The Relation of Radio Sky-Wave Transmission to Ionosphere Measurements*

NEWBERN SMITH†, NONMEMBER, I.R.E.

Summary—*A simple, rapid, graphical method is given for obtaining maximum usable frequencies and effective reflection heights of radio waves, from vertical-incidence measurements of the critical frequencies and virtual heights of the various layers in the ionosphere. The method consists of the use of "transmission curves," which are superimposed on the curve of frequency against virtual height, observed at vertical incidence. The intersection of the curves gives the level of reflection in the ionosphere.*

The factors considered in deriving the transmission curves are variation of virtual height with frequency, effect of the curvature of the ionosphere and earth, influence of the earth's magnetic field, and absorption by or reflection from lower layers in the ionosphere. A chart is included for rapid calculation of the factor sec ϕ_0, used in plotting the transmission curves.

I. Introduction

It IS possible to interpret certain aspects of long-distance radio transmission conditions in terms of measurements of the critical frequencies and virtual heights of the ionosphere made at vertical incidence. The purpose of the present paper is to outline a simple, rapid, graphical method of obtaining, from these vertical-incidence data, the maximum usable frequency over a given path, and the effective heights of reflection of waves incident obliquely upon the ionosphere. Transmission curves of the type to be described have been in use at the National Bureau of Standards since the beginning of 1936, for interpreting radio transmission data and irregularities in terms of the regular ionosphere measurements. More recently they have been used in preparing data for the weekly ionosphere bulletins broadcast over station WWV, and in supplementing local ionosphere measurements by data obtained from observations of the field intensity of distant stations. Part of this work was presented as a paper at the 1936 joint meeting of the Institute of Radio Engineers and the International Scientific Radio Union at Washington.[1]

The treatment of the propagation of radio waves given here is based entirely on the geometrical or ray theory, and considers the ionosphere as nondissipative; i.e., only the refracting strata are considered. It should be remembered that the ray approximation may be subject to a considerable correction in cases where the earth's magnetic field is not negligible. The much-discussed Lorentz polarization term[2,3] has been omitted throughout in the expression for the refractive index.

II. Equivalence Theorem for Plane Ionosphere and Earth

The process of refraction and reflection of radio waves in the ionosphere depends on the fact that the refractive index μ' of the medium decreases with increasing ionic density N. In the absence of a magnetic field, we may write the relation of μ' to N thus

$$\mu' = \sqrt{1 - \frac{f_0^2}{f'^2}} \tag{1}$$

where $f_0 = \sqrt{Ne^2/\pi m}$ and f' is the frequency of the transmitted wave.[4,5]

The propagation of electromagnetic waves in a plane ionosphere, i.e., one whose surfaces of equal ionic density are planes, is illustrated in Fig. 1. By Snell's law, the wave will penetrate the medium until the refractive index is reduced to the value sin ϕ_1, where ϕ_1 is the angle of incidence of the waves upon the ionosphere. At this level the direction of phase propagation is horizontal. This, therefore, is the highest level reached by the wave, or the level of true reflection, at a height z_0 above the lower boundary of the ionosphere. In the case of vertical incidence, reflection occurs where the refractive index is zero.

Let us define the equivalent vertical-incidence frequency f, corresponding to a wave frequency f' over a given transmission path, as the frequency of the wave reflected, at vertical incidence, at the same level as is the actual wave over the given path. It may be easily shown that, in the absence of a magnetic field

$$f = f'\sqrt{1 - \mu_0'^2} \tag{2}$$

where μ_0' is the value of μ' at the level of reflection for the wave of frequency f'.

* Decimal classification: R113.61. Original manuscript received by the Institute, June 27, 1938. Publication approved by the director of the National Bureau of Standards of the U.S. Department of Commerce.

† National Bureau of Standards, Washington, D. C.

[1] N. Smith, S. S. Kirby, and T. R. Gilliland, "Recent correlations between the ionosphere and high-frequency radio transmission." Presented before joint meeting of I.R.E. and U.R.S.I., May 1, 1936.

[2] D. R. Hartree, "Propagation of electromagnetic waves in a stratified medium," *Proc. Camb. Phil. Soc.*, vol. 25, pp. 97–120; January, (1929).

[3] C. G. Darwin, "The refractive index of an ionized medium," *Proc. Roy. Soc.*, vol. 146A, pp. 17–46; August, (1934).

[4] W. H. Eccles, "On the diurnal variations of the electric waves occurring in nature and on the propagation of electric waves around the bend of the earth," *Proc. Roy. Soc.*, vol. 87A, pp. 79–99; August 13, (1912).

[5] J. Larmor, "Why wireless electric rays can bend round the earth," *Phil. Mag.*, vol. 48, pp. 1025–1036; December, (1924).

For the plane ionosphere of Fig. 1, $\mu_0' = \sin \phi_1$ and so

$$f = \frac{f'}{\sec \phi_1},$$

the well-known "secant law." We shall denote by z_v' the height above the lower boundary of the ionosphere of the vertex of the equivalent triangular path (the triangular path having the same base and angle of departure as has the actual path), and by z_v the virtual height above this lower boundary, at vertical incidence, for the equivalent vertical-incidence frequency[6] f. Then

$$z_v' + h = \frac{1}{2} D \cos \phi_1 = \frac{\cos \phi_1}{2} \int_{\text{path}} \frac{ds \sin \phi}{\sin \phi_1}$$

where ϕ is the angle made with the vertical by the element of path ds. Since, by Snell's law, $\mu' \sin \phi = \sin \phi_1$

$$z_v' + h = \frac{\cos \phi_1}{2} \int_{\text{path}} \frac{ds}{\mu'} = \cos \phi_1 \int_0^{z_0} \frac{dz}{\mu' \cos \phi} + h.$$

By Snell's law, also,

$$\mu' \cos \phi = \sqrt{\mu'^2 - \sin^2 \phi_1}. \tag{3}$$

Now

$$\mu'^2 = 1 - \frac{f_0^2}{f'^2} = 1 - \frac{f_0^2}{f^2} \cos^2 \phi_1$$

so that

$$z_v' = \int_0^{z_0} \frac{dz}{\sqrt{1 - \frac{f_0^2}{f^2}}} = z_v. \tag{4}$$

This is the equivalence theorem for the plane earth[7] and is a direct consequence of Breit and Tuve's theorem.[8] It should be noted that ϕ_1 is equal to ϕ_0, the half-vertex angle of the equivalent triangular path.

III. Transmission Curves for Plane Ionosphere and Earth

The virtual height measured at vertical incidence for some frequency f has been shown, on the basis of the simple theory, to be the same as the height of the equivalent triangular path for a higher frequency $f' = f \sec \phi_1$, where ϕ_1 is the angle between the ray entering the ionosphere and the normal to the lower boundary of the ionosphere. This means that for transmission to take place over a given distance D at a given frequency f', the equivalent vertical-incidence frequency f must be returned, at vertical incidence, from a virtual height z_v equal to the height of the equivalent triangular path. In order to determine the z_v and f which will correspond to this transmission, it is necessary to solve simultaneously the vertical-incidence equation,

$$z_v = z_v(f), \tag{5}$$

and the transmission equation,

$$f = \frac{f'}{\sec \phi_1}, \tag{6}$$

where ϕ_1 is determined in terms of z_v by the relation $z_v = z_v'$ and by the geometry of the path. For the case of the plane earth, ϕ_1 is given by the relation

$$D = 2(z_v' + h) \tan \phi_1,$$

as may be seen from Fig. 1.

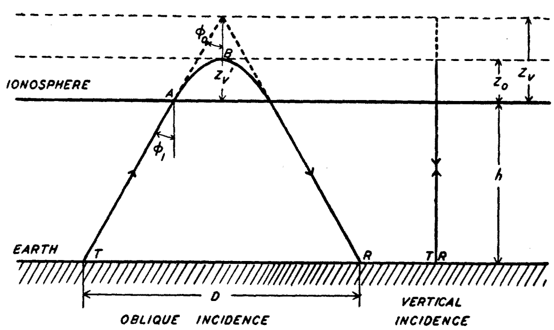

Fig. 1—Vertical-incidence and oblique-incidence transmission over a plane earth. T is the transmitter and R is the receiver. The solid curve is the actual path taken by the waves, and the dotted line shows the equivalent triangular path. z_v is the virtual height at vertical incidence for frequency f; z_0 is the true height of reflection for frequency f at vertical incidence and for frequency $f' = f/\sqrt{1 - \mu_0'^2}$ at oblique incidence; z_v' is the height of the equivalent triangular path; and μ_0' is the value of the refractive index at the level z_0 for the frequency f'.

