Confidence on the Modified Allan Variance And the Time Variance*

Marc A. Weiss[1], F. L. Walls[1], Charles A. Greenhall[2], Todd Walter[3]

[1]Time and Frequency Division, National Institute of Standards and Technology, 325 Broadway, Boulder, Colorado 80303, USA

[2]Jet Propulsion Laboratory, California Institute of Technology, 4800 Oak Grove Dr., Pasadena, California 91109, USA

[3]HEPL Gravity Probe B, Stanford University, Stanford, California 94305-4085, USA

ABSTRACT

This paper presents tabulated factors for calculating confidence intervals for the square root of the Modified Allan Variance and the related Time Variance when the analysis uses full overlapping of the data. Confidence intervals are determined by multiplying factors from the appropriate table times sample deviations. These factors have been calculated from the equivalent degrees of freedom (edf) for the five common power law noise types. Closed form expressions for calculating the edf for these variances and noise types have only recently become available for fully overlapped data analysis. We compute the edf using two different closed form expressions as well as simulations to ensure correctness of the results. We also show the relative advantage of using the modified Allan variance instead of the Allan variance to evaluate frequency or time stability in the presence of white phase modulated noise.

INTRODUCTION

The focus of this paper is the discussion and presentation of factors for determining confidence intervals for the Modified Allan Variance (MVAR) [1] and the related Time Variance (TVAR) [2]. Like the Allan Variance (AVAR) [3], these variances have become standard tools for analyzing high-performance signal sources, signal processing devices, and measurement systems. AVAR, MVAR, and TVAR have been adopted by the IEEE and the ITU for characterizing such systems. This is because dominant noise processes in those systems have nonwhite spectra sometimes called colored noise. The standard sample variance typically is dependent on the data length for nonwhite noise types and is not defined in the limit as the data length approaches infinity. Hence, the standard sample variance is not satisfactory for characterizing the frequency or time stability of high-performance frequency sources and associated equipment.

Five power law noise types are commonly found in such equipment: white phase noise modulation (PM), flicker PM, random walk PM, flicker frequency modulation (FM), and random walk FM. The spectra associated with these noise types obey power laws in the sense that the spectrum of a measurement of time offset x(t) dominated by one of these noise types is of the form

$$S_x(f) = g\, f^{\beta}, \qquad (1)$$

where $\beta = 0, -1, -2, -3, -4$, respectively, for the five noise types. We may also consider the spectrum of the frequency offset y(t), the derivative of x(t), which would be of the form

$$S_y(f) = h\, f^{\alpha}, \qquad (2)$$

where $\alpha = +2, +1, 0, -1, -2$, respectively, for the five noise types. A random walk modulation of phase is the same as a white noise modulation of frequency.

Each of the three variances, AVAR, MVAR, and TVAR, has a different dependence on averaging time τ for the five different noise types. For a given noise type, each variance has a power-law dependence τ. This is illustrated in Figure 1. Thus, using these variances it is possible to measure the dominant noise type for different averaging times, if the spectrum is a sum of power-law spectra.

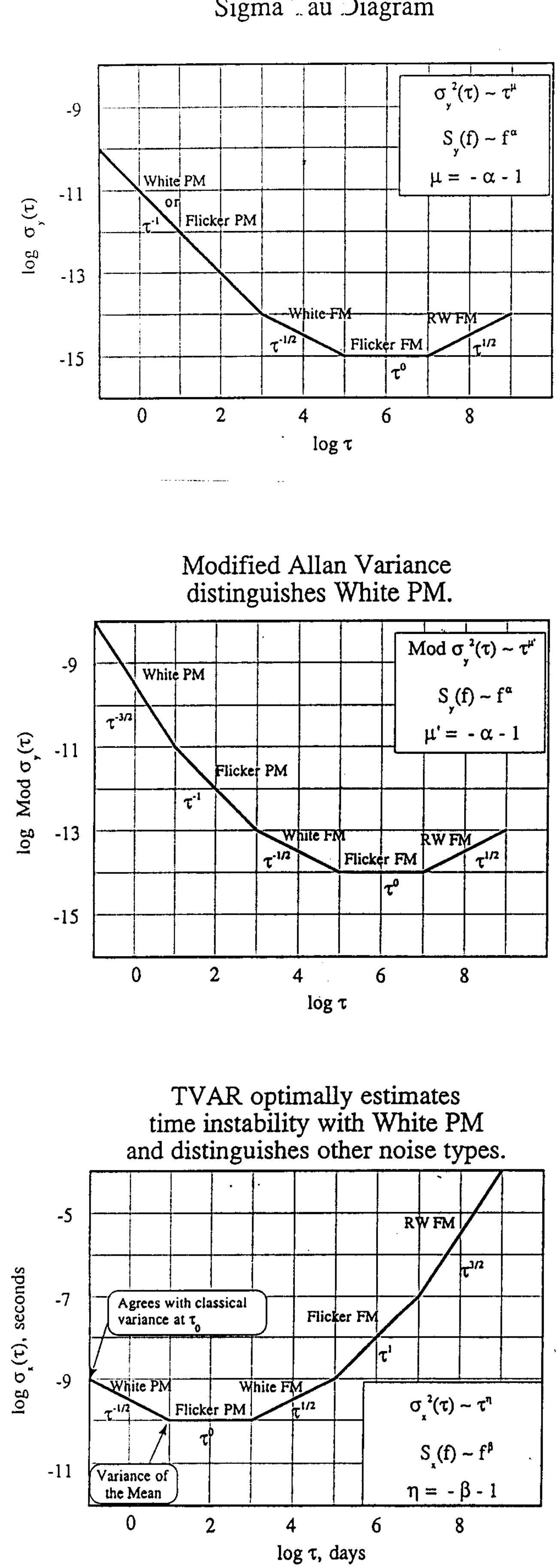


Figure 1 Illustration of the three variances, the Allan, Modified Allan, and Time Variances, and their relation to the power spectrum.

