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CHARACTERIZATION OF FREQUENCY AND PHASE NOISE *

(Study Programme 3B/7)

(1974-1978-1986)

1. Introduction

Techniques to characterize and to measure the frequency and phase instabilities in frequency generators and received radio signals are of fundamental importance to users of frequency and time standards.

In 1964 a subcommittee on frequency stability was formed, within the Institute of Electrical and Electronic Engineers (IEEE) Standards Committee 14 and later (in 1966) in the Technical Committee on Frequency and Time within the Society of Instrumentation and Measurement (SIM), to prepare an IEEE standard on frequency stability. In 1969, this subcommittee completed a document proposing definitions for measures on frequency and phase stabilities. These recommended measures of stabilities in frequency generators have gained general acceptance among frequency and time users throughout the world. Some of the major manufacturers now specify stability characteristics of their standards in terms of these recommended measures.

Models of the instabilities may include both stationary and non-stationary random processes as well as systematic processes. Concerning the apparently random processes, considerable progress has been made [IEEE-NASA, 1964; IEEE, 1972] in characterizing these processes with reasonable statistical models. In contrast, the presence of systematic changes of frequencies such as drifts should not be modelled statistically, but should be described in some reasonable analytic way as measured with respect to an adequate reference standard, e.g., linear regression to determine a model for linear frequency drift. The separation between systematic and random parts however is not always easy or obvious. The systematic effects generally become predominant in the long term, and thus it is extremely important to specify them in order to give a full characterization of a signal's stability. This Report presents some methods of characterizing the random processes and some important types of systematic processes.

Since then, additional significant work has been accomplished. For example, Baugh [1971] illustrated the properties of the Hadamard variance – a time-domain method of estimating discrete frequency modulation sidebands – particularly appropriate for Fourier frequencies less than about 10 Hz; a mathematical analysis of this technique has been made by Sauvage and Rutman [1973]; Rutman [1972] has suggested some alternative time-domain measures while still giving general support to the subcommittee's recommendations; De Prins *et al.* [1969] and De Prins and Cornelissen, [1971] have proposed alternatives for the measure of frequency stability in the frequency domain with specific emphasis on sample averages of discrete spectra. A National Bureau of Standards Monograph devotes Chapter 8 to the "Statistics of time and frequency data analysis" [Blair, 1974]. This chapter contains some measurement methods, and applications of both frequency-domain and time-domain measures of frequency/phase instabilities. It also describes methods of conversion among various time-domain measures of frequency stability, as well as conversion relationships from frequency-domain measures to time-domain measures and vice versa. The effect of a finite number of measurements on the accuracy with which the two-sample variance is determined has been specified [Lesage and Audoin, 1973, 1974 and 1976; Yoshimura, 1978]. Box-Jenkins-type models have been applied for the interpretation of frequency stability measurements [Barnes, 1976; Percival, 1976] and reviewed by Winkler [1976].

Lindsey and Chie [1976] have generalized the r.m.s. fractional frequency deviation and the two-sample variance in the sense of providing a larger class of time-domain oscillator stability measures. They have developed measures which characterize the random time-domain phase stability and the frequency stability of an oscillator's signal by the use of Kolmogorov structure functions. These measures are connected to the frequency-domain stability measure $S_y(f)$ via the Mellin transform. In this theory, polynominal type drifts are included and some theoretical convergence problems due to power-law type spectra are alleviated. They also show the close relationship of these measures to the r.m.s. fractional deviation [Cutler and Searle, 1966] and to the two-sample variance [Allan, 1966]. And finally, they show that other members from the set of stability measures developed are important in specifying performance and writing system specifications for applications such as radar, communications, and tracking system engineering work.

Other forms of limited sample variances have been discussed [Baugh, 1971; Lesage and Audoin, 1975; Boileau and Picinbono, 1976] and a review of the classical and new approaches has been published [Rutman, 1978].

