

CHARACTERIZATION OF CLOCKS AND OSCILLATORS

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This is a collection of published papers assembled as a reference for those involved in characterizing and specifying high-performance clocks and oscillators. It is an interim replacement for NBS Monograph 140, *Time and Frequency: Theory and Fundamentals*, an older volume of papers edited by Byron E. Blair. This current volume includes tutorial papers, papers on standards and definitions, and a collection of papers detailing specific measurement and analysis techniques. The discussion in the introduction to the volume provides a guide to the content of the papers, and tables and graphs provide further help in organizing methods described in the papers.

Key words: Allan variance, clocks, frequency, oscillators, phase noise, spectral density, time, two-sample variance.

A. INTRODUCTION

A.1 OVERVIEW

The papers in this volume are organized into three groups: Introductory and Tutorial Papers, Papers on Definitions and Standards, and Supporting Papers. The three sections (A.2, A.3, and A.4) immediately following this introduction provide overviews of each of the three groups of papers with comments on each paper. The arrangement of the papers in this particular order is somewhat arbitrary, since, for example, the first two papers under Supporting Papers could be included with the Introductory and Tutorial Papers, while paper B.4 could easily be placed with the Supporting Papers. Our rationale for the first group of papers (discussed in more detail in section A.2) is that, taken as a group, they provide reasonably complete coverage of the concepts used in characterizing clocks and oscillators. Several of the papers, taken individually, are good introductory papers, but, for this publication, need to be complemented with additional material to provide coverage of an appropriate range of topics.

The second group (section C) of three papers discussed in section A.3 were specifically written to address definitions and standards. This is a particularly important section, since consistency in specification of performance can only be achieved if manufacturers and users refer to the same measurement and characterization parameters.

The Supporting Papers in section D provide additional discussion of topics introduced in the first group. The first papers in this group (D.1 and D.2) also provide good introductory material which might be used with section B to gain a better understanding of the concepts.

Section A.5 provides a table and graph designed to help the reader select a measurement method to meet a particular need. To make this useful, it was kept simple and must therefore be used with care. Such tabular information can never be arranged well enough to anticipate all

of the wide range of measurement situations which might be encountered. However, it can serve as a starting point for the decision-making process.

Section A.6 contains some new material which should be helpful in understanding the relationship between the Allan variance and the modified Allan variance. Since these ideas are unpublished, we include them here rather than with the papers on those topics. This section is followed by a reading list with references to major articles and books which can be used as supplementary resources. Because some of the papers include extensive reference lists, we have limited our list to works which are either very comprehensive or only recently published. Particularly extensive reference lists are included with papers B.1, B.2, C.1, C.3, D.1, and D.2.

Since notation and definitions have changed over the period bridged by these papers, we have highlighted problem areas on the papers with an asterisk (*). A note directs the reader to the Appendix where the particular problem is discussed. We have also used this device to highlight inconsistencies and the usual typographic and other errors which creep into the literature. The page numbers of the original publications are retained, but we have also used a continuous page numbering to simplify location of items in the volume.

The topical index on page xi organizes much of the material in the papers under a few key subject headings. This index provides a shortcut to locating material on a particular topic.

A.2 COMMENTS ON INTRODUCTORY AND TUTORIAL PAPERS

Paper B.1 in this section, by Howe, Allan, and Barnes, was originally prepared and presented as a tutorial paper and has been used with success as an introductory paper in our annual Time and Frequency Seminar. This paper is now 9 years old, so there are a substantial number of notes which relate to updates in notation. The paper is nevertheless highly readable and introduces many of the key measurement methods, providing circuit diagrams with enough specific detail to be useful in real laboratory situations. Furthermore, it includes discussion and examples on handling of data which are useful for practical application of the concepts. The paper presents a particularly useful discussion of the pitfalls encountered in digitizing data, a problem which is often overlooked.

The second paper (B.2) by Stein is more advanced and those familiar with the general concepts may find it a better starting point. This and other papers in this collection cite earlier IEEE recommendations on measures of frequency stability and, while much of this has not changed, there is a new IEEE standard (paper C.1). In general, the reader should consult the overview and papers of section C if there is any question concerning definitions or terminology. Paper B.2 is quite comprehensive, introducing topics (not covered in paper B.1) such as the modified Allan variance, the delay-line-phase-noise-measurement system, and the use of frequency synthesis to reach frequencies far from normally available reference frequencies.