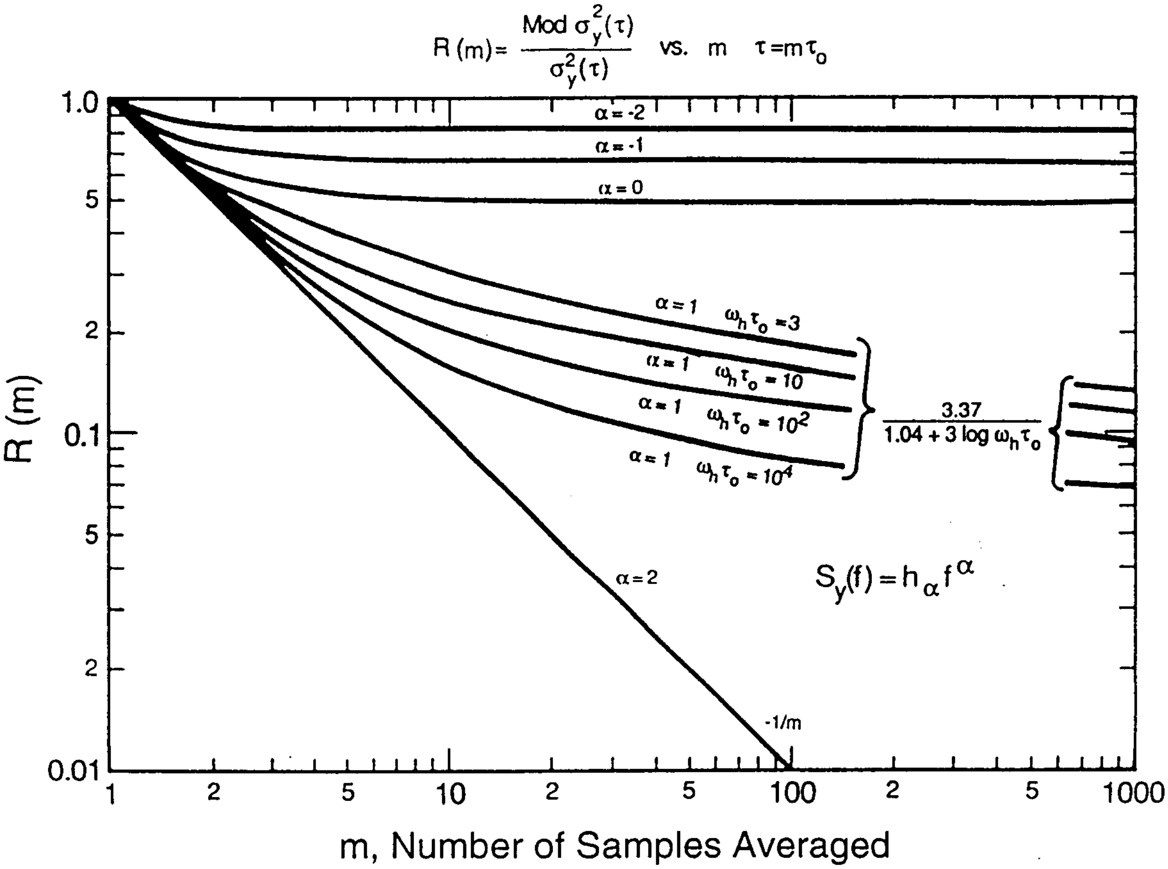


Figure 2 The ratio of the Modified Allan Variance to the Allan Variance as a function of the number of samples in the estimate for different noise types and bandwidths.

There is an advantage to using MVAR over AVAR in the presence of white PM, since MVAR falls off faster for white PM than AVAR. Figure 2 shows the ratio of MVAR to AVAR. We see that for white PM the ratio drops with a constant slope, unlike for the other noise types. This allows MVAR to measure the levels of flicker PM, white FM, flicker FM and random walk FM in the presence of white PM for shorter values of τ than AVAR. Though this advantage is reduced somewhat because the confidence intervals are larger for MVAR than for AVAR, MVAR still provides an improvement over AVAR. Additionally, MVAR can distinguish between white and flicker PM, whereas AVAR cannot. TVAR has all the same advantages as MVAR, the only difference being its power law dependence on τ as indicated in Figure 1 and its level. Often white PM comes from a different part of the system being measured than other noise types. Thus it is valuable to be able to average down white PM optimally to characterize the other noise types.

COMPUTATION OF CONFIDENCE INTERVALS

If s^2 denotes the usual sample variance of n independent and identically distributed Gaussian measurements (i.e., white noise) with actual variance σ^2, then it is well-known that

$$U = \frac{s^2}{\sigma^2}\cdot\nu. \tag{3}$$

has a chi-square distribution with $\nu = n-1$ degrees of freedom [4]. In the classical situation, the degrees of freedom associated with σ^2, are integer values depending only on the number of measurements, and exact confidence limits on the measurement variance are easily calculated using percentiles of the appropriate chi-square distribution. For example, figure 3 shows the chi-square distribution with 10 degrees of freedom, and also depicts the percentiles a and b that are needed to calculate a uncertainty bounds on σ^2 at the p = 0.95 confidence level from a particular s^2 based on 11 Gaussian measurements.

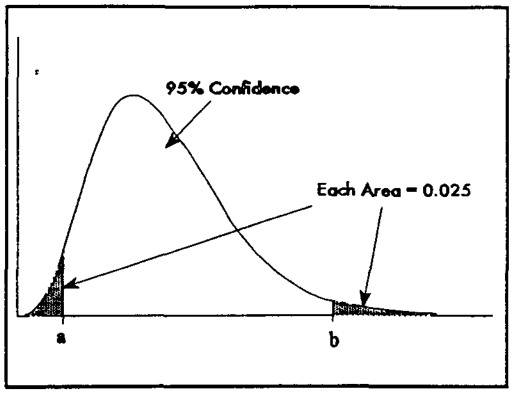


Figure 3 Finding the 95% confidence limits under the chi-squared distribution with 10 degrees of freedom.

