1995 IEEE INTERNATIONAL FREQUENCY CONTROL SYMPOSIUM

CIRCULAR REPRESENTATION OF INFINITELY EXTENDED SEQUENCES

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ABSTRACT All data in a space-ordered or time-ordered series are always observed for a finite distance or time. A sample variance of the data represents a formalized means of capturing the extent of observed variations over a finite interval. Different types of variances abound and for the existence of a "true" particular variance, assumptions are made regarding ergodicity, stationarity and statistical independence of the random variables. This writing is not about variance *per se* but rather about the fact that proper estimations of variance come from understanding the implications of these basic assumptions. In particular, there is usually an underlying assumption of ergodicity. Ergodicity means that we treat one statistical average as an ensemble of smaller statistical averages. If we use the ergodic assumption within a data set, then we acknowledge its generalization as an included assumption for data outside the set, namely before and after, in the case of a time series. Ergodicity implies that any series will have a likelihood of recurrence which inversely depends on the number of independent observations. If only one series is ever observable, absolute recurrence of that series is the consequence under the implied assumption of ergodicity and an assumption that a "true" variance indeed exists. This yields the model that time-ordered data may be treated as circular or wrapped.

Mathematical formalisms such as a sample variance (or any estimator) are based on assumptions involving what might happen if given infinite time. These formalisms can easily be misinterpreted and even misunderstood. Therefore, I clarify the implications of specific infinite-time variance-related concepts in what I hope is a compact, speculative manner and style.

INTRODUCTION

The circular representation of two-dimensional measurements means that one of the coordinates is wrapped so that the endpoints of the coordinate match. This paper focusses on the notable example of measurements with respect to time. If the measurements derive from physical systems described as having periodicity, circularity, or any of the set of wave-like observations, then we assume that periodicity underlies a time-series measurement. Fourier analysis proceeds from this assumption. Building a model of the observation is then driven by a search for the nature of this periodicity which was assumed. Furthermore we might go further and infinitely extend and circularly represent the finite time series so that the measurement is recurrent for all time again because periodicity is an assumed property of the data set. This is precisely what is done for most frequency-domain representations of a time series. There is a vast literature on Fourier analysis techniques. See for example [Percival and Walden, 1992] for a comprehensive review.

The assumption of periodicity as underlying a time series is often physically correct. For example using Fourier analysis to model the motion of an orbiting satellite, the motion of underwater ocean currents, or the current in a resonant electronic circuit seems plausible. But here is a major problem. What if we are not given a basis for assuming that periodicity is intrinsic to the data? What if we are not at liberty to assume any known deterministic cause whatsoever for that matter? Then we resort to a model that can be used to calculate a probability of a future value lying between two calculable limits which is based on past values. It is formally called a stochastic model [Box and Jenkins, 1970]. In other words, in analyzing a time series as non-deterministic functions, we regard it as a realization of a stochastic process, one supposedly dependent on chance within formalized limits given by past information. I claim however that the interpretation of a stochastic process inadvertently includes a fundamental assumption, namely ergodicity. This paper discusses the appropriateness of an assumption of ergodicity and shows that we can infinitely extend and circularly represent finite periodic processes and, in addition, finite random or stochastic processes under such an assumption.

BASIC CONCEPTS

Science is the trained observation and interpretation of a part of something based on the fundamental principle that everything is contained in or generalized from any part of itself. See for example [Poincaré, 1912]. This assumes that order and consistency are a part of all that we observe. Basically, scientists try to predict events based on a model derived from limited observation. The purpose of models is prediction. For example, every measurement of the phase difference x(t) between two clocks, or its derived frequency difference y(t) has a beginning at, say, t_o and an ending at, say, t_M. Furthermore, the values of x(t) or y(t) are always sampled, preferably equispaced, in intervals of, say, Δt. A useful figure of merit is frequency stability, which is an estimate of the variance of y(t) from this limited view of the data. I say "estimate" in the sense that the true variance requires that we somehow obtain all the data for all time, a physical impossibility requiring infinite time and infinitesimal Δt spacing. If we remove systematic trends so that to the best of our available knowledge any value of x(t) or y(t) is unrelated to any other value, then the residuals of x(t) or y(t) are said to be processes that are random, and their variance remains more or less constant and independent of when and for how long we have made observation. There is, therefore, some expectation that the degree of deviation within x(t) or y(t) has fixed limits. The methodology is depicted in figure 1. I will not pursue the issue that the assumption of continuous randomness is, however, idealistic and that "expecting randomness" is a contradiction of itself.