The solution of (5) and (6) is best done graphically. When (5) is plotted there is obtained a curve of virtual height against frequency observed at vertical incidence. This will be called the (z_v, f) curve. Equation (6) gives a family of curves of z_v plotted against f for different values of f' and D. These curves will be called transmission curves. The intersection of a (z_v, f) curve with a transmission curve for a given f' and D gives the height of the equivalent triangular path for transmission of the frequency f' over the distance D, and also gives the equivalent vertical-incidence frequency f for that path.

The solid curve in Fig. 2 is a typical (z_v, f) curve, showing both the E and F_2 layers in the ionosphere. Superimposed on it are the dotted transmission curves for various transmission frequencies and distances. Curves I, II, III, IV, and V are for increasing

[6] The notation on this paper differs somewhat from that in the author's previous papers, in that the primed quantities all refer to oblique-incidence and the unprimed to vertical-incidence transmission.

[7] D. F. Martyn, "The propagation of medium radio waves in the ionosphere," *Proc. Phys. Soc.*, vol. 47, pp. 323–339; March 1, (1935).

[8] G. Breit and M. A. Tuve, "A test of the existence of the conducting layer," *Phys. Rev.*, vol. 28, pp. 554–575; September, (1926).

values of f' and a fairly short distance. For the frequency corresponding to I, transmission is by way of the E layer. For curve II the given distance is within the "skip zone" for E-layer transmission but is transmitted by the F_2 layer. Curve III, which intersects the F_2 curve in two points, shows transmission by

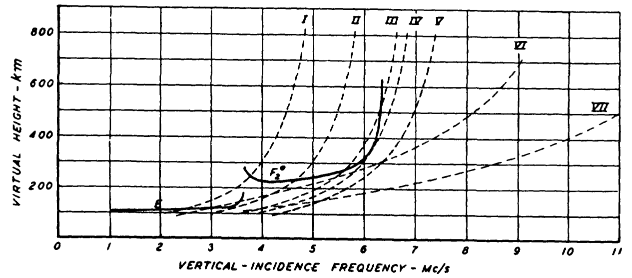

Fig. 2—Transmission curves for various frequencies and distances superimposed on a typical (z_v, f) curve for the o component.

Curve	D Kilometers	f' Kilocycles
I	400	5000
II	400	6000
III	400	6900
IV	400	7000
V	400	7600
VI	800	10,000
VII	800	14,000

two paths (high and low values of z_v) as reported by various observers.[9,10] Curve IV corresponds to the maximum possible frequency that can be used over the given distance. As the F_2-layer critical frequency varies, transmission has been observed to begin and fail abruptly at this point. For curve V the frequency f' is too high, and the given distance is within the skip zone. Curves VI and VII are for a longer distance, for which E-layer transmission takes place at higher frequencies than does F-layer transmission. Curve VII shows the skip effect again.

A type of transmission curve based on the principles described thus far was developed independently by L. V. Berkner of the Department of Terrestrial Magnetism, Carnegie Institution of Washington, and described by him at the joint meeting of the I.R.E. and U.R.S.I. at Washington on April 30, 1937.

IV. Logarithmic Transmission Curves

It will be noted that the transmission curve is merely a plot of $f'/\sec\phi_1$ against z_v. If the frequencies be plotted logarithmically on both the (z_v, f) and transmission curves the transmission curve is but a logarithmic $1/\sec\phi_1$ curve, with the abscissa unity at the frequency f'. It is thus possible to use this logarithmic transmission curve to represent any value of f', for any given distance D, by making the abscissa $\sec\phi_1 = 1$ coincide with the given f' on the (z_v, f) curve sheet.

The variation of f and z_v with f' over a given distance may then be determined by sliding the logarithmic $\sec\phi_1$ transmission curve along the frequency scale, and noting its intersection with the (z_v, f) curve, for every value of f'. The maximum usable frequency over the given distance is then the highest frequency for which the two curves have points in common.

An example of this type of logarithmic $\sec\phi_1$ curve is shown in Fig. 3, for the case where f' = maximum usable frequency.

Although we have neglected the magnetic field of the earth thus far in the development, it is necessary to recognize the fact that the vertical incidence (z_v, f) curve actually possesses two branches, one for the so-called o component or ordinary ray, and the other for the so-called x component, or extraordinary ray. Since the (z_v, f) curve which has been referred to above is for a wave for which $\mu' = 0$ for the equivalent vertical-incidence frequency at the level of reflection, the foregoing development and curves may be applied to the (z_v, f) curve for the o component,

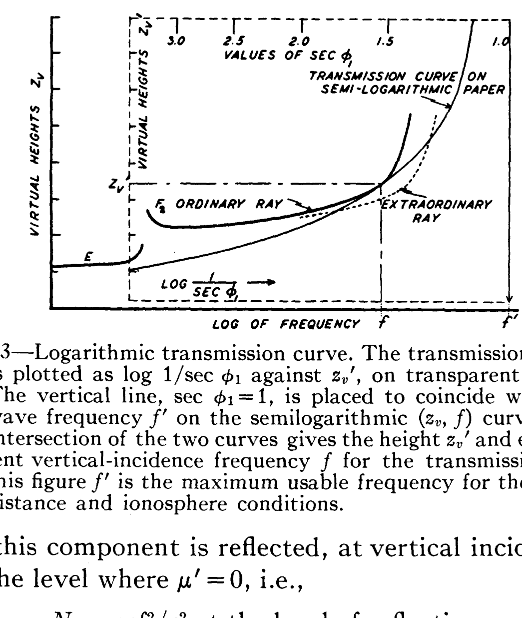

Fig. 3—Logarithmic transmission curve. The transmission curve is plotted as log $1/\sec\phi_1$ against z_v', on transparent paper. The vertical line, $\sec\phi_1 = 1$, is placed to coincide with the wave frequency f' on the semilogarithmic (z_v, f) curve. The intersection of the two curves gives the height z_v' and equivalent vertical-incidence frequency f for the transmission. In this figure f' is the maximum usable frequency for the given distance and ionosphere conditions.

for this component is reflected, at vertical incidence, at the level where $\mu' = 0$, i.e.,

$$N = \pi m f^2 / e^2 \text{ at the level of reflection.}$$

V. Transmission Curves for Flat Ionosphere and Curved Earth

If the earth's curvature be not neglected, but the ionosphere be considered flat, the above developments and equivalence theorem are still valid for the part of the wave's path which lies in the ionosphere.

[9] F. T. Farmer and J. A. Ratcliffe, "Wireless waves reflected from the ionosphere at oblique incidence," *Proc. Phys. Soc.*, vol. 48, pp. 839–849; November, (1936).

[10] W. Crone, K. Krüger, G. Goubau, and J. Zenneck, "Echo measurements over long distances," *Hochfrequenz. und Elektroakustic*, vol. 48, pp. 1–7; July, (1936).

Such a case is illustrated in Fig. 4. This is the case where the wave does not travel very far, horizontally, in a part of the ionosphere where μ' differs appreciably from unity, so that $z_v' - z_0$ is very small compared with $(R+h)/\tan^2\phi_1$. It should be noted that, to the first order, ϕ_1 still equals ϕ_0 and $z_v = z_v'$.

In this case the distance of transmission is connected with the angle of incidence by the relation

$$\tan \phi_1 = \frac{\sin \dfrac{D}{2R}}{\dfrac{z_v + h}{R} + 1 - \cos \dfrac{D}{2R}} \tag{7}$$

and this value of ϕ_1 is the one to be used in the transmission equation (6).

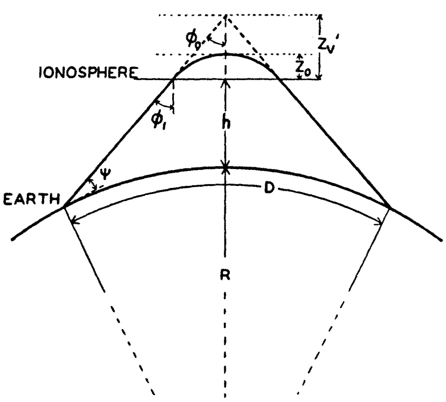

Fig. 4—Geometric relation for determining ϕ_1 in terms of transmission distance D for a curved earth and flat ionosphere. In this and in Figs. 5 and 6 the distance between the ionosphere and the earth is greatly exaggerated for clarity of representation.

The transmission curves can still be plotted logarithmically and applied to the observed (z_v, f) curves to obtain values of z_v' for the oblique transmission, good to the first order of approximation.