The materials in papers B.1 and B.2, aside from differences in level of presentation, are organized in quite different ways. The Howe-Allan-Barnes paper goes directly to the measurement concepts and then describes the means for analyzing the output data and understanding the confidence of the measurements. On the other hand, the Stein paper carefully lays out the theoretical background needed to analyze the data before introducing the measurement concepts. Both papers cover time-domain and frequency-domain measurements.

Paper B.3 by Allan reviews the concepts of the two-sample or Allan variance and the modified Allan variance showing how classical statistical methods fail to usefully describe the time-domain performance of good oscillators. The Allan variance concept is also introduced in

papers B.1 and B.2, and the modified Allan variance is described further in papers D.4 and D.5. The presentation in paper B.3 is particularly useful in that it discusses general aspects of performance of different types of oscillators (quartz, rubidium, hydrogen, and cesium) providing the basis for prediction of time errors, a topic which may prove useful to those who must develop system specifications.

Paper B.4 by Walls, Clements, Felton, Lombardi, and Vanek adds to the discussion of frequency-domain measurements providing information on methods which can be used to increase the dynamic range for both carrier frequency and for Fourier frequencies up to 10 percent from the carrier. With some aerospace hardware now carrying phase-noise specifications, this is an important addition to the literature.

A.3 COMMENTS ON PAPERS ON STANDARDS AND DEFINITIONS

The first paper in this group (paper C.1) outlines the standard terminology now used for fundamental frequency and time metrology. This document was widely circulated for comment during the draft stage and, with its acceptance by IEEE as a standard, supersedes the earlier reference (paper C.2) which had served as the foundation for characterization of frequency stability. This latter paper is included because it is so widely cited, and the reader will probably be confronted with specifications based on its recommendations. Paper C.2 contains additional material on applications of stability measures and measurement techniques including a useful discussion of some of the common hazards in measurements. Paper C.1 restricts itself to very concise statements of the definitions.

The reader will note that the updated terminology in the first paper (C.1) varies in a number of minor ways from the earlier paper (C.2). A notable addition to definitions is the introduction of script "ell", $\ell(f)$, which has become an important measure of phase noise. This quantity was previously defined as the ratio of the power in one sideband, due to phase modulation to the total signal power. For Fourier frequencies far from the carrier, this quantity can be simply related to the usual spectral densities which are the quantities that are generally measured, but the relation breaks down in the important region near the carrier. To resolve this problem, the new standard defines the approximate relation between $\ell(f)$ and spectral density as being exact and applicable for any Fourier frequency.

The third paper in this group (paper C.3), from the 1986 report of the International Radio Consultative Committee (CCIR), presents the definitions and terminology which have been accepted for international use by this body. The material in this particularly readable document is fairly consistent with the IEEE standard and would be useful to those involved in specification of performance for international trade. A number of minor changes to this document have been recommended by different delegations to the CCIR and these will likely be made in their next publication.

A.4 COMMENTS ON SUPPORTING PAPERS

Papers D.1 and D.2 are included in this collection for a number of reasons. First, they provide alternative introductions to the general topic of oscillator characterization. And second, they include material not fully covered by introductory papers, B.1 and B.2.

The first of these (D.1) by Lesage and Audoin covers most of the same ground as papers B.1 and B.2, but, in addition, includes a nice discussion of characterization of frequency stability via filtering of phase or frequency noise, a method which may be especially useful for rapid, automated measurements where high accuracy is not required. Furthermore, this paper discusses characterization of stable laser sources, a topic not covered in any of the other reference papers.

Paper D.2 by Lance, Seal, and Labaar limits itself to discussion of the measurement of phase noise and amplitude-modulation (AM) noise. The paper presents a detailed discussion of delay-line measurement methods which can be used if a second reference oscillator is not available. The delay-line concept is also introduced in B.2 and B.4, but in much less detail. While the delay-line method is less sensitive (for lower Fourier frequency) than two-oscillator methods, it is easier to implement. The paper contains many good examples which the reader will find useful.



A complete discussion of AM noise is beyond the scope of this volume. AM noise is usually ignored in the measurement and specification of phase noise in sources under the assumption that the AM noise is always less than the phase noise. This assumption is generally true only for Fourier frequencies close to the carrier. At larger Fourier frequencies the normalized AM noise can be the same order of magnitude as the phase noise. In systems with active amplitude leveling, the normalized AM noise can be higher than the phase noise. Under this condition the AM to PM conversion in the rest of the system may degrade the overall phase noise performance. For these reasons, we cannot ignore amplitude noise altogether. Paper D.2 provides a useful discussion of amplitude noise. Note 1 in appendix E provides further information on definitions, notation and, in particular, the specification of added phase noise and amplitude noise for signal-handling components.