* See Appendix Note # 23

Frequency and phase instabilities may be characterized by random processes that can be represented statistically in either the Fourier frequency domain or in the time domain [Blackman and Tukey, 1959]. The instantaneous, normalized frequency departure $y(t)$ from the nominal frequency ν_0 is related to the instantaneous-phase fluctuation $\varphi(t)$ about the nominal phase $2\pi\nu_0 t$ by:

$$y(t) = \frac{1}{2\pi\nu_0}\frac{d\varphi(t)}{dt} = \frac{\dot{\varphi}(t)}{2\pi\nu_0} \tag{1}$$

$$x(t) = \frac{\varphi(t)}{2\pi\nu_0}$$

where $x(t)$ is the phase variation expressed in units of time.

2. Fourier frequency domain

In the Fourier frequency domain, frequency stability may be defined by several one-sided (the Fourier frequency ranges from 0 to ∞) spectral densities such as:

$$S_y(f) \text{ of } y(t),\ S_\varphi(f) \text{ of } \varphi(t),\ S_{\dot{\varphi}}(f) \text{ of } \dot{\varphi}(t),\ S_x(f) \text{ of } x(t), \text{ etc.}$$

These spectral densities are related by the equations:

$$S_y(f) = \frac{f^2}{\nu_0^2}\, S_\varphi(f) \tag{2}$$

$$S_{\dot{\varphi}}(f) = 4\pi^2 f^2\, S_\varphi(f) \tag{3}$$

$$S_x(f) = \frac{1}{(2\pi\nu_0)^2}\, S_\varphi(f) \tag{4}$$

Power-law spectral densities are often employed as reasonable models of the random fluctuations in precision oscillators. In practice, it has been recognized that these random fluctuations are the sum of five independent noise processes and hence:

$$S_y(f) = \begin{cases} \sum\limits_{\alpha=-2}^{+2} h_\alpha f^\alpha & \text{for } 0 < f < f_h \\ 0 & \text{for } f > f_h \end{cases} \tag{5}$$

where h_α's are constants, α's are integers, and f_h is the high frequency cut-off of a low pass filter. Equations (2), (3) and (4) are correct and consistent for stationary noises including phase noise. High frequency divergence is eliminated by the restrictions on f in equation (5). The identification and characterization of the five noise processes are given in Table I, and shown in Fig. 1. In practice, only two or three noise processes are sufficient to describe the random frequency fluctuations in a specific oscillator; the others may be neglected.

3. Time-domain

Random frequency instability in the time-domain may be defined by several sample variances. The recommended measure is the two-sample standard deviation which is the square root of the two-sample zero dead-time variance $\sigma_y^2(\tau)$ [von Neumann *et al.*, 1941; Allan, 1966; Barnes *et al.*, 1971] defined as:

$$\sigma_y^2(\tau) = \left\langle \frac{(\bar{y}_{k+1} - \bar{y}_k)^2}{2} \right\rangle \tag{6}$$

where

$$\bar{y}_k = \frac{1}{\tau}\int_{t_k}^{t_k+\tau} y(t)\,dt = \frac{x_{k+1} - x_k}{\tau} \quad \text{and} \quad t_{k+1} = t_k + \tau \quad \text{(adjacent samples)}$$

< > denotes an infinite time average. The x_k and x_{k+1} are time residual measurements made at t_k and $t_{k+1} = t_k + \tau$, $k = 0, 1, 2, \ldots$, and $1/\tau$ is the fixed sampling rate which gives zero dead time between frequency measurements. By "residual" it is understood that the known systematic effects have been removed.

If the initial sampling rate is specified as $1/\tau_0$, then it has been shown [Howe *et al.*, 1981] that in general one may obtain a more efficient estimate of $\sigma_y(\tau)$ using what is called "overlapping estimates". This estimate is obtained by computing equation (7).

$$\sigma_y^2(\tau) = \frac{1}{2(N-2m)\,\tau^2} \sum_{i=1}^{N-2m} (x_{i+2m} - 2x_{i+m} + x_i)^2 \tag{7}$$

where N is the number of original time departure measurements spaced by τ_0, ($N = M + 1$, where M is the number of original frequency measurements of sample time, τ_0) and $\tau = m\tau_0$. The corresponding confidence intervals [Howe *et al.*, 1981], discussed in § 6, are smaller than those obtained by using equation (12), and the estimate is still unbiased.