VI. Equivalence Relations for Curved Ionosphere and Earth

To obtain the variation, with μ', of ϕ, the angle made by an element of the ray path with the vertical, we must consider first the angle a straight line makes with the vertical at various heights above the earth's surface. The geometry of Fig. 5 leads to the relation

$$\sin \phi' = \sin \phi \left(1 + \frac{dz}{R + h + z}\right). \tag{8}$$

Assuming the validity of Snell's law for a ray traversing an infinitely thin layer of the ionosphere, of thickness dz, we can use (8) to obtain the differential equation

$$\frac{d(\mu' \sin \phi)}{\mu' \sin \phi} = - \frac{dz}{R + h + z}.$$

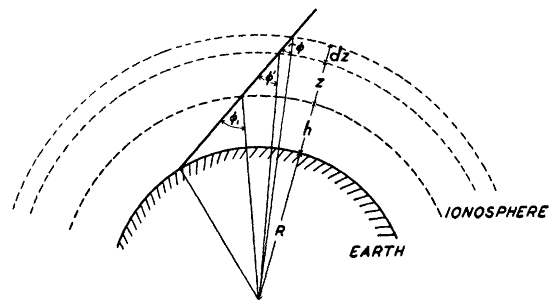

Fig. 5—Variation, with height, of the angle that a straight line makes with the normal to the earth's surface.

Integrating this from the lower boundary of the ionosphere, where $\mu' = 1$, $\phi = \phi_1$, and $z = 0$ (see Fig. 6) up to the level z, we get

$$\mu' \sin \phi = \frac{\sin \phi_1}{1 + \dfrac{z}{R + h}} \tag{9}$$

as the form of Snell's law appropriate to the curved ionosphere.

This means that the wave will penetrate the curved ionosphere until the refractive index is reduced to the value

$$\mu_0' = \frac{\sin \phi_1}{1 + \dfrac{z_0}{R + h}} \tag{10}$$

where ϕ_1 is the angle of incidence of the wave on the lower boundary of the ionosphere, and z_0 is the maximum height of penetration above this lower boundary.

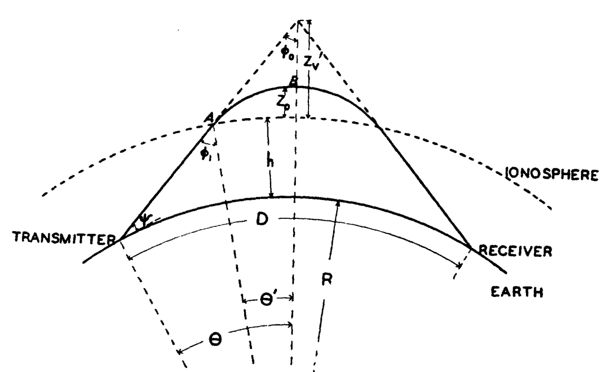

Fig. 6—Transmission through curved ionosphere. z_v' = height of equivalent triangular path, z_0 = true height of reflection, at the level where $\mu' = \sin \phi_1(1 - z_0/(R+h))$, R = radius of earth, D = distance of transmission.

If we express this in terms of ϕ_0, the half-vertex angle of the equivalent triangular path, (10) becomes

$$\mu_0' = \sin\phi_0 \frac{1+\dfrac{z_v'}{R+h}}{1+\dfrac{z_0}{R+h}} \tag{10a}$$

or, if $z_0 \ll R+h$, as is usually the case,

$$\mu_0' = \sin\phi_0\left(1+\frac{z_v'-z_0}{R+h}\right) \tag{10b}$$

instead of the relation $\mu_0' = \sin\phi_0 = \sin\phi_1$, which was true for the flat ionosphere. This means that the level of reflection is lower than would be the case were the ionosphere flat, and a given maximum ionic density will permit a higher frequency to be propagated over a given distance.

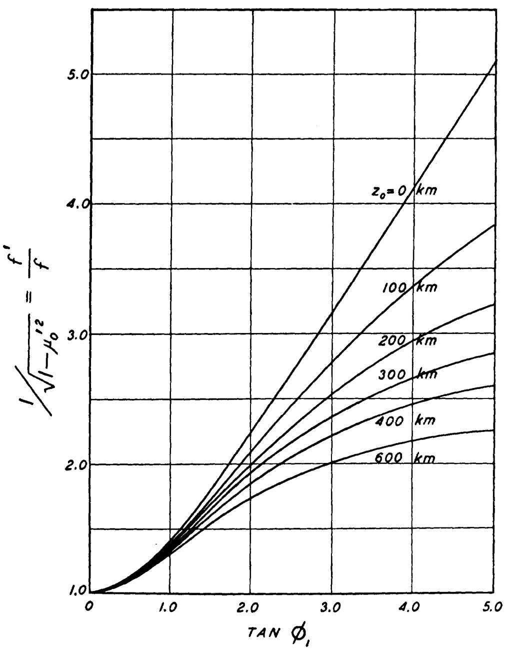

Fig. 7—Variation of $1/\sqrt{1-\mu_0'^2}$ with z_0 and $\tan\phi_1$.

The equivalent vertical-incidence frequency in this case is

$$f = f'\sqrt{1-\mu_0'^2} = f'\sqrt{1-\frac{\sin^2\phi_1}{\left(1+\dfrac{z_0}{R+h}\right)^2}}$$

$$= f'\sqrt{1-\sin^2\phi_0\left[\frac{1+\dfrac{z_v'}{R+h}}{1+\dfrac{z_0}{R+h}}\right]^2}. \tag{11}$$

Under practical conditions z_0 is almost always less than 400 kilometers, and values of z_v' greater than 400 kilometers contribute nothing to transmission over appreciable distances, so that we can always consider z_0 and $z_v' \ll R+h$. Using this approximation (11) becomes

$$f = f'\cos\phi_1\sqrt{1+\frac{2z_0}{R+h}\tan^2\phi_1}$$

$$= f'\cos\phi_0\sqrt{1-\frac{2(z_v'-z_0)}{R+h}\tan^2\phi_0}. \tag{11a}$$

Fig. 7 gives values of $1/\sqrt{1-\mu_0'^2}$ plotted against $\tan\phi_1$ for various values of z_0, assuming $h=100$ kilometers (at the bottom of the E layer). These curves may be used to determine the relation of f to f' for various values of z_0 and ϕ_1. By substituting ϕ_0 for ϕ_1 and $z_0 - z_v'$ for z_0, similar curves may also be plotted to determine the relation of f to f' for various values of z_0, z_v', and ϕ_0.

If, further, $z_0 \ll (R+h)\cot^2\phi_1$ or $(z_v'-z_0) \ll (R+h)\cot^2\phi_0$ we may write (11a) approximately,

$$f = f'\sqrt{1-\mu_0'^2} = f'\cos\phi_1\left[1+\frac{z_0}{R+h}\tan^2\phi_1\right]$$

$$= f'\cos\phi_0\left[1-\frac{z_v'-z_0}{R+h}\tan^2\phi_0\right] \tag{11b}$$

a form which it is convenient to use in some discussions. This approximation leads to results good to 1 per cent or better for E-layer transmission, where z_0 is less than 50 kilometers, and for single-reflection F-layer transmission over distances less than 1500 kilometers, or multireflection F-layer transmission where each reflection covers less than 1500 kilometers. For transmission over distances greater than 1500 kilometers for each reflection it is necessary to use the more exact expression.

The correction term, or frequency Δf which it is necessary to subtract from $f'/\sec\phi_0$ in order to obtain f, is the amount by which the equivalent vertical-incidence frequency for the curved ionosphere differs from that for the flat ionosphere. This is

$$\Delta f = f'[\cos\phi_0 - \sqrt{1-\mu_0'^2}]$$

which is approximately

$$\Delta f = f'\cos\phi_0\left[1-\sqrt{1-2\frac{z_v'-z_0}{R+h}\tan^2\phi_0}\right]$$

and, if $z_v' - z_0 \ll (R+h)\cot^2\phi_0$

$$\Delta f \doteq f'\left(\frac{z_v'-z_0}{R+h}\right)\frac{\sin^2\phi_0}{\cos\phi_0}.$$

This correction term is zero at vertical incidence and for $z_0 = z_v$, i.e., reflection from a sharp boundary at the height z_0. For $z_0 \neq z_v$ it increases rapidly with distance.

The vertical-incidence frequency f derived above may be used to plot transmission curves of z_v' against f, exactly as in the case of the flat ionosphere. If the assumption be made that z_v' is the same as the z_v measured at the frequency f, then these transmission curves could be superimposed on the (z_v, f) curve, to give directly the z_v' for the oblique-incidence case.