A 95% confidence interval is obtained as follows. First the probability 0.95 that U of equation (3) lies between a and b. This condition is equivalent to the following:

$$\begin{gathered} a < \frac{s^2}{\sigma^2}\cdot\nu < b \\ \textit{iff and only iff} \\ \frac{\nu}{b} < \frac{\sigma^2}{s^2} < \frac{\nu}{a} \\ \textit{iff and only iff} \\ \frac{\nu}{b}\cdot s^2 < \sigma^2 < \frac{\nu}{a}\cdot s^2. \end{gathered} \tag{4}$$

While the lower and upper bounds in the final inequality are confidence limits on the actual sample variance, note that the confidence factors ν/b and ν/a needed in the calculations are indepent of the actual data. They give the magnitude of the confidence interval as a function of the number of points used to compute the variance. Hence we can compute these confidence factors for each noise type and various data lengths. The factors $1-\nu/b$ and $\nu/a-1$ give the multipliers for the magnitude of the lower and upper confidence intervals, respectively, and these values are presented at the end of this for each of the five noise types and for varous data lengths. We also use these values to characterize the relative merits of AVAR and MVAR for estimating noise processes in the presence of white PM.

Since the common time and frequency stability measures (AVAR, MVAR, TVAR) are calculated from data arising from non-white noise processes, the confidence limit procedure outlined above is an approximate method [5]. The method is based on approximating the distribution of U in (3) with the chi-square distribution with degrees of freedom

$$\nu = \frac{2(\sigma^2)^2}{Var\ (s^2)}. \qquad (5)$$

where σ^2 represents the appropriate stability measure (e.g., MVAR), s^2 represents its corresponding estimator, and $Var(s^2)$ is the expected variance of the s^2 estimators. The quantity ν here is called the equivalent degrees of freedom, edf, since it need not be integer-valued. Equations for the edf of MVAR and TVAR have only recently been published in closed form [6], and other papers have been submitted for publication [7],[8].

We generated tables of the factors $1-\nu/b$ and $\nu/a-1$ by computing the edf using two different formulas and verified the theoretical results by simulating corresponding noise processes. Both theoretical solutions express MVAR as a function of the discrete autocorrelation function, and evaluate (5) for autocorrelation functions associated with respective noise types. One method uses the conventional definition of MVAR as a variance of second differences of averages, whereas the other technique uses an equivalent definition of MVAR in terms of third differences [9]. While the definition of MVAR is equivalent in the two formulations, the autocorrelation functions differ, so the numerical solutions for calculating the edf differ as well. The two methods agreed in all results.

SIMULATION

To check the theory we simulated power-law noise series and calculated estimated edf's by substituting appropriate estimates of σ^2 and $Var(s^2)$ in (5). We generated data consistent with a model of white PM using a random number generator, computed cumulative sums of the data sequence to simulate white FM, then again computed cumulative sums to simulate random walk FM. We used the Barnes-Jarvis filter [10],[11] to simulate flicker PM, then computed cumulative sums to simulate flicker FM. Series lengths N appearing in tables at the end of this paper were generated in each case.

For each noise type and series length N, we generated 10,000 time series and computed MVAR for each series. For every set of 10,000 MVAR estimates, corresponding sample mean and sample variance of the 10,000 obverved values of MVAR were substituted, respectively, for $\sigma^2$and $Var(s^2)$ in equation (5), thereby producing an estimated value of the edf, ν. Estimated values of the edf. from the simulation agreed with the numerically computed values within a few percent for white PM, white FM, and random walk FM noises. For flicker noise, the match was not as close for small values of m, where $\tau=m\cdot\tau_0$. This is consistent with recent publications showing that the effects of discrete sampling, as reflected for example in the autocorrelation function, produce different effects than the output of the Barnes-Jarvis filter [6].

The edf from the two theoretical methods and from simulation are presented in Table 1 for N=1025 and for white PM (whpm) and flicker PM (flpm) noise, respectively. The table illustrates the agreement.

Table 1
Comparison of Degrees of Freedom
From Three Methods
N = 1025

	Theoretical	Methods	Simulation	
m	whpm	flpm	whpm	flpm
1	526.4	589.9	527.66	630.38
2	477.4	497.0	493.24	496.87
4	298.7	263.1	294.80	257.91
8	158.2	128.4	156.68	126.50
16	78.96	62.33	80.94	61.84
32	38.15	29.85	38.80	29.96
64	17.62	13.73	17.69	13.69
128	7.40	5.74	7.29	5.71
256	2.85	2.07	2.73	2.07

USE OF OVERLAPPING ESTIMATES

There are two common methods for the estimation of the three variances, AVAR, MVAR, and TVAR. One method uses fully overlapping and the other nonoverlapping time intervals. We will explain the meaning of these terms, then discuss their relative advantages. To understand the potential for overlapping estimates we will use the definitions of AVAR, MVAR, and TVAR, and use another common notation for them, $\sigma^2_y(\tau)$, mod.$\sigma^2_y(\tau)$, and $\sigma^2_x(\tau)$, respectively.

Theoretically we can define these variances independent of sampling. Suppose there is a time-difference signal, x(t), between two clocks. Then, at least conceptually, for AVAR we would apply an operator which finds one-half the ensemble average of the mean second difference squared. For MVAR the operator would extract one-half the ensemble average of the square of the time averaged second-differences. This process is illustrated in Figure 4. TVAR is simply $\tau^2\cdot$MVAR/3. Thus,

$$\sigma^2_y(\tau) = \frac{1}{2\tau^2}\left\langle \left(\Delta^2_\tau x\right)^2 \right\rangle$$

$$\mathrm{mod}.\sigma^2_y(\tau) = \frac{1}{2\tau^2}\left\langle \left(\Delta^2_\tau \bar{x}^\tau\right)^2 \right\rangle \quad (6)$$

$$\sigma^2_x(\tau) = \frac{1}{6}\left\langle \left(\Delta^2_\tau \bar{x}^\tau\right)^2 \right\rangle$$

Here, Δ^2_τ is the second difference operator over the interval τ, $< ... >$ is the infinite ensemble average, and $\bar{x}^\tau$ is generated from x by averaging over an interval of length τ.