If dead time exists between the frequency departure measurements and this is ignored in computing equation (6), it has been shown that the resulting stability values (which are no longer the Allan variances), will be biased (except for the white frequency noise) as the frequency measurements are regrouped to estimate the stability for $m\tau_0$ ($m > 1$). This bias has been studied and some tables for its correction published [Barnes, 1969; Lesage, 1983].

A plot of $\sigma_y(\tau)$ versus τ for a frequency standard typically shows a behaviour consisting of elements as shown in Fig. 1. The first part, with $\sigma_y(\tau) \sim \tau^{-1/2}$ (white frequency noise) and/or $\sigma_y(\tau) \sim \tau^{-1}$ (white or flicker phase noise) reflects the fundamental noise properties of the standard. In the case where $\sigma_y(\tau) \sim \tau^{-1}$, it is not practical to decide whether the oscillator is perturbed by white phase noise or by flicker phase noise. Alternative techniques are suggested below. This is a limitation of the usefulness of $\sigma_y(\tau)$ when one wishes to study the nature of the existing noise sources in the oscillator. A frequency-domain analysis is typically more adequate for Fourier frequencies greater than about 1 Hz. This τ^{-1} and/or $\tau^{-1/2}$ law continues with increasing averaging time until the so-called flicker "floor" is reached, where $\sigma_y(\tau)$ is independent of the averaging time τ. This behaviour is found in almost all frequency standards; it depends on the particular frequency standard and is not fully understood in its physical basis. Examples of probable causes for the flicker "floor" are power supply voltage fluctuations, magnetic field fluctuations, changes in components of the standard, and microwave power changes. Finally the curve shows a deterioration of the stability with increasing averaging time. This occurs typically at times ranging from hours to days, depending on the particular kind of standard.

A "modified Allan variance", $MOD\,\sigma_y^2(\tau)$, has been developed [Allan and Barnes, 1981] which has the property of yielding different dependences on τ for white phase noise and flicker phase noise. The dependences for $MOD\,\sigma_y(\tau)$ are $\tau^{-3/2}$ and τ^{-1} respectively. The relationships between $\sigma_y(\tau)$ and $MOD\,\sigma_y(\tau)$ are also explained in [Allan and Barnes, 1981; IEEE 1983, Lesage and Ayi, 1984]. $MOD\,\sigma_y(\tau)$ is estimated using the following equation:

$$MOD\,\sigma_y^2(\tau) = \frac{1}{2\tau^2 m^2 (N - 3m + 1)} \sum_{j=1}^{N-3m+1} \left[\sum_{i=j}^{m+j-1} (x_{i+2m} - 2x_{i+m} + x_i) \right]^2 \tag{8}$$

where N is the original number of time measurements spaced by τ_0, and $\tau = m\tau_0$ the sample time of choice. Properties and confidence of the estimate are discussed in Lesage and Ayi [1984]. Jones and Tryon [1983] and Barnes *et al.* [1982] have developed maximum likelihood methods of estimating $\sigma_y(\tau)$ for the specific models of white frequency noise and random walk frequency noise, which has been shown to be a good model for observation times longer than a few seconds for caesium beam standards.