Actually it is more convenient to use the sec ϕ_0 transmission curves, wherein z_v' is plotted against $f'/\sec \phi_0$, and to refer the correction to the (z_v, f) curve. On the assumption that $z_v' = z_v$, Δf can be calculated, for each point on the (z_v, f) curve, for each sec ϕ_0 transmission curve. This transmission curve can then be superimposed on the (z_v, f) curve displaced toward the higher frequencies by the amount Δf, and the oblique-incidence values of z_v' read off.

The calculation of Δf for each distance from z_v and z_0, without any assumption as to z_v', and for a ϕ_0 calculated for an equivalent triangular path of height z_v, is discussed below. This value of Δf can be used in the same manner as was described in the preceding paragraph, and the sec ϕ_0 curves applied, ϕ_0 being calculated for an equivalent triangular path of height z_v instead of z_v'.

If the (z_v, f) curve and the sec ϕ_0 curves are plotted logarithmically, we may calculate a factor $1 + (\Delta f/f)$ as follows:

$$1 + \frac{\Delta f}{f} = \frac{\cos \phi_0}{\sqrt{1 - \mu_0'^2}} \cdot$$

The frequencies on the (z_v, f) curve then can be easily multiplied by this factor by adding the factor logarithmically to the curve, and the log sec ϕ_0 curves can be applied to the resulting corrected (z_v, f) curve. It should be noted that in this expression the ϕ_0 refers only to an equivalent triangular path of height z_v, and not z_v'. The z_v' enters only into the expression for μ_0'.

A typical (z_v, f) curve corrected in this manner for a given distance (2000 kilometers) is shown in Fig. 8. The virtual heights are lower on the corrected curve and the curve extends out to frequencies higher than the critical frequency for the ordinary ray. The correction to the curve is different for different distances, increasing with the distance. This means that the virtual heights of reflection are lower, for greater distances, than if the curvature of the earth were not considered. The limiting frequency of transmission, or maximum usable frequency, over considerable distances is correspondingly increased, the difference becoming as much as 20 per cent at the greater distances, depending of course on the form of the (z_v, f) curve.

When the (z_v, f) curve has been corrected in the above manner it may be plotted logarithmically as described above, and the logarithmic sec ϕ_0 curves may be applied to give directly the maximum usable frequencies and virtual heights over transmission paths of given lengths.

Since the first draft of this paper was written there has come to the author's attention an excellent unpublished paper "Skip Distance Analysis," by T. L. Eckersley and G. Millington, in the form of a contribution to the November, 1937, London meeting of the Special Radio Wave Propagation Committee held in preparation for the Cairo Radio Conference.[11] In this they undertake an analysis of radio transmission over a curved earth and obtain curves for determining the retardation of sky wave over ground wave at a distance, in terms of vertical-incidence measurements. They limit the analysis, however, to the case where z_0 is very small compared with h. The analysis can thus apply only to E-layer transmission, since for F-layer transmission z_0 must be measured from the lower boundary of the ionosphere in order to include the effect of retardation in the E layer.

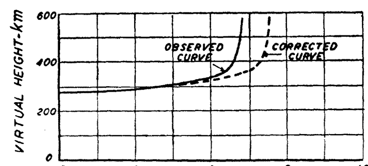

Fig. 8—(z_v, f) curve corrected for a given distance (2000 kilometers) for the effect of the earth's curvature.

Their work, however, suggests a method of approximate analysis for larger values of z_0 and a means for easily evaluating the correction factor $1 + \Delta f/f$ without making the assumption that z_v is equal to z_v'.

The development given by Eckersley and Millington begins with the following relation, for the ray path shown in Fig. 6

$$\theta' = \int_A^B d\theta = \int_A^B \frac{ds \sin \phi}{R + h + z}$$

where ds is an element of the ray path making an angle ϕ with the vertical.

Since, from (9)

$$\sin \phi = \frac{1}{\mu'} \frac{\sin \phi_1}{1 + \dfrac{z}{R + h}}$$

[11] *Added in proof:* G. Millington, "The relation between ionospheric transmission phenomena at oblique incidence and those at vertical incidence," *Proc. Phys. Soc.*, vol. 50, pp. 801–825; September 1, (1938).

this becomes

$$\theta' = \frac{\sin \phi_1}{R+h}\int_A^B \frac{ds}{\mu'\left(1+\frac{z}{R+h}\right)^2}. \quad (12)$$

They also used the relation

$$\int_A^B \frac{ds}{\mu'} = \int_0^{z_0} \frac{dz}{\mu' \cos \phi}$$
$$= \int_0^{z_0} \frac{dz}{\sqrt{\mu'^2 - \frac{\sin^2 \phi_1}{\left(1+\frac{z}{R+h}\right)^2}}}. \quad (13)$$

For the case where there is no magnetic field

$$\mu' = \sqrt{1-\frac{f_0^2}{f'^2}}, \quad \text{where} \quad f_0 = \sqrt{\frac{Ne^2}{\pi m}}, \quad \text{and so}$$

$$\int_A^B \frac{ds}{\mu'} = \int_0^{z_0} \frac{f'dz}{\sqrt{f'^2 - f_0^2 - f'^2 \frac{\sin \phi^2{}_1}{\left(1+\frac{z}{R+h}\right)^2}}}. \quad (14)$$

Putting in this expression the wave frequency f' in terms of the equivalent vertical-incidence frequency given above in (11), (11a), and (11b), and using the value of μ_0' derived in (10),

$$\int_A^B \frac{ds}{\mu'} = \frac{1}{\sqrt{1-\mu_0'^2}}$$
$$\int_0^{z_0} \frac{fdz}{\sqrt{\frac{f^2}{1-\mu_0'^2}\left(1-\frac{\sin^2 \phi_1}{\left(1+\frac{z}{R+h}\right)^2}\right)-f_0^2}}. \quad (15)$$

At this point Eckersley and Millington assumed that they were dealing with a thin layer, and that in consequence (1) θ' was only slightly less than

$$\frac{\sin \phi_1}{R+h}\int_A^B \frac{ds}{\mu'},$$

from equation (12), (2)

$$\int_A^B \frac{ds}{\mu'}$$

was only slightly greater than

$$\frac{1}{\sqrt{1-\mu_0'^2}}\int_0^{z_0} \frac{fdz}{\sqrt{f^2-f_0^2}},$$

from equation (15).

This simplification is justified for E-layer transmission, but is unfortunately not justified for F_2-layer transmission, since the lower boundary of the ionosphere must be taken at the beginning of the E layer in each case. We shall not, therefore, confine our discussion to this limited case. It must also be noted that the simplification introduced by assuming $z_0 \ll (R+h) \cot^2 \phi_1$ also is not valid except at comparatively short distances, where $\phi_1 = 45$ degrees or less. For practical purposes, it is, however, justifiable to assume $z_0 \ll R+h$. Using this relation then, we obtain

$$\theta' = \frac{\sin \phi_1}{R+h}\int_A^B \frac{ds}{\mu'}\left(1-\frac{2z}{R+h}\right). \quad (16)$$

Eckersley and Millington combined the approximate forms of (12) and (15) and introduced the value of $\sqrt{1-\mu_0^2} = \cos \phi_1$, to obtain

$$\theta' = \frac{\tan \phi_1}{R+h}\int_0^{z_0} \frac{fdz}{\sqrt{f^2-f_0^2}}.$$

We shall, however, combine (15) and (16) to obtain the more exact relation

$$\theta' = \frac{\sin \phi_1}{(R+h)\sqrt{1-\mu_0'^2}}$$
$$\int_0^{z_0} \frac{fdz}{\sqrt{f^2(1-A)-f_0^2}}\left(1-\frac{2z}{R+h}\right) \quad (16a)$$

where

$$1-A = \frac{1-\frac{\sin^2 \phi_1}{\left(1+\frac{z}{R+h}\right)^2}}{1-\mu_0'^2} \doteq \frac{1+\frac{2z}{R+h}\tan^2 \phi_1}{1+\frac{2z_0}{R+h}\tan^2 \phi_1}. \quad (16b)$$

If we let

$$1+B = \frac{1}{\sqrt{1-A\left(\frac{f^2}{f^2-f_0^2}\right)}}, \quad (16c)$$

$$\theta' = \frac{\sin \phi_1}{(R+h)\sqrt{1-\mu_0'^2}}$$
$$\int_0^{z_0} \frac{dz}{\sqrt{1-\frac{f_0^2}{f^2}}}(1+B)\left(1-\frac{2z}{R+h}\right). \quad (17)$$

For values of z_0 considerably less than $(R+h)/2 \cot^2 \phi_1$, B is usually a small number, and for $z \rightarrow z_0$ (near the level of reflection) it is also usually small. It may attain, however, relatively large values.