Practically speaking, all that we can do is obtain a data set from t_o to t_M sampled at Δt intervals. We say that we ignore everything else, but in fact we must be definite about what we mean by and the expectations of the "ignored" data. Granted, the goal of statistical analysis in many cases is to model at least two elements, the deterministic and stochastic processes, so that we can somehow know within finite bounds the "unmeasurable" part of the data (outside our measurement time) without actually measuring it. This is what we do with an accurate model. But besides ignoring the turn-on/turn-off effects at t_o, t_M, and Δt, ignoring the unmeasured or unmeasurable data strictly means in the usual sense that we are assuming that it is noiseless with zero mean as shown in figure 2. With surprising regularity, traditional statistical treatments do not address properly the assumptions of (1) noiseless zero mean up to t_o, (2) the effect of turn-on at t_o, (3) sampling at Δt, (4) the effect of turn-off at t_M, and (5) the effect of noiseless zero-mean after t_M.

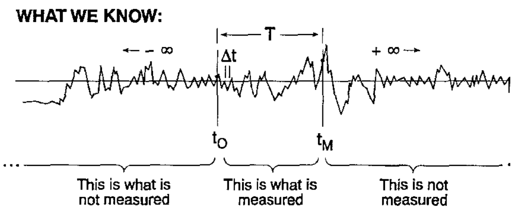

Figure 1

In every sense, "estimating" the variance is akin to estimating what the rest of the unmeasurable data are. So why assume from the outset that the rest of the data are noiseless with zero mean? Obviously this is not a good model. As I will point out, we never rightfully use this model, but worse yet, we erroneously might think we are using it.

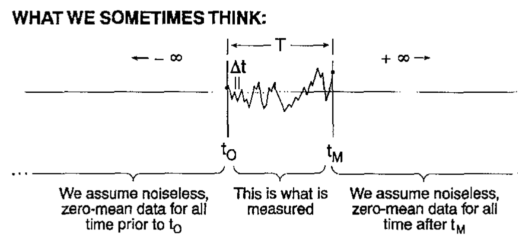

Figure 2

ERGODICITY

We assume that the outcome of flipping a coin ten times is equivalent to flipping ten coins all at once. The property is called "ergodicity." When dealing with a population, this is a reasonable starting point. Applied to a "time series," a stationary random process is defined as "ergodic" if all types of ensemble averages are interchangeable with the corresponding time averages. Thus in "ergodic" processes, the averages across an ensemble are equal to the averages over time of a single function of infinite extent.[1] For example, given the single function y(t), with zero mean, we have

$$\sigma^2 = \lim_{T\to\infty} \frac{1}{T} \int_{-\frac{T}{2}}^{+\frac{T}{2}} y^2(t)\,dt, \qquad (1)$$

where σ^2 is a type of average called the standard variance and the form of eq. (1) yields the variance spectrum. By "spectrum" we mean that y(t) is ordered or arrayed by an independent varying component, in this case "T" which is the time interval from some origin. Its reciprocal (frequency) can be used as the basis for assigning coefficients of a Fourier series to determine the spectral components in y(t) [Bingham, Godfrey, Tukey, 1967].

