker noise behavior must cut off at some low, nonzero frequency ϵ.

It is possible to construct some measure of frequency stability $\langle\chi\rangle$ as the time average of a function $\chi(t)$ of the frequency. This function may or may not depend critically on the cutoff frequency ϵ. Thus, it might be that, if one measures the average value of $\chi(t)$ for some finite time T, this average value $\langle\chi\rangle_T$ will begin to approach $\langle\chi\rangle$ only after T is several times larger than $1/\epsilon$. Such a stability measure is thus said to be "cutoff-dependent" and is an inconvenient measure of frequency stability, since averaging times in excess of several years may be required to obtain a reasonable approximation to $\langle\chi\rangle$.

It is apparent that a necessary condition (not a sufficient condition) on any cutoff independent stability measure $\langle\chi\rangle$ is that $\langle\chi\rangle$ exists in the limit $\epsilon\rightarrow 0$. One can show that such quantities as the variance of frequency fluctuations around a uniform drift of frequency are, in fact, cutoff-dependent and hence not a very useful measure of frequency stability.

Some measures of frequency stability which are *not* cutoff-dependent and have direct use in various applications are: (1) variance of frequency fluctuation for finite sampling and averaging times, and (2) the variance of the nth finite difference of the phase for $n\geq 2$.

GENERAL PROBLEM

As stated, it has been established that quartz crystal oscillators are frequency-modulated by a "flicker" or $1/\omega$ type of noise spectrum. It even has been shown that this flicker-noise type of spectrum extends to 1 cycle per year and probably even lower. While this spectral region is not in the realm of "short-term" frequency fluctuations, it does have a very profound influence on their definition and measurement.

This can be seen by considering a crystal oscillator whose frequency fluctuations Ω from a nominal value have a (power) spectral density given by

$$G_\Omega(\omega) = g(\omega) + (h/|\omega|), \qquad (1)$$

where $g(\omega)$ predominates for the higher values of ω and thus gives rise to the "short-term" frequency fluctuations. The second term on the right is the flicker noise term. The total mean square of the instantaneous frequency fluctuation $\langle(\Omega)^2\rangle$ is then given by

$$\langle(\Omega)^2\rangle = 2\int_{\epsilon}^{\infty} G_{\Omega}(\omega)\, d\omega, \qquad \epsilon \geq 0. \tag{2}$$

Since any physical device must emit a finite frequency, Ω *must* be bounded; and thus the integral in Equation 2 must exist. This requires that $\epsilon \neq 0$ in order to insure the existence of

$$\int_{\epsilon}^{\infty} (h/|\omega|)\, d\omega.$$

From the preceding comments, it is apparent that this "cutoff" frequency ϵ is not known but is certainly less than 1 cycle per year. Thus, any meaningful measure of frequency stability should, in effect, be cutoff-independent; otherwise, averaging times exceeding several years must be employed.

It is possible to consider some function $\chi(t)$ obtained from the frequency (or phase) of the oscillator:

$$\chi(t) = X[f(t)]. \tag{3}$$

The expectation value of $\chi(t)$ is then given by

$$\langle\chi\rangle = \lim_{T\to\infty} T^{-1}\int_{-T/2}^{T/2} \chi(t)\, dt. \tag{4}$$

In principle, it is possible to obtain the Fourier transform of Equation 3 and substitute this in the integral of Equation 4. In this situation, the only quantities $\chi(t)$ which have physical significance (in the sense of being easily measurable) are quantities which do not depend critically on ϵ as $\epsilon \to 0$. In other words, $\lim_{\epsilon\to 0^+}\langle\chi\rangle$ must exist for meaningful quantities. Table 11–1 shows several functions of the frequency which do *not* exist as $\epsilon \to 0^+$. Physically, this can be pictured as follows: One can measure the quantity

$$\langle\chi\rangle_T = T^{-1}\int_{-T/2}^{T/2} \chi(t)\, dt \tag{5}$$

for some given time T, then extend the averaging time to NT and obtain $\langle\chi\rangle_{NT}$. If the sequence $\{\langle\chi\rangle_{NT}\}$ is considered, one might find that

$$\lim_{N\to\infty}\{\langle\chi\rangle_{NT}\} \to \infty.$$

For any finite N and T, the quantity $\langle\chi\rangle_{NT}$ certainly may exist. In the limit, however, the quantity may or may not exist.