In general the evaluation of B involves a knowledge of the distribution of ionic density, and a precise evaluation of the quantity θ' involves graphical integration for each case considered. We shall for the present consider two principal cases, (a) $z_0 \to 0$ and (b) $z_0 = z_v$, the virtual height at vertical incidence for the frequency $f = f'\sqrt{1-\mu_0'^2}$, and discuss some cases of linear distributions of ionic density, where $z/z_0 = f_0^2/f^2$.

(a) $z_0 \ll R + h \cot^2 \phi_1$.

This case applies to E-layer transmission at any distance or to F-layer transmission at short distances (ϕ_1 small).

From (11b) we have

$$f = f' \cos \phi_1 \left[1 + \frac{z_0}{R+h} \tan^2 \phi_1\right], \tag{18a}$$

and from (16b)

$$1 - A = 1 + 2\left(\frac{z - z_0}{R+h}\right)\tan^2 \phi_1.$$

From (16c)

$$1 + B = 1 + \left(\frac{z_0}{R+h}\right)\frac{1 - \dfrac{z}{z_0}}{1 - \dfrac{f_0^2}{f^2}} \tan^2 \phi_1,$$

if we assume that

$$\left(1 - \frac{z}{z_0}\right) \Big/ \left(1 - \frac{f_0^2}{f^2}\right)$$

is nowhere so large as to make B comparable with unity. This is equivalent to assuming that f_0^2/f^2 does not approach 1 much more rapidly than does z/z_0, an assumption that is valid except quite close to a critical frequency. Transmissions involving equivalent vertical-incidence frequencies close to the critical frequency are of interest, however, only when the distance of transmission is short, and in this case ϕ_1 is small, so that B is a small number, anyhow.

Thus

$$\theta' = \frac{\tan \phi_1}{R+h}\left[1 - \frac{z_0}{R+h}\tan^2 \phi_1\right]\int_0^{z_0} \frac{dz}{\sqrt{1 - \dfrac{f_0^2}{f^2}}}$$

$$\left\{1 + \frac{z_0}{R+h}\tan^2 \phi_1 \frac{\left(1 - \dfrac{z}{z_0}\right)}{\left(1 - \dfrac{f_0^2}{f^2}\right)} - \frac{2z}{R+h}\right\}.$$

Now

$$\int_0^{z_0} \frac{dz}{\sqrt{1 - \dfrac{f_0^2}{f^2}}} = z_v,$$

the virtual height measured at vertical incidence for the equivalent vertical-incidence frequency f. Since the terms involving $z_0/(R+h)$ are small, we may write

$$\theta' = \frac{z_v \tan \phi_1}{R+h}\left\{1 - \frac{z_0}{R+h}\tan^2 \phi_1 (1 - C) - C'\right\} \tag{18b}$$

where

$$C = \frac{1}{z_v}\int_0^{z_0} \frac{dz}{\sqrt{1 - \dfrac{f_0^2}{f^2}}} \frac{1 - \dfrac{z}{z_0}}{1 - \dfrac{f_0^2}{f^2}}$$

and

$$C' = \frac{2}{z_v(R+h)}\int_0^{z_0} \frac{z\,dz}{\sqrt{1 - \dfrac{f_0^2}{f^2}}}.$$

For a linear distribution of ionic density

$$1 - \frac{z}{z_0} = 1 - \frac{f_0^2}{f^2}, \quad z_v = 2z_0, \quad \text{and } C = 1,$$

and

$$\theta' = \frac{\tan \phi_1}{R+h}\int_0^{z_0} \frac{dz}{\sqrt{1 - \dfrac{z}{z_0}}}\left[1 - \frac{2z}{R+h}\right]$$

$$= \frac{z_v \tan \phi_1}{R+h}\left[1 - \frac{2}{3}\left(\frac{z_v}{R+h}\right)\right]. \tag{19}$$

For z_0 vanishingly small $C' \to 0$ and (18) reduces simply to

$$\theta' = \frac{z_v \tan \phi_1}{R+h} \tag{19a}$$

(b) $z_0 \to z_v$.

This case applies to reflection from a fairly sharp boundary at the level z_0, the refractive index being nearly unity up to nearly this level. An example of this would be reflection from the sporadic-E region. The angular distance of the part of the ray path where μ departs appreciably from unity is small and so the ionosphere can be considered as essentially flat. This case was treated in section V. The results will be summarized here for completeness. Here $z_v = z_v'$ and

$$\tan \phi_0 = \frac{\sin \theta'}{\frac{z_v}{R+h} + 1 - \cos \theta'}. \tag{20}$$

In this case

$$\sqrt{1-\mu_0'^2} = \cos \phi_0 \left(1 - \frac{z_v - z_0}{R+h} \tan^2 \phi_0\right). \tag{20a}$$

For $z_v - z_0$ vanishingly small compared with $(R+h)\cot^2 \phi_0$, this reduces to the case for the plane ionosphere, where $\sqrt{1-\mu_0'^2} = \cos \phi_0$.

(c) $z_0 = \frac{1}{2} z_v$, i.e., a linear distribution of ionic density.

Here $1 - f_0^2/f^2 = 1 - z/z_0$ and (16a) becomes

$$\theta' = \frac{\sin \phi_1}{(R+h)\sqrt{1-\mu_0'^2}} \int_0^{z_0} \frac{dz\left(1 - \frac{2z}{R+h}\right)}{\sqrt{1 - A - \frac{z}{z_0}}}.$$

Putting in the value of $1-A$ and integrating

$$\theta' = z_v \frac{\sin \phi_1 \sqrt{1 + \frac{2z_0}{R+h}\tan^2 \phi_1}}{(R+h)\sqrt{1-\mu_0'^2}} \left(1 - \frac{2}{3}\frac{z_v}{R+h}\right). \tag{21}$$

Now since we may consider $z_0 \ll R+h$ we can write, as in (11b),

$$\sqrt{1-\mu_0'^2} = \cos \phi_1 \sqrt{1 + \frac{2z_0}{R+h}\tan^2 \phi_1}$$

and thus (21) becomes

$$\theta' = \frac{z_v \tan \phi_1}{R+h}\left[1 - \frac{2}{3}\frac{z_v}{R+h}\right] \tag{19}$$

just as in the case where $z_0 \ll (R+h)\cot^2 \phi_1$. We must, however, write for the equivalent vertical-incidence frequency in this case

$$f = f' \cos \phi_1 \sqrt{1 + \frac{2z_0}{R+h}\tan^2 \phi_1} \tag{21a}$$

instead of the value

$$f = f' \cos \phi_1 \left[1 + \frac{z_0}{R+h}\tan^2 \phi_1\right] \tag{18a}$$

which we could use when $z_0 \ll (R+h)\cot^2 \phi_1$.

The three examples just treated give an idea of what happens on transmission through the ionosphere, and of the angular distance θ' the wave travels in the ionosphere as a function of angle of incidence ϕ_1 (or vertex angle of equivalent triangular path ϕ_0), of true height of reflection z_0, and of virtual height measured at vertical incidence z_v. We must now consider the part of the path from the earth to the lower boundary of the ionosphere. If we consider the incident ray to traverse an angular distance $(\theta - \theta')$ in going from the earth to the lower boundary of the ionosphere, the geometry of Fig. 6 tells us that

$$\tan \phi_1 = \frac{\sin(\theta - \theta')}{\frac{h}{R} + 1 - \cos(\theta - \theta')}. \tag{22}$$

For a given ϕ_1, then, we may solve this equation for $(\theta - \theta')$, and, by adding this angle to the angle θ' already computed, we obtain the entire angular distance θ traversed by the wave from the ground to the point of reflection in the ionosphere. For $h = 100$ kilometers, as we are assuming, it is sufficiently accurate to replace $\sin(\theta - \theta')$ by $(\theta - \theta')$, and $1 - \cos(\theta - \theta')$ by $1/2\,(\theta - \theta')^2$ in this equation. We can thus solve for $(\theta - \theta')$, obtaining

$$\theta - \theta' = \cot \phi_1 - \sqrt{\cot^2 \phi_1 - \frac{2h}{R}}. \tag{22a}$$

We can now write the expression for the total distance of transmission D in terms of ϕ_1 (or ϕ_0) z_0 and z_v'

$$D = 2R\left(\cot \phi_1 - \sqrt{\cot^2 \phi_1 - \frac{2h}{R}} + \theta'\right) \tag{23}$$

where θ' is computed as above, from (16a).