For computing estimates of these variances, we assume stationarity of the second difference of the data and compute time averages of the square of the second differences. In forming the second differences in AVAR, as well as the $\bar{x}^\tau$ in MVAR and TVAR, we can choose where to start the next second difference relative to the last one. This allows the option of overlapping the second differences. Figure 4 illustrates a second difference starting with $\bar{x}_n^\tau$. The next second difference can begin from a point either τ_0 or τ increased from this one. Equations (7) and (8) below give the computational definitions.

If there are N data points with an even spacing of τ_0, then the variances are functions of $\tau = m \cdot \tau_0$. Suppose we have sampled our time difference signal x(t) with a time series $\{x_n\}$. Then the estimates of AVAR, MVAR, and TVAR using fully overlapping samples would be computed as in equation (7) below.

The computational equations for estimates of AVAR, MVAR, and TVAR using nonoverlapping samples take the form in equation (8) below where [A] is the greatest integer less than A.

The difference between the fully overlapping and non-overlapping estimates is in how the summation parameter n increases. Either we would increase by the minimum spacing, or by m intervals. The latter means that we would use different data for intervals of length τ included in each second difference. Figure 5 plots the ratio of the confidence factors for the use of non-overlapping over fully overlapping samples for the Allan Variance. The table shows that the use of fully overlapping estimates is better primarily for noise types of white and flicker PM.

ADVANTAGES OF MVAR OVER AVAR

AVAR and MVAR are standard tools for the characterization of frequency standards and measurement systems. Often we measure the frequency stability between frequency standards through measurement systems whose noise may exceed the performance of the standards in short-term. The noise types of measurement systems are typically white and flicker PM, whereas the noise types of the frequency standards are often white, flicker and random walk FM. If the noise type of the measurement system is white PM, we can average it down faster to potentially determine the performance of frequency standards if we use MVAR instead of AVAR. This is because the ratio of the Modified Allan Deviation over the Allan Deviation for white PM is proportional to $\tau^{-1/2}$, where τ is the averaging time, the Modified Allan Deviation (MDEV) is the square root of MVAR, and the Allan Deviation (ADEV) is the square root of AVAR. Figure 2 shows a plot of the ratio of MVAR over AVAR for different noise types and bandwidths. Note that flicker PM is bandwidth dependent. This Figure shows that the advantage of using MVAR is primarily in the case of white PM. Flicker PM shows some advantage, though the loss of confidence, as discussed below, destroys any improvement of MVAR over AVAR in this case.

We illustrate the advantage of MVAR over AVAR in Table 3 for the case of a data set of length N=1025 dominated by a white PM noise, using a confidence interval for a probability of 68%. This advantage should be understood as the combination of two effects. MVAR has an advantage over AVAR because it drops faster for white PM, but it loses some of that advantage because the confidence intervals for MVAR are poorer than for AVAR. The advantage of MVAR over AVAR due to its faster averaging of white PM is presented for our particular example as the ratio of the deviations, ADEV over MDEV, in column 2 of Table 3. The advantage of AVAR over MVAR is due to its smaller confidence intervals. We define σ(ADEV) and σ(MDEV) as the root-mean-square of the upper and lower confidence intervals for ADEV and MDEV respectively, using a probability of 0.68. Thus, σ(ADEV) and σ(MDEV) are the equivalent of standard deviations around ADEV and MDEV respectively. The ratio of σ(ADEV) over σ(MDEV) is shown in column 3 of Table 3, illustrating the advantage of using AVAR. The advantage of MVAR over AVAR is presented in column 4. This is the ratio of column 2 over column 3.

$$\sigma^2_y(\tau) = \frac{1}{2(N-2m)\tau^2}\sum_{n=1}^{N-2m}\left(x_{n+2m} - 2x_{n+m} + x_n\right)^2,$$

$$\text{mod.}\sigma^2_y(\tau) = \frac{1}{2(N-3m+1)\tau^2}\sum_{n=1}^{N-3m+1}\left(\frac{1}{m}\sum_{k=0}^{m-1}\left(x_{n+k+2m} - 2x_{n+k+m} + x_{n+k}\right)\right)^2, \qquad (7)$$

$$\sigma^2_x(\tau) = \frac{\tau^2}{3}\text{mod.}\sigma^2_y(\tau).$$

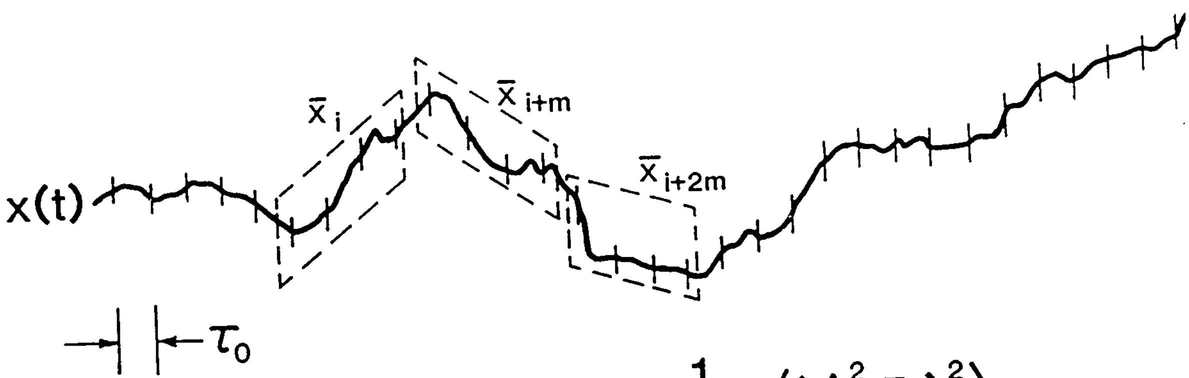
$$\sigma^2_y(\tau) = \frac{1}{2[(N-2m-1)/m]\tau^2}\sum_{n=0}^{[(N-2m-1)/m]}\left(x_{nm+1+2m} - 2x_{nm+1+m} + x_{nm+1}\right)^2,$$