Let $y(t|n)$ represent the value of the nth sample function drawn from the population y at time t. If y is ergodic, then we have (by definition):

$$\sigma^2 = \lim_{T\to\infty} \frac{1}{T} \int_{-\frac{T}{2}}^{+\frac{T}{2}} [y(t|n)]^2\,dt \qquad (2)$$

for any specific time series n in the population y. This must be equal to the variance at one specific instant t across the ensemble of sample functions (that is, all values of y at one instant t). This variance can be written as

$$\sigma^2 = \lim_{M\to\infty} \frac{1}{M} \sum_{n=1}^{M} [y(t|n)]^2 = \lim_{T\to\infty} \frac{1}{T} \int_{-\frac{T}{2}}^{+\frac{T}{2}} y^2(t)\,dt. \qquad (3)$$

eq (1) and eq (2) say that the true value of the standard variance of the entire data is equivalent to the average of variances of finite length = T as shown in figure 3. One activity observed forever has an equivalence to a finite observation as shown in figure 4.

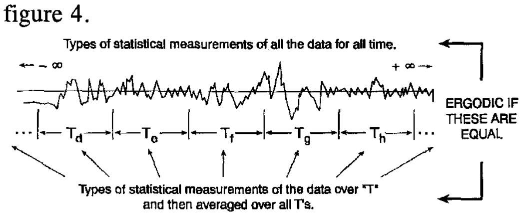

Figure 3

The equality of eq (1) and eq (2) is the ergodic hypothesis when expressed as eq (3). By "equality," I mean that within a statistical uncertainty given by say, a standard deviation over many samples which test the hypothesis, I cannot tell that the expressions are not equal in the limit given.

We say the finite observations are statistically independent if the measurement times are different, that is, if no two measurements over T overlap. Then the time average over one infinite record is equivalent to an ensemble average of finite records of length T. Of course, the concept of a true variance is an idealization since it implies that we have collected data infinitely, a physical impossibility. Interestingly, ergodic theory originated from the opposite view that an ensemble of infinite members can be viewed as one member for all time as shown in figure 4.

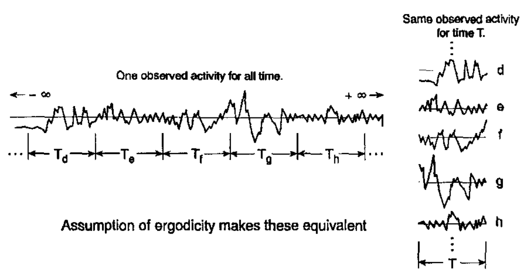

Figure 4

Ergodic theory originated in the mid-1920's from experiments in classical statistical mechanics. At that time, surmising the statistical properties of a single particle was virtually impossible, and experiments involved a large number of particles or members of an ensemble. It was verifiable by experiment that in most cases and under proper conditions, ensemble statistics were approximately equal to time-averaged statistics for a single particle or member. Mathematicians attempted to prove this for arbitrary data rather than for observations in statistical mechanics. Ergodic theory was the subject of widespread mathematical interest for a fairly brief period of time from the 1930's to 1950's. Ergodicity was never proved to be true in all situations but was assumed in many situations. Physicists accepted it based on seemingly sufficient mathematical tractability, and mathematicians accepted it because of experimental results. Judging from events, ergodicity (eq (3)) was eagerly accepted as a way to replace phase statistics by time statistics. By the 1960's, ergodic theory was an accepted model and was not pursued seriously thereafter, and the remaining unproven aspects were left more or less hanging [Halmos, 1956].

The conditions for ergodicity are related to the uniformity in the data and measurement procedures. The only test for ergodicity is that any given statistical average is uniformly the same over any interval throughout a given interval. There are many decidedly nonergodic situations. For example, we cannot combine the time readings of an ensemble of different clocks and view them as one clock unless the readings have identical statistical averages. In any application nonergodicity of one time-series just means that the random numbers concerned are, in fact, an artificial union of a number of distinct and otherwise (ergodic) stationary sequences [Yaglom, 1987]. An important consideration in frequency metrology is whether the power-law of the fractional RMS frequency stability changes with scale (or averaging time τ). If so, the series is clearly non-ergodic.