The time T required for the sky wave to travel from the transmitter to the receiver is, if c = velocity of the wave in vacuum

$$T = \frac{2}{c}\left(R\frac{\sin(\theta - \theta')}{\sin \phi_1} + \int_A^B \frac{ds}{\mu}\right).$$

If we put in the value of $\int_A^B ds/\mu$ obtained on the basis of the above analysis, and replace $\sin(\theta - \theta')$ by its value in terms of ϕ_1, we may express T in terms of the quantities ϕ_1 (or ϕ_0), z_0, and z_v, as was done with D.

The expression for the height z_v' of the equivalent triangular path can now be written. From the geometry of Fig. 6,

$$\tan \phi_0 = \tan(\phi_1 - \theta') = \frac{\sin \theta'}{\frac{z_v'}{R+h} + 1 - \cos \theta'}$$

so that

$$z_v' = (R+h)[\sin \theta' \cot(\phi_1 - \theta') - 1 + \cos \theta']. \tag{24}$$

The relation between z_v, z_v', z_0, ϕ_1, ϕ_0, θ', D, f', and f may be summarized for the cases discussed here.

(a) $z_0 \ll 1/2(R+h)\cot^2\phi_1$.

$$f = f'\cos\phi_1\left[1+\frac{z_0}{R+h}\tan^2\phi_1\right]$$

$$\theta' = \frac{\tan\phi_1}{R+h}\left[1-\frac{z_0}{R+h}\tan^2\phi_1\right]\int_0^{z_0}\frac{dz}{\sqrt{1-\dfrac{f_0^2}{f^2}}}\left\{1+\frac{z_0}{R+h}\tan^2\phi_1\left[\frac{1-\dfrac{z}{z_0}}{1-\dfrac{f_0^2}{f^2}}\right]-\frac{2z}{R+h}\right\}$$

$$T = \frac{2}{c}\left[\frac{R(\theta-\theta')}{\sin\phi_1}+\frac{1}{\cos\phi_1}\int_0^{z_0}\frac{dz}{\sqrt{1-\dfrac{f_0^2}{f^2}}}\left\{1-\frac{z_0}{R+h}\tan^2\phi_1\left[1-\frac{1-\dfrac{z}{z_0}}{1-\dfrac{f_0^2}{f^2}}\right]\right\}\right]$$

$$\theta-\theta' = \cot\phi_1-\sqrt{\cot^2\phi_1-\frac{2h}{R}}$$

$$D = 2R\theta$$

$$z_v' = (R+h)[\sin\theta'\cot(\phi_1-\theta')-1+\cos\theta']$$

$$z_v = \int_0^{z_0}\frac{dz}{\sqrt{1-\dfrac{f_0^2}{f^2}}}$$

(a′) $z_0=0$.

$$f = f'\cos\phi_1$$

$$\theta' = \frac{z_v\tan\phi_1}{R+h}$$

$$T = \frac{2}{c}\left[\frac{R(\theta-\theta')}{\sin\phi_1}+\frac{z_v}{\cos\phi_1}\right]$$

$$\theta-\theta' = \cot\phi_1-\sqrt{\cot^2\phi_1-\frac{2h}{R}}$$

$$D = 2R\theta$$

$$z_v' = (R+h)[\sin\theta'\cot(\phi_1-\theta')-1+\cos\theta']$$

(b) linear gradient of ionic density ($z_0=1/2\ z_v$).

$$f = f'\cos\phi_1\sqrt{1+\frac{z_v}{R+h}\tan^2\phi_1}$$

$$\theta' = \frac{z_v\tan\phi_1}{R+h}\left[1-\frac{2}{3}\frac{z_v}{R+h}\right]$$

$$T = \frac{2}{c}\left[\frac{R(\theta-\theta')}{\sin\phi_1}+\frac{z_v}{R+h}\right]$$

$$\theta-\theta' = \cot\phi_1-\sqrt{\cot^2\phi_1-\frac{2h}{R}}$$

$$D = 2R\theta$$

$$z_v' = (R+h)[\sin\theta'\cot(\phi_1-\theta')-1+\cos\theta']$$

(c) $z_0=z_v$.

$$f = f'\cos\phi_0$$

$$\tan\phi_0 = \frac{\sin\theta'}{\dfrac{z_v}{R+h}+1-\cos\theta'} = \frac{\sin\theta}{\dfrac{z_v+h}{R}+1-\cos\theta}$$

$$T = \frac{2}{c}\frac{(R+h)\sin\theta}{\sin\phi_0}$$

$$D = 2R\theta$$

$$z_v' = z_v.$$

Case (a′) corresponds to the special case treated by Eckersley and Millington, except that, for simplicity of analytical computation, they took $R+h=R$ in the expression for θ', which introduces an error of about $0.016h$ per cent (h expressed in kilometers, not a serious error for $h=100$ kilometers, as we have assumed. This error becomes appreciable, however, if it is attempted to extend this treatment, as they have done, to a height of 400 kilometers or so.

Martyn's equivalence theorem, discussed above, tells us that for a plane ionosphere

$$z_v' = z_v$$

$$T = \frac{2}{c}\frac{z_v}{\cos\phi_0} = \frac{2}{c}\frac{z_v}{\cos\phi_1}$$

$$D' = 2z_v\tan\phi_0 = 2z_v\tan\phi_1$$

$$f = f'\cos\phi_0 = f'\cos\phi_1$$

where

ϕ_1 = angle of incidence of waves upon the ionosphere

ϕ_0 = half the vertex angle of the equivalent triangular path

z_v' = height of equivalent triangular path

T = time the wave spends in the ionosphere

D' = horizontal distance the wave travels in the ionosphere.

We see that the relation $z_v=z_v'$ is valid in case (c) but not in any other case. The relation $T=(2/c)(z_v/\cos\phi_1)$ is valid in cases (a′) and (b), but not otherwise in general. The relation $D'=2z_v\tan\phi_1$ holds approximately only in case (a′). And finally

$f=f' \cos \phi_0$ in case (c), $f=f' \cos \phi_1$ in case (a′), and f is a complicated function in every other case. It is therefore concluded that the equivalence theorem, in the form given, cannot be applied to the curved-earth problem.

VII. Transmission Curves for Curved Ionosphere and Earth

Referring again to the summaries of cases (a′), (b), and (c), we may for a given value of z_v take a set of values of z_0, for each of which can be calculated the variation of D with ϕ_1, in each of the three cases. We may also, for these values of z_v and z_0, calculate the variation of f'/f with ϕ_1. By eliminating ϕ_1 graphically, we can determine the variation of f'/f with D for a given z_v and z_0, or, what is the same thing, the variation of f'/f with z_v for a given D and z_0. We can plot a family of transmission curves, with f as abscissas and z_v as ordinates, for each value of D and several values of z_0, corresponding to each of these cases.

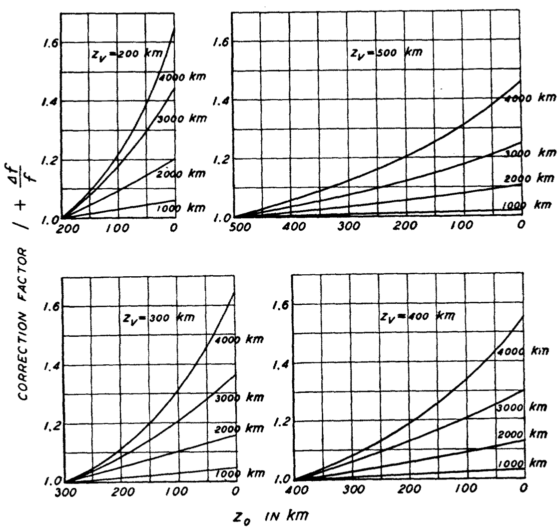

Fig. 9—Approximate factors by which the equivalent vertical-incidence frequency f must be multiplied to give the frequency $f'/\sec \phi_0$ which is to be used in conjunction with the log sec ϕ_0 curves.

Such curves may be superimposed on the (z_v, f) curve obtained at vertical incidence, and the (z_0, f) curve deduced therefrom, to give directly the wave frequency f' corresponding to reflection at a given level z_0 which is characterized by a given z_v. The value of z_0 appropriate to reflection at the given height determines which curve of the family for a given D is to be used, and the point of reflection is determined as the intersection of this transmission curve with the (z_v, f) curve.

The value of z_v' depends only on D and ϕ_1, and so lines of equal z_v' may be plotted on the curve sheet for each D, so that z_v', as well as f' may be read off directly.

Because of the approximations made in case (b) and the assumption of a linear variation of ionic density with height this case is of only special significance. It will be assumed, until further investigation determines more precisely the variation of conditions with z_0, that the curves vary smoothly between those calculated for $z_0=0$ and those calculated for z_0+z_v.