The next contribution (D.3) provides substantially more detail on the extension of the time-domain, dual-mixer concept for highly accurate time and time-interval measurements. The basic dual-mixer ideas are included in papers B.1 and B.2.

Paper D.4, published recently, provides the first quantitative treatment of confidence estimates for phase-noise measurements. To the best of our knowledge, this is the only available treatment of this important subject. We expect to see additional papers on this topic in the future.

Paper D.5 discusses the modified Allan variance in more detail than the introductory papers B.1, B.2, and B.3. It is followed by the Lesage-Ayi paper (D.6) which provides analytical expressions for the standard set of power-law noise types and also includes discussion of the uncertainty of the estimate of the modified Allan variance.

Linear frequency drift in oscillators is treated by Barnes in paper D.7. As noted in this paper, even with correction for drift, the magnitude of drift error eventually dominates all time uncertainties in clock models. Drift is particularly important in certain oscillators (e.g., quartz oscillators) and a proper measure and treatment of drift is essential. As with other topics treated by this group of papers, introductory papers B.1, B.2, and B.3 present some discussion of frequency drift, but D.7 is included because it contains a much more comprehensive discussion of the subject.

The final paper (D.8) by Barnes and Allan contains the most recent treatment of measurements made with dead times between them. Paper C.2 introduced the use of bias functions, B_1 and B_2 , which can be used to predict the Allan variance for one set of parameters based on another set (for the power-law noise models). This last paper extends those ideas, introducing a third bias function, B_3 , which can be used to translate the Allan variance between cases where

dead time is accumulated at the end and where dead time is distributed between measurements, a useful process for many data-acquisition situations.

A.5 GUIDE TO SELECTION OF MEASUREMENT METHODS

The table and graph in this section are quick guides to performance limits of the different methods as well as indications of advantages and disadvantages of each. The ideas presented here are drawn from papers B.1 and B.4, but we have modified and expanded the material to make it more comprehensive. The table has been kept simple, so the suggested methods should be viewed as starting points only. They cannot possibly cover all measurement situations.

For a given measurement task, it is often best to start with a quick, simple measurement which will then help to define the problem. For example, faced with the need to characterize an oscillator, a good starting point might be to feed the output of that oscillator along with the output of a similar, but more stable, oscillator into a good mixer and then look at the output. If the two can be brought into quadrature by tuning one of the oscillators or by using a phase-locked loop, then the output can be fed to a spectrum analyzer to get an immediate, at least qualitative idea, of the performance of the oscillator. This mixing process, which brings the fluctuations to baseband where measurement is much more straightforward, is basic to many of the measurement methods. A large number of measurement problems can probably be resolved with this simple, single-conversion, heterodyne arrangement. If the simplest approach is insufficient, then some of the more advanced methods outlined below can be used.

There are many ways to go about categorizing the various measurement methods. Since this volume is aimed at practical measurements, we choose to use the characteristics of the measurement circuit as the basis for sorting. In this arrangement we have (1) direct measurements where no signal mixers are used, (2) heterodyne measurements where two unequal frequencies are involved, and (3) homodyne measurements where two equal frequencies are involved. These methods are listed below.

- I. Direct Measurements
 - 1. Measurements at the Fundamental Frequency
 - 2. Measurements after Multiplication/Division
- II. Heterodyne Measurements
 - 1. Single-Conversion Methods
 - 2. Multiple-Conversion Methods
 - 3. Time-Difference Method
 - a. Dual-Mixer, Time-Difference Method
- III. Homodyne Measurements
 - 1. Phase-Lock-Loop Methods (two oscillators)
 - a. Loose-Phase-Lock-Loop Method
 - b. Tight-Phase-Lock-Loop Method
 - 2. Discriminator Methods (single oscillator)
 - a. Cavity-Discriminator Method
 - b. Delay-Line Method

Table 1. Guide to Selection of Measurement Methods.