TABLE 11–1.—*Cutoff-Dependent Quantities*

Name	Expression
Auto-covariance function of the phase fluctuations	$\lim_{T\to\infty} T^{-1}\int_{-T/2}^{T/2} \phi(t)\phi(t+\tau)\, dt$
Auto-covariance function of the frequency fluctuations	$\lim_{T\to\infty} T^{-1}\int_{-T/2}^{T/2} \bar{\Omega}(t,\tau)\bar{\Omega}(t+\tau',\tau)\, dt$,
	where $\bar{\Omega}(t,\tau) = [\phi(t+\frac{1}{2}\tau) - \phi(t-\frac{1}{2}\tau)]/\tau$
Standard deviation of the frequency fluctuations	$\lim_{T\to\infty}\left\{T^{-1}\int_{-T/2}^{T/2} [\bar{\Omega}(t,\tau)]^2\, dt - \left[T^{-1}\int_{-T/2}^{T/2} \bar{\Omega}(t,\tau)\, dt\right]^{-2}\right\}$

MEANINGFUL QUANTITIES

The method of (power) spectral densities is a powerful and often meaningful way of encompassing a broad range of measurements. It certainly has application to the case of flicker noise (Reference 1). Occasionally, however, the quantities of physical interest are not simply related to the spectrum or the spectrum contains more information than is needed. Thus, other measures of frequency stability have been devised. Two additional methods are considered here.

RMS FREQUENCY FLUCTUATIONS

As was stated in Table 11-1, the quantity

$$\sigma_f^2(\tau)=\lim_{T\to\infty}\left\{T^{-1}\int_{-T/2}^{T/2}\left[\frac{\phi(t+\tau)-\phi(t)}{\tau}\right]^2 dt\right.$$

$$\left.-\left[T^{-1}\int_{-T/2}^{T/2}\left(\frac{\phi(t+\tau)-\phi(t)}{\tau}\right)dt\right]^2\right\} \quad (6)$$

is cutoff-dependent. However, if one does not pass to the limit $T\to\infty$ but specifies T and τ, the integral most certainly exists even in the limit $\epsilon\to 0$. Thus, one measure of frequency stability is the function

$$\sigma_f^2(\tau,\ T)=T^{-1}\int_{-T/2}^{T/2}\left[\frac{\phi(t+\tau)-\phi(t)}{\tau}\right]^2 dt$$

$$-\left[T^{-1}\int_{-T/2}^{T/2}\left(\frac{\phi(t+\tau)-\phi(t)}{\tau}\right)dt\right]^2, \quad (7)$$

which unfortunately depends on *two* parameters τ and T.

THE METHOD OF FINITE DIFFERENCES

It is of value here to digress from short-term stability and consider how quartz crystal oscillators are used in clock systems—a problem in long-term stability. Typically, an oscillator is used as sort of a "fly wheel" in a clock system between regular calibrations with a frequency standard. Thus, one measures an average frequency $\bar{\Omega}$ during some interval $t-\frac{1}{2}\tau$ to $t+\frac{1}{2}\tau$, say. One predicts, then, that (on the average) the total phase accumulated by the oscillator $\Delta\Phi$ in the larger interval $t-\frac{1}{2}T$ to $t+\frac{1}{2}T$ is given by

$$\Delta\Phi\approx T\bar{\Omega}=T\left[\frac{\phi(t-\frac{1}{2}\tau)-\phi(t+\frac{1}{2}\tau)}{\tau}\right] \quad (8)$$

While this may be true "on the average," the frequency fluctuations of the oscillator cause some error $\delta\Phi$, given by

$$\delta\Phi=\Delta\Phi-T\bar{\Omega},$$

$$\delta\Phi=\phi(t+\tfrac{1}{2}T)-\phi(t-\tfrac{1}{2}T)-(T/\tau)[\phi(t+\tfrac{1}{2}\tau)-\phi(t-\tfrac{1}{2}\tau)]. \quad (9)$$

If one now sets $T=3\tau$ and defines the variable ϕ_n defined on the discrete range of the integer n by the relation

$$\phi_n\equiv\phi(t_0+n\tau)$$

and if one writes $t=t_0+\frac{1}{2}T$ in Equation 9, $\delta\Phi$ can be written in the simpler form

$$\delta\Phi=\Delta^3\phi_n, \quad (10)$$

where $\Delta^3\phi_n$ is the third finite difference of the variable ϕ_n (see Table 11-2). Thus the precision of an oscillator used in this system is related to the quantity

$$\langle(\delta\Phi)^2\rangle=\langle(\Delta^3\phi_n)^2\rangle, \quad (11)$$

which certainly must exist if the clock is any good at all.