A family of curves for each distance is rather cumbersome for rapid use. It is, as was said above, more convenient to use the log sec ϕ_0 transmission curves, and apply a correction to the (z_v, f) curve by multiplying each vertical-incidence frequency by the factor $1+\Delta f/f$, where

$$1+\frac{\Delta f}{f}=\frac{\cos \phi_0}{\sqrt{1-\mu_0'^2}}.$$

This factor is obtained, for a given D, z_v, and z_0, by determining corresponding values of D and $\sqrt{1-\mu_0'^2}$ for arbitrary values of ϕ_0 or ϕ_1. It is unity for $z_v=z_0$ and is quite easily obtained for $z_0=0$. For intermediate values of z_0 it will be assumed that the factor $1+\Delta f/f$ varies in a manner similar to that determined from the relation (7b) with $z_v'=z_v$; i.e.,

$$\frac{\cos \phi_0}{\sqrt{1-\mu_0'^2}}=\sqrt{1-\frac{2(z_v'-z_0)}{R+h}\tan^2 \phi_0}$$

but drawn through the values for $z_0=0$ and $z_0=z_v$ determined in the more precise analysis, rather than those indicated on the assumption that $z_v=z_v'$. Cos ϕ_0 is here calculated for an equivalent triangular path of height z_v.

Fig. 9 gives the approximate factors by which f must be multiplied to give $f'/\sec \phi_0$ for values of z_v from 200 to 500 kilometers and for distances up to 4000 kilometers.

VIII. Effect of the Earth's Magnetic Field

The presence of the earth's magnetic field introduces some complications in the use of these transmission curves. These complications are often of minor importance compared with some of the unknown factors (e.g., the geographic uniformity of the ionosphere over the transmission path), especially over long distances, but the effect of the earth's field must at times be taken into account. The anisotropy of the ionosphere due to this field causes the effect of the field on radio transmission to vary with the length, direction, and geographic location of the transmission path.

One effect of the field is to cause the received signal to be split in general into two main components, the one with the lower maximum usable frequency known as the *o* component and the other as the *x* component. The refractive index for a frequency f, in the presence of a magnetic field H, whose components along and transverse to the direction of phase propagation are, respectively, H_L and H_T, is given[12] by

$$\mu' = 1 - \frac{f_0^2}{f'^2 - \dfrac{f'^2 f_T^2}{2(f'^2 - f_0^2)} \pm \sqrt{\left(\dfrac{f'^2 f_T^2}{2(f'^2 - f_0^2)}\right)^2 + f'^2 f_L^2}}$$

where

$$f_0 = \sqrt{\frac{Ne^2}{\pi m}}$$

$$f_T = \frac{e}{2\pi mc} H_T$$

$$f_L = \frac{e}{2\pi mc} H_L$$

N = ionization density.

The upper sign refers to the *o* component; the lower to the *x* component. The frequency of a wave whose *x* component is returned from a given ionization density, at a given height, is different from the frequency of the wave whose *o* component is returned from the same level. This frequency separation is in general a function both of frequency and distance, and may often become negligible at great distances.

For practical calculation it may be assumed that only the field and direction of wave propagation in the region near the level of reflection will appreciably affect the propagation of the wave. This assumption is probably better for the *o* than for the *x* component, and is, it must be emphasized, only a good approximation.

With this limitation, therefore, the value of μ'^2 may be calculated for a given transmission frequency and transmission path. If we put $\mu' = \sin\phi_0$ and deduce z_v from $\sin\phi_0$ as was done in the previous paper, we may plot transmission curves, of virtual height against effective normal-incidence frequency, for the *o* and *x* components. When these curves are applied to the corrected normal-incidence-frequency—virtual-height curves, they may be expected to give reasonably good results. An example of this type of transmission curve is shown in Fig. 10. The curve marked *x* gives transmission conditions for the *x* component, and that marked *o* for the *o* component. The frequency used is well over the gyrofrequency $f_H = eH/2\pi mc$ so that the *x* component is returned from a lower level than is the *o* component and has a higher limiting frequency.

It is not now justifiable to plot the transmission curves logarithmically, since the form of the curves will vary with the transmission frequency. For practical purposes, however, a logarithmic curve may be used within a limited range of frequencies about the frequency for which the curve is plotted; a practical limit might be, say, within ±15 per cent of this frequency.

The logarithmic sec ϕ_0 transmission curves may be used in estimating the maximum usable frequency

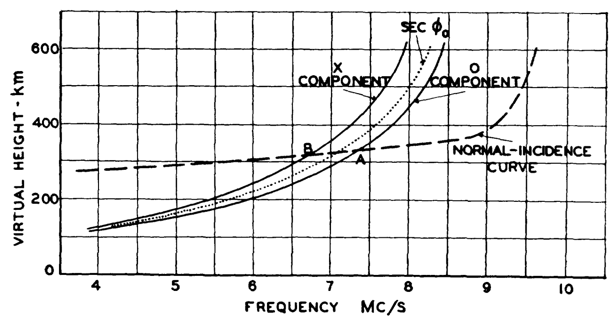

Fig. 10—Transmission curve, including effect of earth's magnetic field at level of reflection. Dotted line = secant-law curve ($f' = f\sec\phi_0$) for the given distance D and wave frequency f'. Dashed line = (z_v, f) curve for the *o* component, this being the component which, at vertical incidence is a measure of the ionization density N. The *o* component of the frequency f' over the given distance is reflected at A, and the *x* component at B.

for each component over a given path by adding or subtracting the separation between the limiting frequencies for the two components, evaluated at that frequency and distance. This separation is in general a function only of the transmission frequency and the quantity sec ϕ_0 and may be estimated within the limits of experimental error in most cases.

Fig. 11 gives the frequency to be added to or subtracted from the maximum usable frequency given by the logarithmic sec ϕ_0 transmission curves for the cases $H_L = 0$, $H_T \doteq 0$, and $H_T = H_L$. The $H_T = 0$ case is uninteresting save for single-hop transmission over the magnetic equator, close to the magnetic meridian. For transmission in the continental United States H_L is much less than H_T and, indeed, is negligible over east-west paths, so that such transmission we may consider as essentially transverse transmission. In this case

$$\mu'^2 = 1 - \frac{f_0^2}{f^2 - \delta\dfrac{f^2 f_H^2}{f^2 - f_0^2}}$$

where $\delta = 0$ for the *o* component and 1 for the *x* component, and the logarithmic sec ϕ_0 transmission

[12] E. V. Appleton, "Wireless studies of the ionosphere," *Jour. I.E.E.* (London), vol. 71, pp. 642–650; October, (1932).

curves give the correct maximum usable frequency for the o component. For this reason the o component lies along the sec ϕ_0 axis in the $H_L=0$ case in Fig. 11.

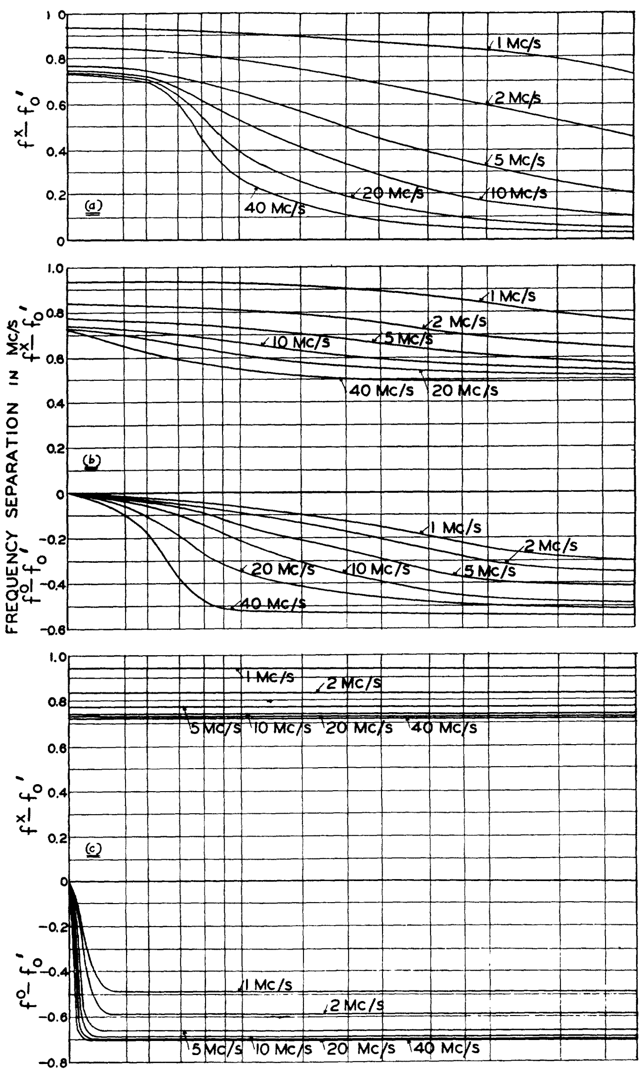

Fig. 11—$f^x - f_0'$ and $f^0 - f_0'$ plotted against sec ϕ_0. f^x = frequency whose x component is reflected at the same level as is the o component of f^0. $f_0' = f'$ cos ϕ_0. $H=\sqrt{H_T^2+H_L^2}=0.5$ gauss. The value of f' is given, in megacycles, on each curve.
(a) $H_L=0$. Here $f^0-f_0'=0$ and the curves for the o component lie on the sec ϕ_0 axis.
(b) $H_L=H_T\neq 0$.
(c) $H_T \doteq 0$. Here f^x-f_0' and f^0-f_0' are independent of sec ϕ_0, save near sec $\phi_0=1$.