$$\text{mod.}\sigma^2_y(\tau) = \frac{1}{2[(N-3m)/m]\tau^2}\sum_{n=0}^{[(N-3m)/m]}\left(\frac{1}{m}\sum_{k=0}^{m-1}\left(x_{nm+1+k+2m} - 2x_{nm+1+k+m} + x_{nm+1+k}\right)\right)^2, \qquad (8)$$

$$\sigma^2_x(\tau) = \frac{\tau^2}{3}\text{mod.}\sigma^2_y(\tau).$$

Table 3
Comparison of AVAR and MVAR
Noise Type = White PM
Confidence = 68%, Data Length = 1025

m	ADEV / MDEV	σ(ADEV) / σ(MDEV)	column 2 / column 3
1	1.00	1.00	1.00
2	1.41	1.09	1.30
4	2.00	1.33	1.51
8	2.83	1.83	1.54
16	4.00	2.59	1.55
32	5.66	3.78	1.50
64	8.00	6.02	1.33
128	11.31	10.24	1.10
256	16.00	35.08	0.46



$$\text{Mod. } \sigma_y^2(\tau) = \frac{1}{2\tau^2}\langle(\Delta^2 \bar{x}_i)^2\rangle$$

$$\tau = m\tau_0$$

Figure 4 An illustration of the computation of the Modified Allan Variance.

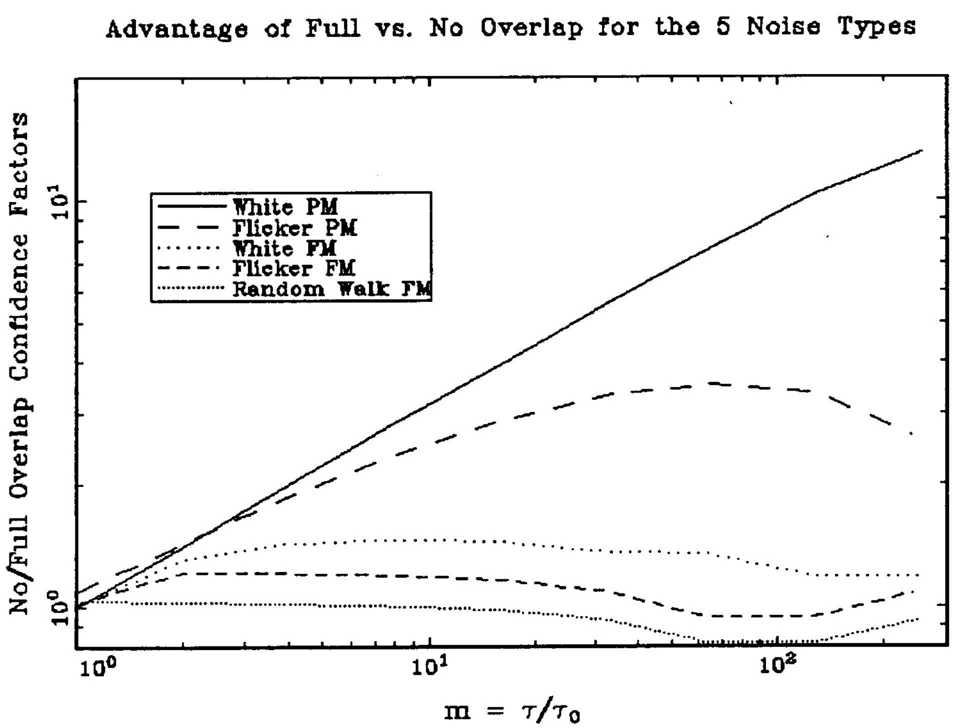


Figure 5 Ratio of the confidence factors for the Allan Variance of data with N= 1025 points using no overlapping data divided by factors using full overlapping data. The factors are larger using no overlapping data leading to poorer confidence limits for the use of no overlap primarily for white and flicker PM.

TABLES OF CONFIDENCE LIMITS

Tables 4-8 give confidence limits for each of the five noise types respectively: white PM, flicker PM, white FM (the same as random walk PM), flicker FM, and random walk FM. In each table we consider cases for data lengths N= 5, 9, 17, 33, 65, 129, 257, 513, and 1025. We cover the possible values of m as a power of 2, where the averaging time is $\tau = m \cdot \tau_0$. Thus m is the number of the minimum intervals between data for the averaging time. Column 3 of each table lists the degrees of freedom from theoretical derivations. Columns 4 and 5 list the lower and upper confidence limits for 68% probability, respectively. Columns 6 and 7 list the same for 95% probability. The values we present in the tables for the lower limits are 1.0 minus the factor needed to multiply times the sample deviation to give the lower limit value as a percent. For the upper limits we present the multiplicative factor minus 1.0. Thus we give the percent change from the sample deviation to give the lower and upper limits corresponding to the given probability. That is, if A and B are the factors such that A·s is the upper limit and B·s is the lower limit for probability p, then the values we present are σ_l for the lower limit and σ_u for the upper limit where

$$\begin{aligned} \sigma_l &= 1.0 - A, \\ \sigma_u &= B - 1.0. \end{aligned} \qquad (9)$$

CONCLUSIONS

MVAR shows benefits over AVAR when the dominant noise type is white PM. In particular, MVAR averages down the white PM faster than AVAR, revealing other noise types for shorter τ values. Even though the confidence is generally poorer for MVAR than for AVAR, the advantage in averaging white PM still prevails, though its main benefit occurs for τ values more toward the middle of the range from the minimum data spacing to the maximum allowable. In addition, while the use of fully overlapping samples always improves the degrees of freedom in estimating AVAR, MVAR, and TVAR, the improvement is best for noise types of white and flicker PM. For white, flicker, and random walk FM, the advantage in the use of fully overlapping samples is insignificant.