4. Conversion between frequency and time domains

In general, if the spectral density of the normalized frequency fluctuations $S_y(f)$ is known, the two-sample variance can be computed [Barnes *et al.*, 1971; Rutman, 1972]:

$$\sigma_y{}^2(\tau) = 2 \int_0^{f_h} S_y(f) \frac{\sin^4 \pi\tau f}{(\pi\tau f)^2}\, \mathrm{d}f \tag{9}$$

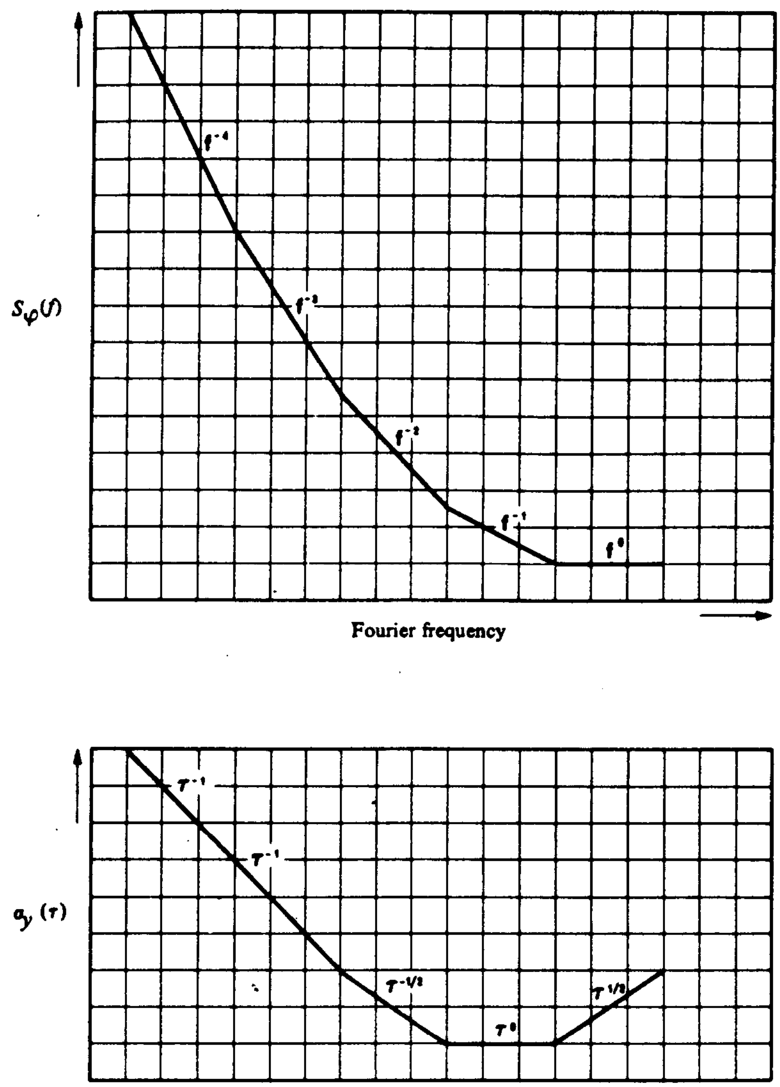

FIGURE 1 – *Slope characteristics of the five independent noise processes*

(log scale)

Specifically, for the power law model given by equation (5), the time-domain measure also follows the power law as derived by Cutler from equations (5) and (9).

$$\sigma_y^2(\tau) = h_{-2}\,\frac{(2\pi)^2}{6}\,\tau + h_{-1}\,2\log_e 2 + h_0\,\frac{1}{2\tau} + h_1\,\frac{1.038 + 3\log_e(2\pi f_h\tau)}{(2\pi)^2\,\tau^2} + h_2\,\frac{3f_h}{(2\pi)^2\,\tau^2} \qquad (10)$$

Note. – The factor 1.038 in the fourth term of equation (10) is different from the value given in most previous publications.

The values of h_α are characteristics of oscillator frequency noise. One may note for integer values (as often seems to be the case) that $\mu = -\alpha - 1$, for $-3 \lesssim \alpha \lesssim 1$, and $\mu \simeq -2$ for $\alpha \gtrsim 1$ where $\sigma_y^2(\tau) \sim \tau^\mu$.

These conversions have been verified experimentally [Brandenberger *et al.*, 1971] and by computation [Chi, 1977]. Table II gives the coefficients of the translation among the frequency stability measures from time domain to frequency domain and from frequency domain to time domain.