Measurement Method	Time Accuracy ^a	Time Stability (1 day)	Frequency Accuracy ^a	Frequency Stability ^b $\sigma_y(\tau)$	Advantages	Disadvantages
I. Direct Measurements						
1. At the Fundamental Frequency	See Note # 2 in Appendix	Limited by time base stability	Limited by time base accuracy	Limited by time base stability	Very simple to perform.	Extremely limited resolution not appropriate for high stability oscillators. ^c
2. After Multiplication/Division	See Note # 2 in Appendix	Limited by time base stability	Limited by time base accuracy	Limited by time base stability	Very simple to perform; for frequency multiplication factor N, noise increases in dB by 20logN.	Provides only modest extension of above method and thus suffers similar limitations. ^c
II. Heterodyne Measurements						
1. Single-Conversion Methods 2. Multiple Conversion Methods	—	See Greenhall ^d	$\sim 10^{-16}$ at 10 MHz	$\sim 10^{-7}/(\nu_0\tau)$ which at 10 MHz is $\sim 10^{-14}/\tau$	Measurement noise can typically be made less than oscillator instabilities for $\tau \approx 1$ s and longer.	Minimum τ determined by period of beat frequency, typically not adjustable; cannot compare oscillators near zero beat; additional information needed to tell which oscillator is high/low in frequency; dead time often associated with measurements.
3. Time-Difference Method	~ 100 ps	~ 20 ps at 10 MHz	$\sim 10^{-16}$ is $\sim 10^{-14}/\tau$	$\sim 10^{-7}/(\nu_0\tau)$ which at 10 MHz problem;	Wide bandwidth input allows a variety of signals; simple to use; cycle ambiguity almost never a measures time, time stability, frequency, and frequency stability.	Using best available equipment, measurement noise is typically greater than oscillator instabilities for τ less than several seconds, hence is often limited to long-term measurements.
a. Dual-Mixer Time-Difference Method	~ 100 ps	~ 5 ps	$\sim 10^{-16}$ at 10 MHz	$\sim 10^{-7}/(\nu_0\tau)$ which at 10 MHz is $\sim 10^{-14}/\tau$	No dead time; may choose sample time (1 ms to as large as desired); oscillators may be at zero beat or different; measurement bandwidth easily changed; measures time, time stability, frequency, and frequency stability.	More complex than other methods, and hence more susceptible to extraneous signal pickup; e.g., ground loops; the time difference is modulo the beat period, e.g., 200 ns at 5 MHz.

^aAccuracy of the measurement cannot be better than the stability of the measurement. Accuracy is limited by the accuracy of the reference oscillator.

^bThis is for a measurement bandwidth of 10^4 Hz; ν_0 = frequency; τ = measurement time.

^cThis assumes use of a simple frequency counter.

^dSee C. Greenhall, *41st Annual Frequency Control Symposium, 1987*, pp. 126-129.

The dash (—) means that the method is not generally appropriate for this quantity.

Table 1. Guide to Selection of Measurement Methods. (continued)

Measurement Method	Time Accuracy ^a	Time Stability (1 day)	Frequency Accuracy ^a	Frequency Stability ^b $\sigma_y(\tau)$	Advantages	Disadvantages
III. Homodyne Methods					Particularly useful for measurement of phase noise.	Generally no used for measurement of time.
1. Phase-Lock-Loop Methods					Assure continuous quadrature of signal and reference.	Care needed to assure that noise of interest is outside loop bandwidth.
a. Loose-Phase-Lock Loop	—	—	Depends on calibration of varicap	$\sim 10^{-7}/(\nu_0\tau)$ which at 10 MHz is $\sim 10^{-14}/\tau$	Useful for short-term time stability analysis as well as spectrum analysis and the detection of periodicity in noise as spectral lines; excellent sensitivity.	Long-term phase measurements (beyond several seconds are not practical.
b. Tight-Phase-Lock Loop	—	—	Depends on calibration of varicap	$\sim 10^{-7}/(\nu_0\tau)$ which at 10 MHz is $\sim 10^{-14}/\tau$	Measurement noise typically less than oscillator instabilities for $\tau = 1$ s and longer; good measurement system bandwidth control; dead time can be made small or negligible.	Need voltage controlled reference oscillator; frequency sensitivity is a function of varicap tuning curve, hence not conducive to measuring absolute frequency differences.
2. Discriminator Methods					Requires no reference oscillator.	Substantially less bandwidth than two-oscillator, homodyne methods; sensitivity low at low Fourier frequency.
a. Cavity Discriminator	—	—	Depends on characteristics of discriminator	Depends on characteristics of discriminator	Requires no reference oscillator; very easy to set up and high in sensitivity; practical at microwave frequencies.	Requires more difficult calibration to obtain any accuracy over even modest range of Fourier frequencies; accurate only for Fourier frequencies less than $0.1 \times$ bandwidth.
b. Delay Line	—	—	—	Depends on characteristics of delay line	Requires no reference oscillator; dynamic range set by properties of delay line; practical at microwave frequencies.	Substantially less accurate than two-oscillator, homodyne methods; cumbersome sets of delay lines needed to cover much dynamic range; considerable delay needed for measurements below 100 kHz from carrier.