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Table 4
White PM Confidence Factors
for MDEV and TDEV
Confidence factors times sample deviations
give magnitudes of
lower and upper confidence intervals
for 68% and 95% probabilities

N	m	edf	lower 68%	upper 68%	lower 95%	upper 95%
5	1	1.862	26.50%	152.4%	48.73%	606.7%
9	1	3.885	22.20%	69.42%	40.42%	194.1%
9	2	2.636	24.57%	101.8%	44.82%	330.4%
17	1	7.988	17.70%	38.29%	32.47%	91.70%
17	2	6.216	19.24%	46.49%	35.17%	116.1%
17	4	2.951	23.90%	90.54%	43.54%	279.3%
33	1	16.21	13.67%	23.32%	25.40%	51.69%
33	2	13.66	14.59%	26.14%	27.02%	58.81%
33	4	7.293	18.25%	41.03%	33.44%	99.64%
33	8	3.011	23.78%	88.71%	43.31%	271.4%
65	1	32.67	10.32%	14.99%	19.42%	31.84%
65	2	28.61	10.90%	16.25%	20.47%	34.74%
65	4	16.64	13.53%	22.92%	25.16%	50.70%
65	8	7.508	18.07%	40.13%	33.13%	97.00%
65	16	2.970	23.86%	89.95%	43.47%	276.8%
129	1	65.58	7.656%	9.956%	14.57%	20.61%
129	2	58.53	8.046%	10.63%	15.29%	22.07%
129	4	35.43	9.974%	14.27%	18.80%	30.20%
129	8	17.49	13.27%	22.18%	24.70%	48.89%
129	16	7.488	18.09%	40.21%	33.16%	97.24%
129	32	2.920	23.96%	91.52%	43.66%	283.6%
257	1	131.4	5.607%	6.748%	10.77%	13.75%
257	2	118.4	5.879%	7.146%	11.28%	14.59%
257	4	73.04	7.300%	9.363%	13.92%	19.32%
257	8	37.57	9.731%	13.78%	18.36%	29.09%
257	16	17.64	13.23%	22.06%	24.62%	48.59%
257	32	7.444	18.13%	40.39%	33.22%	97.77%
257	64	2.885	24.04%	92.66%	43.80%	288.6%
513	1	263.1	4.066%	4.634%	7.863%	9.342%
513	2	238.1	4.261%	4.889%	8.232%	9.867%
513	4	148.3	5.305%	6.315%	10.20%	12.84%
513	8	77.76	7.101%	9.037%	13.55%	18.62%
513	16	38.07	9.677%	13.67%	18.26%	28.84%
513	32	17.64	13.23%	22.06%	24.62%	48.59%
513	64	7.414	18.15%	40.52%	33.26%	98.13%
513	128	2.864	24.08%	93.36%	43.88%	291.7%
1025	1	526.4	2.928%	3.212%	5.693%	6.430%
1025	2	477.4	3.068%	3.381%	5.960%	6.773%
1025	4	298.7	3.831%	4.331%	7.416%	8.718%
1025	8	158.2	5.150%	6.097%	9.910%	12.38%
1025	16	78.96	7.053%	8.959%	13.46%	18.45%
1025	32	38.15	9.669%	13.66%	18.25%	28.81%
1025	64	17.62	13.24%	22.08%	24.63%	48.63%
1025	128	7.396	18.17%	40.59%	33.29%	98.35%
1025	256	2.854	24.10%	93.70%	43.92%	293.2%

Table 5
Flicker PM Confidence Factors
for MDEV and TDEV
Confidence factors times sample deviations
give magnitudes of
lower and upper confidence intervals
for 68% and 95% probabilities

N	m	edf	lower 68%	upper 68%	lower 95%	upper 95%
5	1	2.020	26.08%	137.8%	47.82%	518.7%
9	1	4.300	21.56%	63.35%	39.27%	172.0%
9	2	2.676	24.49%	100.2%	44.65%	322.9%
17	1	8.903	17.04%	35.33%	31.33%	83.34%
17	2	6.427	19.03%	45.27%	34.81%	112.3%
17	4	2.560	24.75%	105.1%	45.15%	345.8%
33	1	18.12	13.09%	21.68%	24.38%	47.65%
33	2	14.18	14.39%	25.49%	26.66%	57.15%
33	4	6.394	19.06%	45.45%	34.86%	112.9%
33	8	2.362	25.21%	114.8%	46.06%	393.9%
65	1	36.56	9.844%	14.01%	18.56%	29.60%
65	2	29.74	10.73%	15.87%	20.16%	33.86%
65	4	14.63	14.22%	24.96%	26.37%	55.81%
65	8	6.066	19.39%	47.40%	35.44%	118.9%
65	16	2.223	25.55%	123.1%	46.75%	436.7%
129	1	73.45	7.282%	9.333%	13.88%	19.26%
129	2	60.89	7.909%	10.39%	15.04%	21.55%
129	4	31.18	10.52%	15.42%	19.78%	32.82%
129	8	14.17	14.39%	25.50%	26.67%	57.18%
129	16	5.882	19.59%	48.59%	35.78%	122.6%
129	32	2.145	25.75%	128.3%	47.15%	465.0%
257	1	147.2	5.323%	6.341%	10.24%	12.89%
257	2	123.2	5.774%	6.991%	11.08%	14.26%
257	4	64.31	7.722%	10.07%	14.69%	20.85%
257	8	30.48	10.62%	15.64%	19.96%	33.32%
257	16	13.90	14.49%	25.84%	26.86%	58.03%
257	32	5.796	19.68%	49.18%	35.94%	124.5%
257	64	2.104	25.86%	131.2%	47.37%	481.4%
513	1	294.8	3.855%	4.362%	7.461%	8.780%
513	2	247.8	4.182%	4.785%	8.083%	9.653%
513	4	130.6	5.622%	6.771%	10.80%	13.79%
513	8	63.12	7.785%	10.18%	14.81%	21.09%
513	16	30.03	10.68%	15.78%	20.08%	33.64%
513	32	13.78	14.54%	25.99%	26.94%	58.41%
513	64	5.757	19.72%	49.45%	36.02%	125.3%
513	128	2.084	25.91%	132.7%	47.47%	489.8%
1025	1	589.9	2.773%	3.026%	5.395%	6.052%
1025	2	497	3.010%	3.310%	5.849%	6.629%
1025	4	263.1	4.066%	4.634%	7.863%	9.342%
1025	8	128.4	5.666%	6.834%	10.88%	13.93%
1025	16	62.33	7.828%	10.25%	14.89%	21.25%
1025	32	29.85	10.71%	15.84%	20.13%	33.78%
1025	64	13.73	14.56%	26.05%	26.97%	58.58%
1025	128	5.740	19.74%	49.57%	36.05%	125.7%
1025	256	2.073	25.94%	133.6%	47.53%	494.5%