The slope characteristics of the five independent noise processes are plotted in the frequency and time domains in Fig. 1 (log log scale).

5. Measurement techniques

The spectral density of phase fluctuations $S_\varphi(f)$ may be approximately measured using a phase-locked loop and a low frequency wave analyzer [Meyer, 1970; Walls *et al.*, 1976]. A double-balanced mixer is used as the phase detector in a lightly coupled phase lock loop. The measuring system uses available state-of-the-art electronic components; also a very high quality oscillator is used as the reference. For very low Fourier frequencies (well below 1 Hz), digital techniques have been used [Atkinson *et al.*, 1963; De Prins *et al.*, 1969; Babitch and Oliverio, 1974]. New methods of measuring time (phase) and frequency stabilities have been introduced with picosecond time precision [Allan and Daams, 1975], and of measuring the Fourier frequencies of phase noise with 30 dB more sensitivity than the previous state of the art [Walls *et al.*, 1976].

Several measurement systems using frequency counters have been used to determine time-domain stability with or without measurement dead time [Allan, 1974; Allan and Daams, 1975]. A system without any counter has also been developed [Rutman, 1974; Rutman and Sauvage, 1974]. Frequency measurements without dead time can be made by sampling time intervals instead of measuring frequency directly. Problems encountered when dead time exists between adjacent frequency measurements have also been discussed and solutions recommended [Blair, 1974; Allan and Daams, 1975; Ricci and Peregrino, 1976]. Discrete spectra have been measured by Groslambert *et al.* [1974].

6. Confidence limits of time domain measurements

A method of data acquisition is to measure time variations x_j at intervals τ_0. Then $\sigma_y(\tau)$ can be estimated for any $\tau = n\tau_0$ (n is any positive integer) since one may use those x_j values for which j is equal to nk. An estimate for $\sigma_y(\tau)$ can be made from a data set with M measurements of $\bar{y}_j$ as follows:

$$\hat{\sigma}_y(n\tau_0) = \hat{\sigma}_y(\tau) \simeq \left| \frac{1}{2(M-1)} \sum_{j=1}^{M-1} (\bar{y}_{j+1} - \bar{y}_j)^2 \right|^{1/2} \tag{11}$$

or equivalent

$$\hat{\sigma}_y(\tau) \cong \left| \frac{1}{2\tau^2(M-1)} \sum_{j=1}^{M-1} (x_{j+2} - 2x_{j+1} + x_j)^2 \right|^{1/2} \tag{12}$$

Thus, one can ascertain the dependence of $\sigma_y(\tau)$ as a function of τ from a single data set in a very simple way. For a given data set, M of course decreases as n increases.

To estimate the confidence interval or error bar for a Gaussian type of noise of a particular value $\sigma_y(\tau)$ obtained from a finite number of samples [Lesage and Audoin, 1973] have shown that:

$$\text{Confidence Interval } I_\alpha \simeq \sigma_y(\tau) \cdot \kappa_\alpha \cdot M^{-1/2} \text{ for } M > 10 \tag{13}$$

where:

M: total number of data points used in the estimate,

α: as defined in the previous section,

$\kappa_2 = \kappa_1 = 0.99$,

$\kappa_0 = 0.87$,

$\kappa_{-1} = 0.77$,

$\kappa_{-2} = 0.75$.

As an example of the Gaussian model with $M = 100$, $\alpha = -1$ (flicker frequency noise) and $\sigma_y(\tau = 1 \text{ second}) = 10^{-12}$, one may write:

$$I_\alpha \simeq \sigma_y(\tau) \cdot \kappa_\alpha \cdot M^{-1/2} = \sigma_y(\tau) \cdot (0.77) \cdot (100)^{-1/2} = \sigma_y(\tau) \cdot (0.077), \tag{14}$$

which gives:

$$\sigma_y(\tau = 1 \text{ second}) = (1 \pm 0.08) \times 10^{-12} \tag{15}$$

A modified estimation procedure including dead-time between pairs of measurements has also been developed [Yoshimura, 1978], showing the influence of frequency fluctuations auto-correlation.