^aAccuracy of the measurement cannot be better than the stability of the measurement. Accuracy is limited by the accuracy of the reference oscillator.

^bThis is for a measurement bandwidth of 10^4 Hz; ν_0 = frequency; τ = measurement time.

The dash (—) means that the method is not generally appropriate for this quantity.

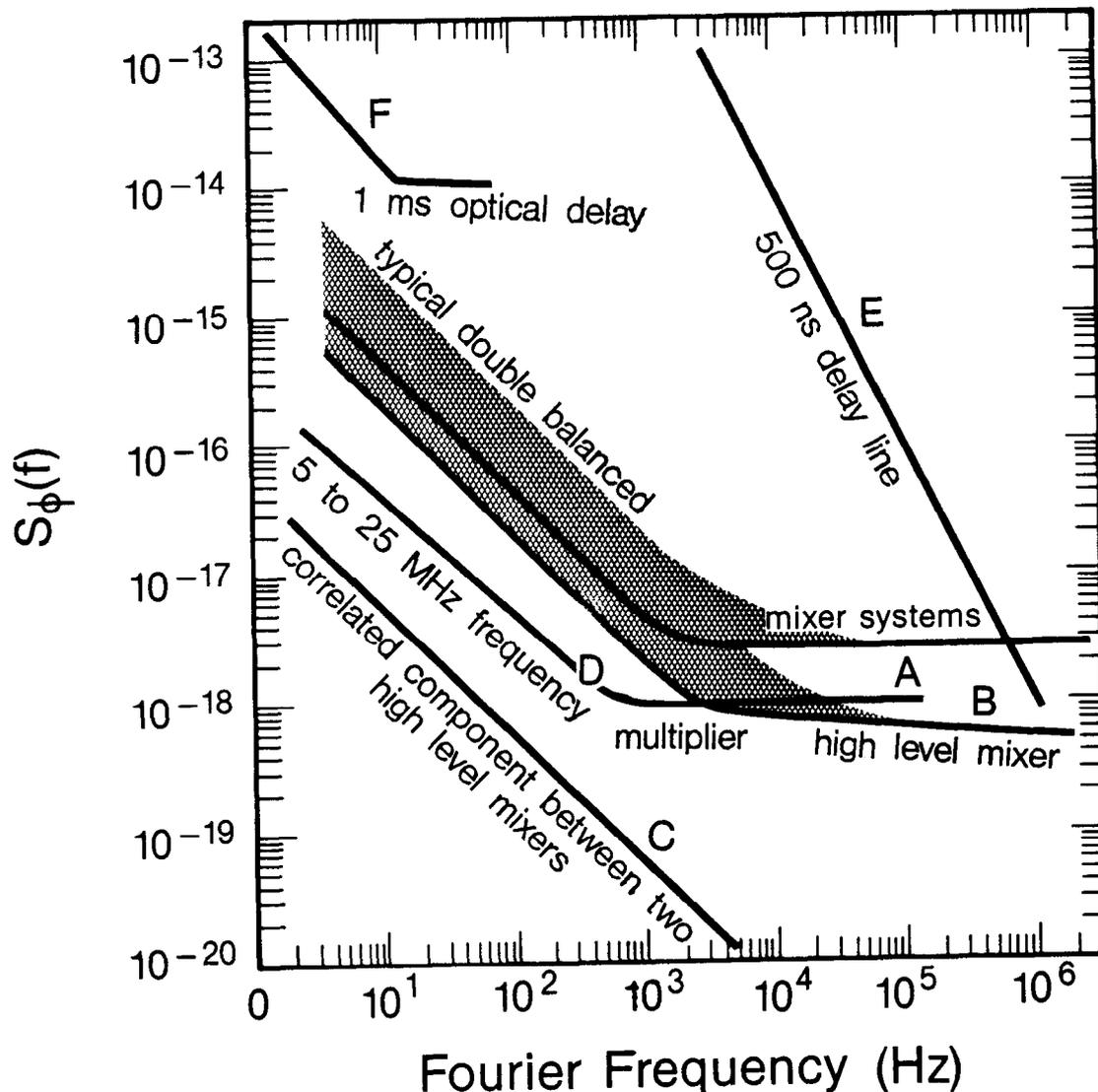


Figure 1. Comparison of nominal lower noise limits for different frequency-domain measurement methods.