Indeed, if one considers Equation 10 in the case of flicker noise (Reference 2), not only is $\Delta^3\phi_n$ a stationary function, but the correlation with $\Delta^3\phi_{n+k}$ for $k\geq 3$ is so small that the convergence of the quantity

$$N^{-1}\sum_{n=1}^{N}(\Delta^3\phi_n)^2$$

for large N is essentially that of a random un-

TABLE 11-2.—*Finite Phase Differences*

Variable	Definition	
ϕ_n	ϕ_n	$\phi(t_0+n\tau)$
$\Delta\phi_n$	$\phi_{n+1}-\phi_n$	$\phi_{n+1}-\phi_n$
$\Delta^2\phi_n$	$\Delta\phi_{n+1}-\Delta\phi_n$	$\phi_{n+2}-2\phi_{n+1}+\phi_n$
$\Delta^3\phi_n$	$\Delta^2\phi_{n+1}-\Delta^2\phi_n$	$\phi_{n+3}-3\phi_{n+2}+3\phi_{n+1}-\phi_n$

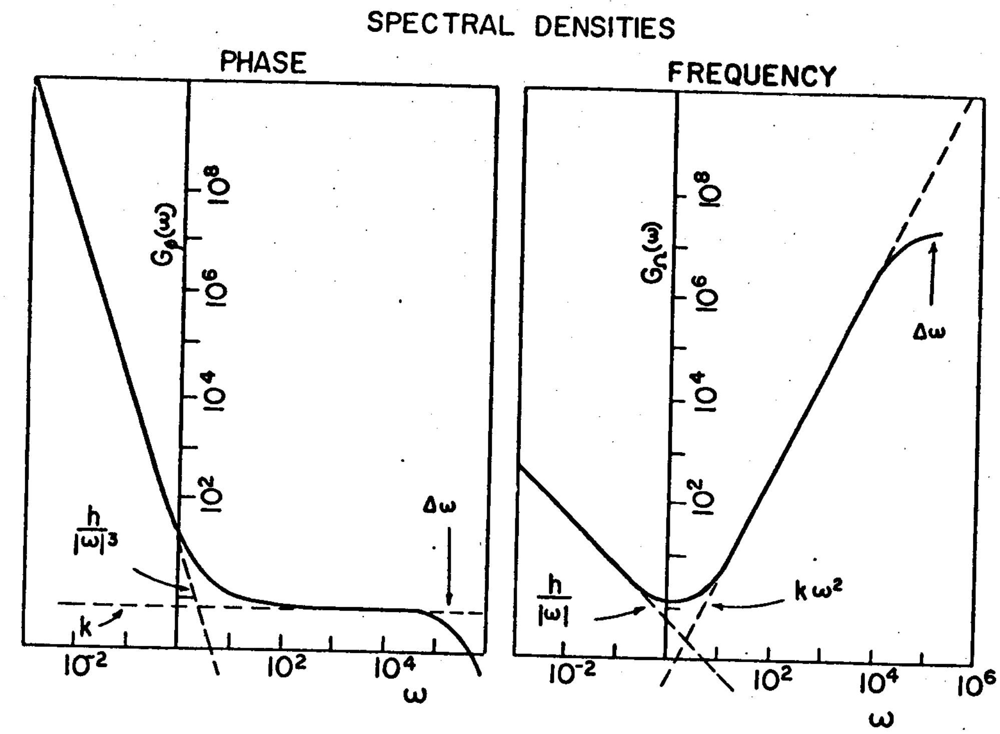

FIGURE 11-1.