7. Conclusion

The statistical methods for describing frequency and phase instability and the corresponding power law spectral density model described are sufficient for describing oscillator instability on the short term. Equation (9) shows that the spectral density can be unambiguously transformed into the time-domain measure. The converse is not true in all cases but is true for the power law spectra often used to model precision oscillators.

Non-random variations are not covered by the model described. These can be either periodic or monotonic. Periodic variations are to be analyzed by means of known methods of harmonic analysis. Monotonic variations are described by linear or higher order drift terms.

TABLE I – *The functional characteristics of five independent noise processes for frequency instability of oscillators*

Description of noise process	Slope characteristics of log log plot			
	Frequency-domaine		Time-domaine	
	$S_y(f)$	$S_\varphi(f)$ or $S_x(f)$	$\sigma^2(\tau)$	$\sigma(\tau)$
	α	$\beta = \alpha - 2$	μ	$\mu/2$
Random walk frequency	−2	−4	1	½
Flicker frequency	−1	−3	0	0
White frequency	0	−2	−1	−½
Flicker phase	1	−1	−2	−1
White phase	2	0	−2	−1

$S_y(f) = h_\alpha f^\alpha$ $\qquad \sigma^2(\tau) \sim |\tau|^\mu$

$S_\varphi(f) = \nu_0^2 h_\alpha f^{\alpha-2} = \nu_0^2 h_\alpha f^\beta \quad (\beta = \alpha - 2)$ $\qquad \sigma(\tau) \sim |\tau|^{\mu/2}$

$S_x(f) = \frac{1}{4\pi^2} h_\alpha f^{\alpha-2} = \frac{1}{4\pi^2} h_\alpha f^\beta$

TABLE II – *Translation of frequency stability measures from spectral densities in frequency domain to variance in time domain and vice versa (for $2\pi f_h \tau \gg 1$)*

Description of noise process	$\sigma_y^2(\tau) =$	$S_y(f) =$	$S_\varphi(f) =$
Random walk frequency	$A\left[f^2 S_y(f)\right]\tau^1$	$\frac{1}{A}\left[\tau^{-1}\sigma_y^2(\tau)\right]f^{-2}$	$\frac{\nu_0^2}{A}\left[\tau^{-1}\sigma_y^2(\tau)\right]f^{-4}$
Flicker frequency	$B\left[f S_y(f)\right]\tau^0$	$\frac{1}{B}\left[\tau^0\sigma_y^2(\tau)\right]f^{-1}$	$\frac{\nu_0^2}{B}\left[\tau^0\sigma_y^2(\tau)\right]f^{-3}$
White frequency	$C\left[f^0 S_y(f)\right]\tau^{-1}$	$\frac{1}{C}\left[\tau^1\sigma_y^2(\tau)\right]f^0$	$\frac{\nu_0^2}{C}\left[\tau^1\sigma_y^2(\tau)\right]f^{-2}$
Flicker phase	$D\left[f^{-1} S_y(f)\right]\tau^{-2}$	$\frac{1}{D}\left[\tau^2\sigma_y^2(\tau)\right]f^1$	$\frac{\nu_0^2}{D}\left[\tau^2\sigma_y^2(\tau)\right]f^{-1}$
White phase	$E\left[f^{-2} S_y(f)\right]\tau^{-2}$	$\frac{1}{E}\left[\tau^2\sigma_y^2(\tau)\right]f^2$	$\frac{\nu_0^2}{E}\left[\tau^2\sigma_y^2(\tau)\right]f^0$

$$A = \frac{4\pi^2}{6} \qquad D = \frac{1.038 + 3\log_e(2\pi f_h\tau)}{4\pi^2}$$

$$B = 2\log_e 2 \qquad E = \frac{3f_h}{4\pi^2}$$

$$C = 1/2$$

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