In practice, we cannot realize the exact limit of eq (3) and must resort to a finite version. This is a source of uncertainty, and, by all rights, eq (3) should be an approximation given as

$$\frac{1}{M}\sum_{n=1}^{M}[y(t|n)]^2 \approx \frac{1}{T}\int_{-\frac{T}{2}}^{+\frac{T}{2}} y^2(t)\,dt \ . \qquad (4)$$

But then questions are raised regarding the degree and conditions of the approximation. Experimentalists working under deadlines eventually say that close enough is good enough. But the assumption of equality in eq (4) has dramatic implications when applied to any data. And in problems involving time-series data, ergodicity has been regularly assumed, perhaps properly or perhaps not. For one important statistical measure, the variance, the assumption of ergodicity applied to any finite observation of ordered deviates is central to many other properties based on the variance.

Our observed time series shown in figure 2 is an ensemble consisting of one measurable member, namely from t_o to t_M. We might think all other members are zero-valued. Any infinite time average of variance of such an ensemble (represented as either eq (1) or eq (2) will naturally approach 0 because we are dividing a finite sample by ∞. But in the case here, I am assuming an infinite sample, which in the limit is supposed to converge to a nonzero value, its so-called true value. Hence our assumption that all other members are zero must not be correct.

What assumption regarding the unobservable events is or should be made when computing any type of nonzero average for a finite record over T (since the infinite average cannot be zero in every case)? The possibilities are that the unobservable events (those outside our observation window) are either different (possibly zero) or the same. Zero as shown is not possible based on our assumption that a nonzero variance must still exist when we extend our observation period to infinite time. If the unobservable events are arbitrarily different, then I argue that there is less and less we can say about the events as we imagine the observation period to be lengthened. And over infinite time, we know nothing. Saying with great mathematical abstraction the degree to which these outside events "behave like" or "are similar to" the observed events is a trap of sorts in which we are denying the obvious; that is, we unknowingly project some expectation of "sameness" onto the unobserved events. At a fundamental level, we are predisposed to thinking that a future possibility (our imagination) is similar to or consistent with the record of the past (our memory). But imagination which is dependent on memory in any way is misleading. That which is imagined becomes nothing more than a goal or expectation.

In short, if the unobservable (future) events are different, there is nothing we can say about them if the events derive from complicated systems. This is the basis of randomness. The assumption then is that we commonly (and inadvertently) assume that the measurements are identical to the one we observe no matter when we observe (a condition of stationarity). This is shown in figure 5 as the same observed measurements from t_o to t_M and is repeated for all time with length T prior to t_o and after t_M. This assumption satisfies the conclusion that any finite, nonzero average is exactly its equivalent infinite average. Of course, any number of different processes can yield an identical, specific average. But it is not our privilege to simply create ad hoc any random process yielding same specific averages. Hence, we conclude that the observation is everywhere recurrent under an ergodic hypothesis given by eq (3) and shown in figure 6.

Without provocation, we accept that conclusions are derived and built from assumptions. I must point out, though, that this is not what I am doing here.

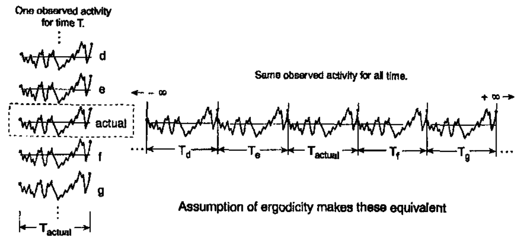

Figure 5

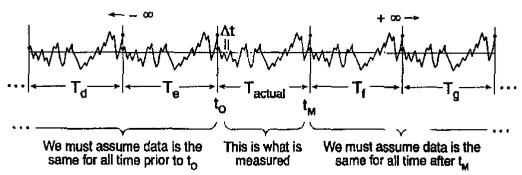

Figure 6

Here I am simply deriving a different view of an assumption from an existing assumption. That the data in an observation window repeat themselves outside the observation window is an assumption derived from the common assumption of ergodicity. Furthermore, one assumption is not so much derived from the other as both are the same assumption. Implicit throughout subjects on data analysis is the pervasive notion of extended periodicity, yet admitting this as applied directly to the observed data seems somehow untenable. Frankly, the notion is very tenable and leads to the practical consequence that we can represent the data as circular as shown in figure 7.