- Curve A. The noise limit (resolution), $S_{\phi}(f)$, of typical double-balanced mixer systems at carrier frequencies from 0.1 MHz to 26 GHz.
- Curve B. The noise limit, $S_{\phi}(f)$, for a high-level mixer.
- Curve C. The correlated component of $S_{\phi}(f)$ between two channels using high-level mixers.
- Curve D. The equivalent noise limit, $S_{\phi}(f)$, of a 5 to 25 MHz frequency multiplier.
- Curve E. Approximate phase noise limit for a typical delay-line system which uses a 500 ns delay line.
- Curve F. Approximate phase noise limit for a delay-line system which achieves a 1 ms delay through encoding the signal on an optical carrier and transmitting it across a long optical fiber to a detector.

Table 1 organizes the measurement methods in the above manner giving performance limits, advantages and disadvantages for each. Figure 1 following the table provides limitations for phase noise measurements as a function of Fourier frequency. The reader is again reminded that the table is highly simplified giving nominal levels that can be achieved. Exceptions can be found to almost every entry.

A.6 RELATIONSHIP OF THE MODIFIED ALLAN VARIANCE TO THE ALLAN VARIANCE

In sorting through this set of papers and other published literature on the Allan Variance, we were stimulated to further consider the relationship between the modified Allan variance and the Allan variance. The ideas which were developed in this process have not been published, so we include them here. Paper D.6, "Characterization of Frequency Stability: Analysis of the Modified Allan Variance and Properties of Its Estimate," by Lesage and Ayi adds new insights and augments the Allan and Barnes paper (D.5), "A Modified Allan Variance with Increased Oscillator Characterization Ability." In this section we extend the ideas presented in these two papers and provide further clarification of the relationship between the two variances, both of which are sometimes referred to as two-sample variances.

Figure 2 shows the ratio $[\text{mod } \sigma_y(\tau)/\sigma_y(\tau)]^2$ as a function of n , the number of time or phase samples averaged together to calculate $\text{mod } \sigma_y(\tau)$. This ratio is shown for power-law noise spectra (indexed by the value of α) running from f^2 to f^{-2} . These corrected results have a somewhat different shape for $\alpha = -1$ than those presented in either paper D.5 or paper D.6. Furthermore, this figure also shows the dependence on bandwidth for the case where $\alpha = 1$. For all other values of α shown, there is no dependence on bandwidth. Table 2 gives explicit values for the ratio as a function of n for low n as well as the asymptotic limit for large n . For $\alpha = 1$, the asymptotic limit of the ratio is considerably simplified from that given in papers D.5 and D.6. With these results it is possible to easily convert between $\text{mod } \sigma_y(\tau)$ and $\sigma_y(\tau)$ for any of the common power-law spectra.

The information in figure 2 and table 2 was obtained directly from the basic definitions of $\sigma_y(\tau)$ and $\text{mod } \sigma_y(\tau)$ using numerical techniques. The results of the numerical calculations were checked against those obtained analytically by Allan and Barnes (D.5) and Lesage and Ayi (D.6). For $\alpha = 2, 0$ and -2 the results agree exactly. For $\alpha = 1$ and -1 the analytical expressions are really obtained as approximations. The numerical calculations are obviously more reliable. Details of our calculations can be found in a NIST report [1]. A useful integral expression (not commonly found in the literature) for $\text{mod } \sigma_y^2(\tau)$ is

$$\text{mod } \sigma_y^2(\tau) = \frac{2}{\pi^4 \pi^2 \tau_0^2} \int_0^{f_h} \frac{s_y(f) \sin^6(\pi \tau f)}{f^2 \sin^2(\pi \tau_0 f)} df.$$

Figure 2 and table 2 show that, for the fractional frequency fluctuations, $\text{mod } \sigma_y(\tau)$ always yields a lower value than $\sigma_y(\tau)$. In the presence of frequency modulation (FM) noise with $\alpha \geq 0$, the improvement is very significant for large n . This condition has been examined in detail by Bernier [2]. For white FM noise, $\alpha = 0$, the optimum estimator for time interval τ is to use the value of the times or phases separated by τ to determine frequency. This is analogous to the algorithm for calculating $\sigma_y(\tau)$ which yields the one-sigma uncertainty in the estimate