Table 6
White FM Confidence Factors for
MDEV and TDEV
Confidence factors times sample deviations
give magnitudes of
lower and upper confidence intervals
for 68% and 95% probabilities

N	m	edf	lower 68%	upper 68%	lower 95%	upper 95%
5	1	2.250	25.49%	121.4%	46.61%	427.6%
9	1	4.900	20.73%	56.55%	37.80%	148.5%
9	2	2.703	24.43%	99.17%	44.54%	318.0%
17	1	10.23	16.23%	31.96%	29.90%	74.08%
17	2	6.618	18.85%	44.24%	34.49%	109.2%
17	4	2.356	25.23%	115.2%	46.09%	395.6%
33	1	20.89	12.38%	19.78%	23.11%	43.06%
33	2	14.67	14.20%	24.92%	26.34%	55.70%
33	4	6.096	19.36%	47.22%	35.39%	118.3%
33	8	2.097	25.88%	131.8%	47.40%	484.3%
65	1	42.22	9.262%	12.86%	17.51%	27.01%
65	2	30.82	10.57%	15.53%	19.88%	33.07%
65	4	14.01	14.45%	25.70%	26.78%	57.68%
65	8	5.759	19.72%	49.43%	36.01%	125.3%
65	16	1.951	26.26%	143.8%	48.21%	553.9%
129	1	84.89	6.829%	8.601%	13.04%	17.68%
129	2	63.14	7.784%	10.17%	14.81%	21.08%
129	4	29.89	10.71%	15.82%	20.12%	33.75%
129	8	13.52	14.65%	26.33%	27.12%	59.28%
129	16	5.612	19.88%	50.49%	36.30%	128.6%
129	32	1.875	26.47%	151.1%	48.65%	598.4%
257	1	170.2	4.979%	5.859%	9.589%	11.88%
257	2	127.8	5.678%	6.852%	10.90%	13.96%
257	4	61.69	7.864%	10.31%	14.95%	21.38%
257	8	29.11	10.82%	16.08%	20.33%	34.34%
257	16	13.32	14.73%	26.60%	27.27%	59.96%
257	32	5.545	19.96%	50.99%	36.43%	130.2%
257	64	1.836	26.57%	155.2%	48.89%	623.9%
513	1	340.9	3.600%	4.038%	6.976%	8.116%
513	2	257.1	4.110%	4.692%	7.947%	9.460%
513	4	125.3	5.729%	6.927%	11.00%	14.12%
513	8	60.31	7.942%	10.45%	15.10%	21.68%
513	16	28.82	10.87%	16.18%	20.41%	34.57%
513	32	13.24	14.76%	26.70%	27.33%	60.24%
513	64	5.513	19.99%	51.23%	36.49%	131.0%
513	128	1.817	26.63%	157.2%	49.00%	637.2%
1025	1	682.2	2.586%	2.805%	5.036%	5.604%
1025	2	515.7	2.957%	3.247%	5.748%	6.500%
1025	4	252.5	4.145%	4.737%	8.013%	9.554%
1025	8	122.7	5.784%	7.007%	11.10%	14.29%
1025	16	59.85	7.968%	10.49%	15.15%	21.78%
1025	32	28.72	10.88%	16.22%	20.44%	34.65%
1025	64	13.21	14.77%	26.75%	27.35%	60.35%
1025	128	5.498	20.01%	51.35%	36.52%	131.4%
1025	256	1.807	26.65%	158.4%	49.06%	644.3%

Table 7
Flicker FM Confidence Factors for
MDEV and TDEV
Confidence factors times sample deviations
give magnitudes of
lower and upper confidence intervals
for 68% and 95% probabilities