A finite observation period T is always a high-pass filter that is insensitive to any change slower than T. A normal starting point is that the ergodic hypothesis requires stationarity of data. Determining convergence, divergence, or stationarity of data requires that we are at liberty to change T and see the resultant change, if any, in a statistical average. As profound as it seems, however, stationarity, is a condition of our chosen models and not actual data [Barnes, et al., 1971]. The assumption of stationarity is satisfied here by the fixed finite period of observation. The circular representation derives wholly from the ergodic assumption, and stationarity is maintained. Therefore, in the end, it does not matter if we decide that the actual data are stationary or not. The circular model is stationary.

As is always true, it is up to the reader to assign meaning to a representation or model. At least three

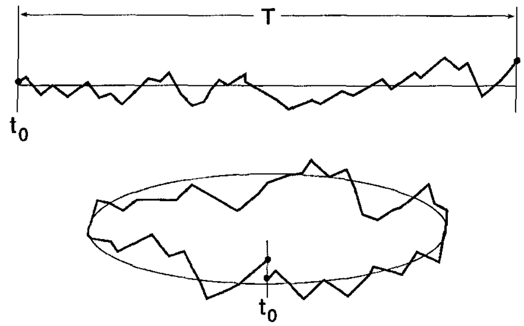

Figure 7

aspects are up for consideration regarding the circular model: (1) Is the model rich enough to include other accepted similar models? (2) Is the model plausible and satisfying in some sense? (3) Is it practical, and how can it work for us? I would like to add a fourth consideration, simplicity. It behooves us to pick simple models.

OTHER MODELS

That we can rightfully assume that a finite time series of random variables recur to an arbitrarily long extent are corroborated by several related, notable examples:

1. Mathematical reasoning proceeds from the particular to the general as inductive reasoning. In the asymptotic limit of inductive reasoning, we proceed from the finite to the infinite and must revert to "reasoning by recurrence"; that is, any finite succession can only be deemed as recurring to an infinite extent [Poincaré, 1912].

2. Any spatial or temporal measures in which the endpoints can be matched without discontinuity are said to be "circular." All analysis can proceed from an assumption of circularity. Endpoint mismatches affect statistical averages only by a scaling factor. See [Bloomfield, 1976] and the discussion on smooth functions.

3. A finite, bounded standing wave can be viewed as counter-propagating travelling waves of arbitrarily long extent. The extended distance to any "virtual" boundary is always a rational multiple of the actual

boundary. See, for example, [Churchill, 1963].

4. For a finite observation period, the method of complex demodulation states that data can be viewed as having in-phase and phase-quadrature components. We must extend the time series by at least half its length to completely account for the phase-quadrature component [Bloomfield, 1976].

5. In computing the power spectrum, commonly used digital processing instruments treat a finite time series as a "block" which is periodic and infinitely extended because all variances still properly converge if given arbitrarily more independent blocks [Welch, 1967].

6. Formulations involving the Fourier transform assume that both discrete continuous functions $\{y_k\}$ and their discrete Fourier transforms $\{Y_k\}$ are circular (periodic) in the sense that evaluations of the functions outside the range $k=1, 2, 3,\ldots,M$ will yield results modulo M [Otnes and Enochson, 1978].