Table 2. Ratio of mod $\sigma_y^2(\tau)$ to $\sigma_y^2(\tau)$ versus n for common power-law noise types $S_y(f) = h_\alpha f^\alpha$. n is the number of time or phase samples averaged to obtain mod $\sigma_y^2(\tau = n\tau_0$ where τ_0 is the minimum sample time. ω_h is 2π times the measurement bandwidth f_h .

$$R(n) = \frac{\text{mod } \sigma_y^2(\tau)}{\sigma_y^2(\tau)} \quad \text{vs. } n \quad \tau = n\tau_0$$

n	$\alpha = -2$	$\alpha = -1$	$\alpha = 0$	$\alpha = +1$				$\alpha = +2$
				$\omega_h \tau_0 = 3$	$\omega_h \tau_0 = 10$	$\omega_h \tau_0 = 100$	$\omega_h \tau_0 = 10^4$	
1	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
2	0.859	0.738	0.616	0.568	0.543	0.525	0.504	0.500
3	0.840	0.701	0.551	0.481	0.418	0.384	0.355	0.330
4	0.831	0.681	0.530	0.405	0.359	0.317	0.284	0.250
5	0.830	0.684	0.517	0.386	0.324	0.279	0.241	0.200
6	0.828	0.681	0.514	0.349	0.301	0.251	0.214	0.167
7	0.827	0.679	0.507	0.343	0.283	0.235	0.195	0.143
8	0.827	0.678	0.506	0.319	0.271	0.219	0.180	0.125
10	0.826	0.677	0.504	0.299	0.253	0.203	0.160	0.100
14	0.826	0.675	0.502	0.274	0.230	0.179	0.137	0.0714
20	0.825	0.675	0.501	0.253	0.210	0.163	0.119	0.0500
30	0.825	0.675	0.500	0.233	0.194	0.148	0.106	0.0333
50	0.825	0.675	0.500	0.210	0.176	0.134	0.0938	0.0200
100	0.825	0.675	0.500	0.186	0.159	0.121	0.0837	0.0100
Limit	0.825	0.675	0.500	$\left(\frac{3.37}{1.04 + 3 \ln \omega_h \tau} \right)$				1/n