N	m	edf	lower 68%	upper 68%	lower 95%	upper 95%
5	1	2.606	24.64%	103.1%	44.95%	336.3%
9	1	5.843	19.63%	48.86%	35.85%	123.5%
9	2	2.615	24.62%	102.7%	44.91%	334.5%
17	1	12.33	15.16%	28.04%	28.02%	63.69%
17	2	6.656	18.82%	44.04%	34.43%	108.6%
17	4	2.086	25.90%	132.6%	47.46%	488.9%
33	1	25.29	11.46%	17.54%	21.48%	37.74%
33	2	14.85	14.14%	24.72%	26.22%	55.19%
33	4	5.883	19.58%	48.59%	35.78%	122.6%
33	8	1.814	26.63%	157.6%	49.02%	639.3%
65	1	51.23	8.525%	11.48%	16.16%	23.95%
65	2	31.27	10.51%	15.39%	19.76%	32.76%
65	4	13.61	14.61%	26.21%	27.06%	58.97%
65	8	5.563	19.94%	50.85%	36.39%	129.8%
65	16	1.683	27.00%	173.8%	49.85%	747.6%
129	1	103.1	6.259%	7.716%	11.98%	15.79%
129	2	64.12	7.732%	10.08%	14.71%	20.89%
129	4	29.10	10.82%	16.09%	20.33%	34.35%
129	8	13.14	14.80%	26.84%	27.40%	60.60%
129	16	5.426	20.09%	51.91%	36.67%	133.2%
129	32	1.619	27.18%	183.2%	50.28%	813.7%
257	1	206.9	4.549%	5.272%	8.777%	10.66%
257	2	129.8	5.638%	6.793%	10.83%	13.84%
257	4	60.11	7.953%	10.47%	15.12%	21.72%
257	8	28.37	10.94%	16.34%	20.54%	34.94%
257	16	12.97	14.88%	27.08%	27.53%	61.21%
257	32	5.363	20.17%	52.41%	36.80%	134.8%
257	64	1.587	27.27%	188.3%	50.50%	850.9%
513	1	414.4	3.282%	3.642%	6.369%	7.306%
513	2	261.3	4.079%	4.651%	7.888%	9.377%
513	4	122.1	5.797%	7.026%	11.12%	14.33%
513	8	58.83	8.028%	10.60%	15.26%	22.00%
513	16	28.13	10.98%	16.42%	20.60%	35.14%
513	32	12.90	14.91%	27.18%	27.58%	61.47%
513	64	5.333	20.20%	52.66%	36.86%	135.6%
513	128	1.571	27.32%	190.9%	50.61%	870.7%
1025	1	829.4	2.354%	2.534%	4.589%	5.056%
1025	2	524.1	2.935%	3.219%	5.704%	6.445%
1025	4	246.2	4.195%	4.802%	8.107%	9.687%
1025	8	119.8	5.848%	7.100%	11.22%	14.49%
1025	16	58.46	8.050%	10.63%	15.30%	22.09%
1025	32	28.04	10.99%	16.46%	20.63%	35.21%
1025	64	12.87	14.92%	27.23%	27.60%	61.58%
1025	128	5.318	20.22%	52.78%	36.89%	136.0%
1025	256	1.564	27.34%	192.1%	50.66%	879.7%

Table 8
Random Walk FM Confidence Factors for MDEV and TDEV
Confidence factors times sample deviations give
magnitudes of lower and upper confidence intervals
for 68% and 95% probabilities

N	m	edf	lower 68%	upper 68%	lower 95%	upper 95%
5	1	3	23.80%	89.04%	43.35%	272.9%
9	1	7	18.50%	42.35%	33.88%	103.5%
9	2	2.067	25.95%	134.0%	47.57%	497.2%
17	1	15	14.08%	24.55%	26.13%	54.77%
17	2	5.492	20.02%	51.39%	36.54%	131.5%
17	4	1.603	27.23%	185.7%	50.39%	832.0%
33	1	31	10.55%	15.48%	19.83%	32.95%
33	2	12.41	15.12%	27.91%	27.96%	63.37%
33	4	4.677	21.03%	58.85%	38.33%	156.3%
33	8	1.429	27.73%	218.6%	51.64%	1090.%
65	1	63	7.791%	10.19%	14.82%	21.11%
65	2	26.26	11.29%	17.14%	21.17%	36.80%
65	4	10.98	15.82%	30.40%	29.18%	69.89%
65	8	4.400	21.41%	62.07%	39.01%	167.5%
65	16	1.353	27.94%	237.3%	52.22%	1252.%
129	1	127	5.694%	6.875%	10.93%	14.01%
129	2	53.96	8.336%	11.14%	15.82%	23.19%
129	4	23.61	11.79%	18.31%	22.06%	39.56%
129	8	10.57	16.04%	31.22%	29.56%	72.10%
129	16	4.285	21.58%	63.54%	39.31%	172.7%
129	32	1.318	28.04%	247.0%	52.50%	1341.%
257	1	255	4.126%	4.713%	7.977%	9.503%
257	2	109.4	6.093%	7.466%	11.68%	15.26%
257	4	48.89	8.699%	11.80%	16.48%	24.65%
257	8	22.93	11.93%	18.65%	22.30%	40.36%
257	16	10.42	16.12%	31.54%	29.71%	72.96%
257	32	4.233	21.66%	64.24%	39.45%	175.2%
257	64	1.301	28.09%	252.0%	52.64%	1388.%
513	1	511	2.970%	3.262%	5.773%	6.532%
513	2	220.2	4.419%	5.098%	8.531%	10.30%
513	4	99.44	6.361%	7.872%	12.18%	16.13%
513	8	47.67	8.794%	11.97%	16.66%	25.04%
513	16	22.72	11.97%	18.76%	22.38%	40.62%
513	32	10.36	16.15%	31.67%	29.77%	73.31%
513	64	4.208	21.70%	64.58%	39.51%	176.4%
513	128	1.292	28.11%	254.8%	52.72%	1415.%
1025	1	1023	2.128%	2.273%	4.151%	4.530%
1025	2	441.8	3.183%	3.521%	6.181%	7.060%
1025	4	200.5	4.616%	5.362%	8.903%	10.85%
1025	8	97.16	6.428%	7.975%	12.30%	16.34%
1025	16	47.34	8.820%	12.02%	16.70%	25.14%
1025	32	22.65	11.98%	18.80%	22.41%	40.70%
1025	64	10.33	16.17%	31.74%	29.80%	73.48%
1025	128	4.196	21.71%	64.74%	39.55%	177.0%
1025	256	1.288	28.12%	256.0%	52.75%	1427.%