PLAUSIBILITY

At first glance, it appears that we cannot use this model of recurrence because our experience is inconsistent with a hypothesis that activity recurs; that is, if we wait, the same thing happens. So an experience-based model suggests that something different can (in fact, usually does) happen. But it is known in physics and mathematics that acting exclusively and exactly on the appearance of our experience leads to inconsistency in the extreme. For example, pure deductive logic can lead to contradictory results from one set of assumptions and one set of rules [Gödel, 1931]. In fact, it is the principle of causality which is under consideration. Any measurement is not the effect of what precedes it but rather is the effect of a set of conditions. It has been argued that what we see as activity actually remains fixed, and it is only a set of conditions, principally the observer's viewpoint, which changes randomly. Thus he incorrectly perceives that the observation itself is changing. It follows that an inconsistency within a model is not an inconsistency at all but rather perhaps only a different view of the same model. Recurrent or nonrecurrent behavior is within the observer's subjective perception [Hofstadter, 1979]. All the same, specifically, no one can honestly say whether a true variance for a time series exists, yet we take on faith that we are "estimating" it in the following way. Virtually every thesis on the estimation of variance starts with three statements taken as fact: (1) a "true" variance exists, when (2) time is infinitely extended, and (3) we do not know (1) because we cannot realize (2). We argue that we could know (1) if our observation were infinite in time. But what constitutes an infinite observation time? Is it from the big bang (call it t_o) to now (call it t_M)? No, that itself is a finite comprehension. Is every event prior to the moment of the big bang classed as uninteresting or 0 as in figure 2? Then all variances (for that matter, all statistical averages) approach 0, and the concept of a "true" variance is a hoax. Hence, whatever we determine as a variance for any observation period is not an estimate at all but is in fact the true (and only) variance for that observation time, however short or long the time may be.

The only correct answer to the question of the existence of a true variance in the statistically pure sense is to say, "I don't know." Furthermore, if in any way it is implicit that the estimate of a variance for an observation time T is taken to be the true variance (which is a reasonable assertion), then the variables measured cannot be treated as independent of those not measured at any other imagined observation time. In summary, all statistical tools applied to a time series based on the assumption of ergodicity will, taken as a whole, require that the actual finite measurement repeat itself for all times t.

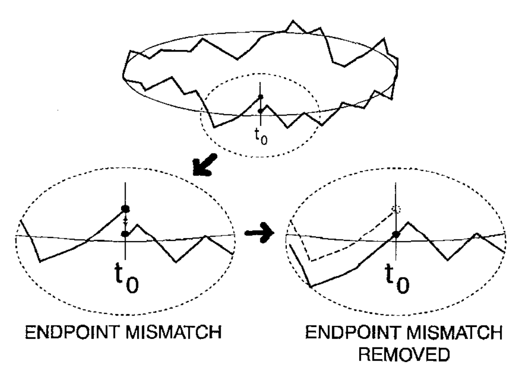

Figure 8

PRACTICALITY

With the circular representation presented here, residual data (that is, data with trends removed) which are wrapped often will have the endpoints miss each other, that is, not match. This is shown in figure 8. The discontinuity of the endpoint mismatch may produce components outside of the physical, system-related, or measurement bandwidth and will create an artifact and a commensurate error in a statistical average such as variance. Furthermore, endpoint mismatch affects overall scaling. More accurate results are obtained if the ends do not miss each other when data is wrapped. For computing statistical averages, we can arrange coordinates so that the ends will match [Howe, to be published].[2]

For any observation period starting at time t_o, continuing with length T, and ending at t_M, the next point after t_M (if our period is extended by an interval Δt) is t_o. In other words, the epoch which marks the end of our observation allows us to predict the next point, the variable at t_o since t_o starts the next repetition. Since this next variable is predictable, it is not a random variable, and the occurrence of the end of the actual period forces a correlation of t_M with t_o The next point is the future, and is outside our window of observation, but at any instant, the next point is surmised to be the one at the beginning time t_o. Furthermore any piece of the total coordinate of time is self-contained and can be treated as isolated from any events outside of that piece. Any observation is treated as circular as shown in figure 7. Now in the extreme case, suppose t_o marks the moment of creation or the big bang and duration T extends to now. Therefore, t_M marks the present (now). The end then becomes the beginning. This is not to say that all events repeat themselves nor that periodicity exists everywhere. Quite the contrary. Random events are not redundant and repeated; each event is unique and need not have any determinable cause (satisfying the observer, at any rate). That periodicity exists everywhere (including outside our observation window) would be a man-made assumption.[3] Let me emphatically state that the conclusion that the end and beginning are one derives from a reasonable assumption that all of the time we can possibly comprehend is still a finite observation time. This model of time correlates "now" with our chosen time origin, or if you wish, the so-called "beginning of time." This concept suitably derives from the question, "Do unobservable events exist?" For the purpose here, I assume they do not, since there can be no cognizance of such events. This is opposed to many physicists' views on realism and takes the stand that a thing does not exist for the observer before his measurement of it. I believe that the observer, however, does not give it its existence, so the future is not given any attribute prior to its observation. Needless to say, there is a great deal left for further philosophical discussion.[4] However, if we accept the hypothesis of ergodicity and accept that a true variance indeed exists, then we are at liberty to treat finite time-series data $\{x_k\}$ as circular. We can wrap the data so that $x_{M+1} = x_1$, and this is a practical tool. Any resultant mismatch of x_1 and x_M produces undesirable effects and ought to be treated as a separate element in our overall model of the observed process being measured.