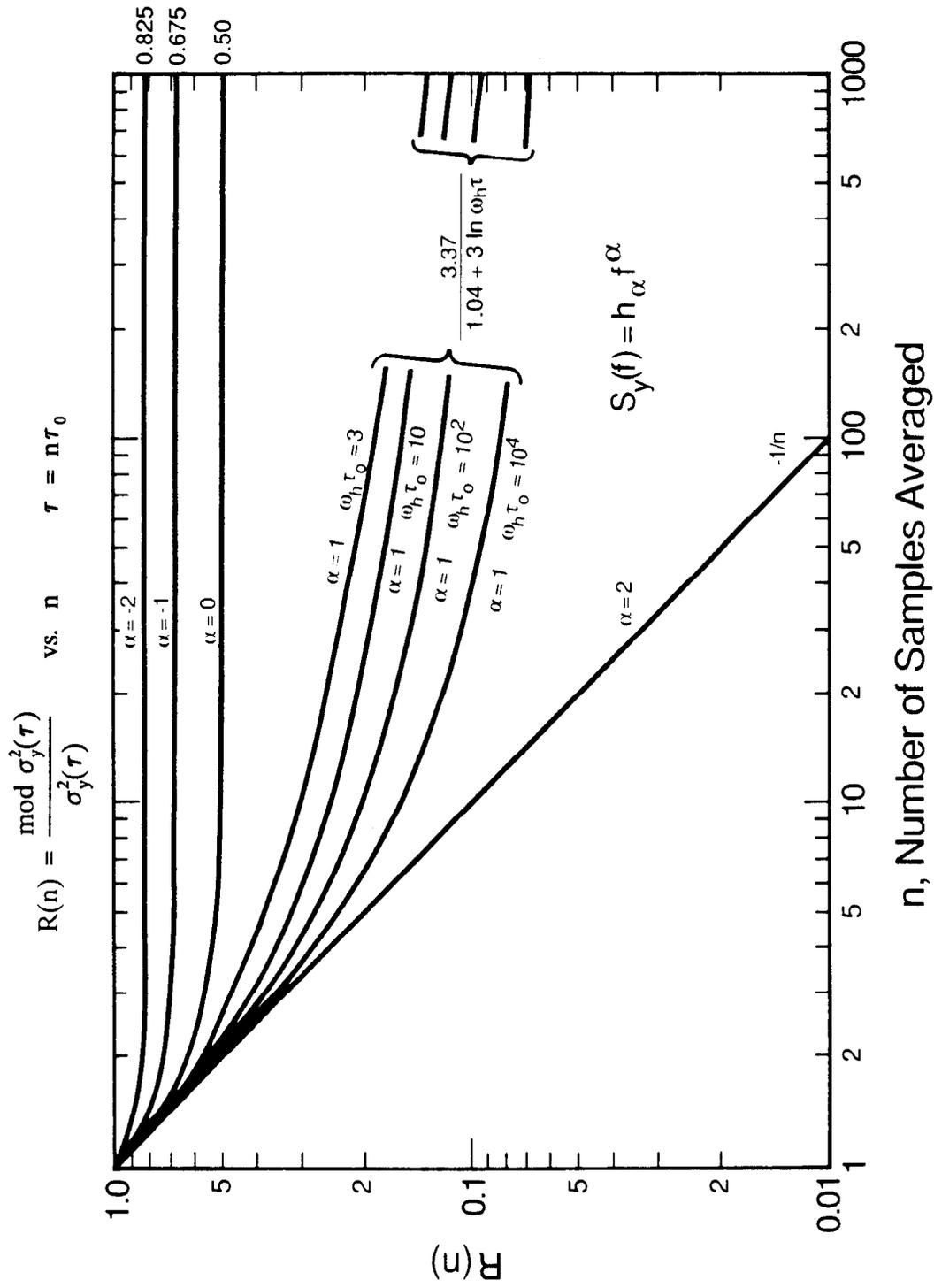


Figure 2. Ratio of $\text{mod } \sigma_y^2(\tau)$ to $\sigma_y^2(\tau)$ as a function of the number of time samples averaged together to calculate either variance.

of the frequency measured in this manner over the interval τ . The estimate of frequency, obtained by averaging the phase or time data, is degraded by about 10 percent from that obtained by using just the end points [3]. This is analogous to the algorithm for calculating $\text{mod } \sigma_y(\tau)$ which, in this case, underestimates the uncertainty in measuring frequency by $\sqrt{2}$. For white phase modulation (PM) noise, $\alpha = 2$, the optimum estimator for frequency is obtained by averaging the time or phase data over the interval τ . This is analogous to the algorithm for calculating $\text{mod } \sigma_y(\tau)$ which yields the one-sigma uncertainty in the estimate for frequency measured in this manner. This estimate for frequency is \sqrt{n} better than that provided by $\sigma_y(\tau)$.

Based on these considerations, it is our opinion that $\text{mod } \sigma_y(\tau)$ can be profitably used much more often than it is now. The presence of significant high-frequency FM or PM noise in the measurement system, in an oscillator, or in an oscillator slaved to a frequency reference, is very common. The use of $\text{mod } \sigma_y(\tau)$ in such circumstances allows one to more quickly assess systematic errors and long-term frequency stability. In other words, a much more precise value for the frequency or the time of a signal (for a given measurement interval) can be derived using $\text{mod } \sigma_y(\tau)$ when n is large.

The primary reasons for using $\sigma_y(\tau)$ are that it is well known, it is simple to calculate, it is the most efficient estimator for FM noise ($\alpha \leq 0$), and it has a unique value for all τ . The advantages of $\text{mod } \sigma_y(\tau)$ are cited in the above paragraph. There are some situations where a study of both $\sigma_y(\tau)$ and $\text{mod } \sigma_y(\tau)$ can be even more revealing than either one. The disadvantages of using $\text{mod } \sigma_y(\tau)$ are that it is more complex to calculate and thus requires more computer time and it has not been commonly used in the literature, so interpretation of the results is more difficult to reconcile with published information. Another concern sometimes raised is that $\text{mod } \sigma_y(\tau)$ does not have a unique value in regions dominated by FM noise ($\alpha \geq 0$). With rapidly increasing computer speeds, the computational disadvantage is disappearing. The corrected and expanded information presented in figure 2 and table 2 addresses the concern about uniqueness.

The primary disadvantage of using $\sigma_y(\tau)$ is that the results can be too conservative. That is, if the level of high-frequency FM noise is high, then the results are biased high, and it can take much longer (often orders of magnitude longer) to characterize the underlying low-frequency performance of the signal under test.

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