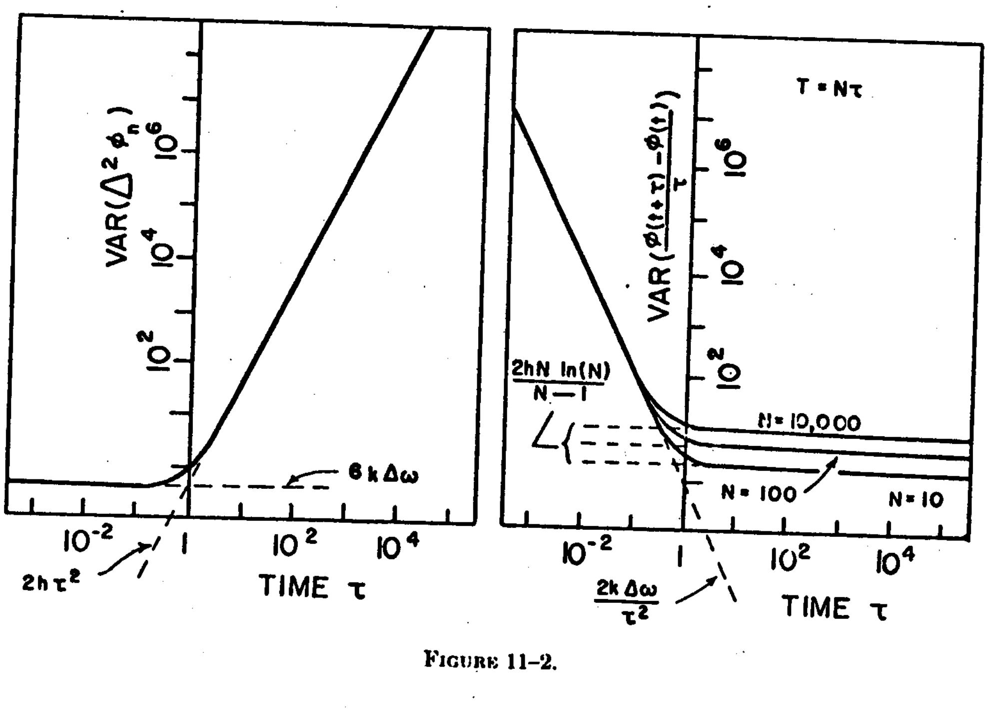

FIGURE 11-2.

correlated variable. Similarly one finds in general that $\Delta^k\phi_n$ is a stationary, cutoff-independent function for $k \geq 2$.

CONCLUSIONS

It is of value now to tie these types of frequency stability together by applying each method to the same theoretical model of an oscillator; in particular, consider an oscillator whose frequency is modulated by a flicker noise. Also, let the system generate an essentially white noise over its bandpass. Since this noise will appear to be half AM and half FM, the spectral densities of frequency and phase will appear as indicated in Figure 11-1. On the basis of this model, the graphs of Figure 11-2 were obtained.

It is interesting to note that the effects of flicker noise on the variance of frequency fluctuations depend on the total averaging time (see Equation 7). Thus, if one were to let $N = T/\tau = 2$ (the smallest possible number for a variance), the oscillator would "appear" better than for any other N. However, one should take the ensemble average of many variances for $N = 2$ to obtain an acceptably precise figure for the variance.

The asymptotes of the curve showing the variance of the second difference differ from the asymptotes of the frequency curve by a function of N and a factor of τ^2. For this specialized case the three numbers k, h, and $\Delta\omega$ serve as a complete measure of frequency stability instead of giving values of a continuous function, such as spectral distribution or variance of the nth finite difference. For the variance of the frequency fluctuations, one has a stability measure which is a function of *two* continuous variables. It also is worth noting that this model, in fact, fits very well a broad class of commercially available oscillators.

One is led to the conclusion that there are some commonly quoted measures of frequency stability which are very impractical. The autocovariance function of phase and frequency and the total rms frequency fluctuations are not useful concepts in the definition of frequency stability, since it is very difficult to obtain them experimentally. While the rms frequency fluctuations for specified sample and averaging times is a meaningful quantity, it is very inconvenient to have a stability measure be a function of two variables. Thus, it is suggested that the more meaningful concepts are (power) spectral densities of phase and frequency fluctuations and the variances of the second and higher finite differences of the phase.

With the difficulty of defining an rms frequency, one sees also the difficulty of measuring the true (power) spectral density of the output voltage of an oscillator. Indeed, as one makes his analyzer narrower in bandwidth and takes longer to sweep the line, the spectrum looks worse and worse.

REFERENCES

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