Is there a coordinate system or reference frame in which time-ordered events can be viewed as nonrepeating or noncircular? This can be answered with an analogous geometrical question regarding space-ordering: Is there a reference frame in which a line (of infinite length and consisting of an infinite number of points) can be viewed on its end, so to speak, as a single point? No. We assume that a line cannot be viewed on end. If it is nowhere viewable as anything except a line, Einstein suggested that it comes back on itself. Every end of the line is also its beginning. We define absolute straightness as a minimum distance between two points in space. We are predisposed to thinking that the line cannot curve around; it is straight, and infinitely long. If space is not curved, how can this be? If space is curved, then how and to what extent is it curved so that everything comes back on itself (since the line becomes a geodesic of unknown dimension)? Perhaps it is not describable by finite human comprehension except to say that infinite space is somehow circular in that it comes back on itself, but any piece is somehow straight by every conceivable test. And all time vs. finite periods may be similarly viewed.[5]

CONCLUSION

Predicting time-dependent measures means predicting a future which is always outside qof our observation window. Observables have varying degrees of reproducibility, hence predictable reoccurrence. There are accurate models for fairly simple physical realizations. Randomness, however, is associated with nondeterministic, complex systems in which simple trends have been modeled and removed. This paper has described the appropriateness of wrapping a finite time-dependent measurement as a tool for the determination of statistical averages such as the variance of non-deterministic processes. It behooves us to pick uncomplicated models. Conditions which satisfy an assumption of ergodicity yield the very simple model that time-ordered data may be treated as circular or wrapped. An endpoint mismatch must be removed because it affects statistical averages by a transient and by a scaling factor which can be a dominant source of errors. Future analysis will include the circular representation (wrapping the data) if appropriate. A remaining topic is whether the ergodic assumption is reliable for arbitrary constant power-law processes such as the kinds used to characterize frequency stability.

ACKNOWLEDGEMENTS

I gratefully acknowledge review and insightful discussion of this manuscript by J. A. Barnes, J. L. Jespersen, J. Levine, D. B. Percival, and G. M. R. Winkler.

FOOTNOTES

[1]The general case of nth-order function yields [Panter, 1965]:

$$\tilde{x}^n = \int_{-\infty}^{\infty} x^n p_n(x,t)dt$$

which is equal to the time average of x^n ($p_n(x,t)$ is the probability density of x^n and is assumed to be uniform-random and normalized to T):

$$\bar{x}^n = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T}[x(t)]^n dt$$

so that $\tilde{x}^n = \bar{x}^n$.

[2]It is thoroughly recognized that the start and stop of the observation window can adversely affect the outcome of particular analyses. For example, an assumption that periodicity is within some set of data might be physically correct. Therefore, Fourier analysis is suitable. Smoothly tapering the ends of a series to zero is often done to reduce errors at the expense of analysis resolution in the periodic extension of the observation, a condition brought about in the implementation of the Fourier transform integral.

[3]Joseph Fourier in 1822 rigorously treated the suggestion of Pythagoras, Kepler, Galileo, and Newton that wave or cyclical motion was everywhere in nature. So far it holds that all observable activity can be broken down into wave-like building blocks. Many types of analysis proceed from this assumption.

[4]Certainly we must reconcile whether the existence of anything depends exclusively on its finite observation.

[5]A long-standing view is held that space and time are sensible geometries for all human experience, but that they are different forms, one for the outer sense (space) and one for the inner (time) [Kant, 1781].

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