For a maximum usable frequency for the o component below the gyrofrequency $f_H=\sqrt{f_T^2+f_L^2}$ the x component always has a maximum usable frequency above f_H. This is only important for E-layer transmission and only in cases where the x component is not too highly absorbed at frequencies near f_H. This case must not be confused with the case of a transmission frequency f' less than f_H in which case the x component is reflected from a level above the level where the o component is reflected.

Another effect of the anisotropy of the ionosphere due to the earth's field is to cause a difference in the directions of phase and energy propagation in the medium. This results in the wave's being reflected, not at the level where the direction of phase propagation is horizontal, but where the direction of energy flow (group direction) is horizontal. This effect has not been considered in these curves, and is probably not important to the degree of accuracy to which these calculations are carried.

IX. Behavior of the Wave Below the Point of Reflection

The development thus far has been given for any distribution of ionization in the ionosphere below the level of reflection. The effect on the reflection of the passage of the waves through the lower ionosphere, and for that matter through the lower atmosphere itself, has been taken care of by using the normal-incidence virtual height in the calculation, since this effect enters into the measurement of this height. Nothing has been said thus far, however, about phenomena that happen below the level of reflection.

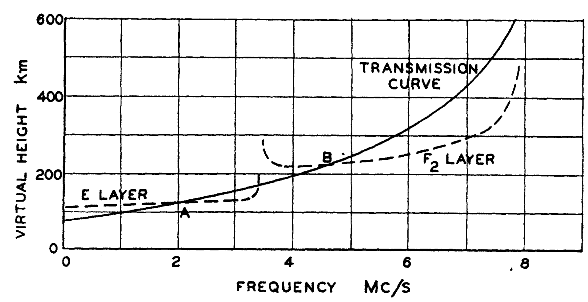

Fig. 12—Transmission by way of both E and F_2 layers.
A = point of reflection on E layer.
B = point of reflection on F_2 layer.
Solid line = transmission curve.
Dashed line = (z_v, f) curve.

An example of such a complication is given in Fig. 12. Here the transmission curve crosses both the E and the F_2 virtual-height curves. Therefore transmission may be expected by way of each layer. The angle of incidence is not the same for each layer, since the virtual heights of the layers are different. The wave reflected from the higher layer has a smaller angle of incidence upon the lower layer than does the wave reflected from the lower layer. Under certain conditions, then, it can penetrate the lower layer, and the waves can reach the receiver by way of the higher as well as the lower layer.

When the critical frequency of the lower layer is higher than the equivalent vertical-incidence frequency for the wave which would be returned from the upper layer, the wave will not penetrate the lower layer, but will reach the receiver by way of the lower layer only. Furthermore, if this equivalent vertical-incidence frequency is not below the critical frequency for the lower layer, but does lie in a region of appreciable absorption in this layer, the wave will be returned from the higher layer, but will be appreciably absorbed in so doing. Experimental observations indicate the absorption of the actual wave to be roughly the same as the absorption of a wave of the equivalent vertical-incidence frequency measured at vertical incidence.

It is possible to plot, on the sec ϕ_0 transmission curves, lines corresponding to the values of sec ϕ_1 (see Fig. 6) for different heights. Such a set of "absorption lines" is shown in Fig. 13. When the transmission curve is superimposed on the (z_v, f) curve the behavior of a wave below the reflection level may be estimated by the region of the (z_v, f) curve sheet through which the absorption line through the reflection point passes. If this line passes through a region of absorption or cuts a lower layer the wave will be absorbed or will not penetrate through to the higher layer.

These absorption lines are the lines sec ϕ_1 = constant for a plane earth and for short distances on a curved earth. They curve toward larger values of sec ϕ_1 for lower heights in the case of greater distances over a curved earth, and approach the transmission curve itself for great distances. An example of the use of the transmission curves and absorption lines is shown in Fig. 14, which gives vertical-incidence curves and the corresponding frequency-height curves derived therefrom for the transmitted wave over a distance. From these latter curves can be deduced, of course, the equivalent paths or group retardation of the waves over the transmission path.

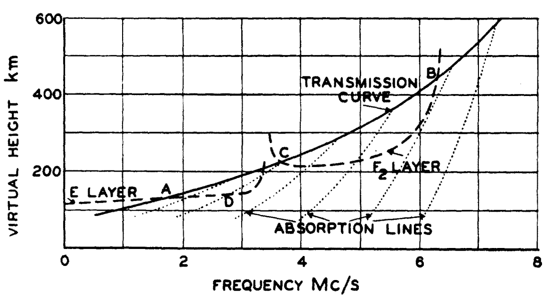

Fig. 13—Absorption lines on transmission curve.
Dashed line = (z_v, f) curve.
Dotted lines = absorption lines.
E reflection takes place at point A and F_2 reflection at point B.
F_2 reflection at C is shielded by the E layer at D.

X. Angle of Departure and Arrival of the Waves

Disregarding any possible asymmetry of the wave trajectory due to the earth's magnetic field the angles of departure of the waves from the transmitter and of arrival of the waves at the receiver are equal, assuming one ionosphere reflection. The amount of energy which is radiated from the transmitter at a given angle from the horizontal depends, especially for low angles, on the design of the transmitting antenna, and to a great extent on the nature of the terrain surrounding the transmitter. Similar factors affect the energy absorbed from the wave at the receiving station.

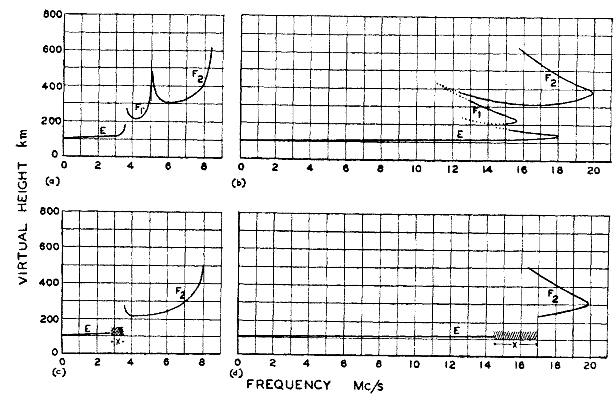

Fig. 14—Virtual heights at vertical and oblique incidence.
(a) (z_v, f) curve showing E, F_1, and F_2 layers.
(b) oblique-incidence heights corresponding to (a). Dotted lines show parts of curves shielded by E layer. Note transmission of some frequencies by several layers. This shows how the F_1-layer reflections are relatively unimportant for transmission.
(c) (z_v, f) curve showing absorption (X) above critical frequency for the E layer.
(d) oblique-incidence heights corresponding to (c). Absorption here (X) completely blocks out some frequencies.
All curves are for the *o* component.

For a station located on or near the ground no energy will be radiated below the horizontal, and but little until the angle ψ of departure of the waves above the horizontal becomes appreciable. The minimum value of ψ at which sufficient energy is radiated (or received) to produce a readable signal varies with the terrain, the power of the transmitter, and the sensitivity of the receiver. Over sea water ψ can be very nearly zero; over land the minimum value may be several degrees. A fair average approximation may be that ψ must exceed about $3\frac{1}{2}$ degrees.

A simple geometrical calculation gives ψ in terms of sec ϕ_0 for various distances, and these may be noted on the transmission curves. The point of reflection must then correspond to an angle ψ greater than the minimum value assumed for the terrain at the receiver and transmitter.

The effect of this limitation is to limit the maximum distance for single-reflection transmission to about 1750 kilometers (over land) for the E layer and about 3500 to 4000 kilometers (over land) for F_2-layer transmission. Where the transmission path exceeds these values in length calculations must be made on the basis of multireflection transmission.

The condition of the part of the ionosphere traversed by the wave determines the behavior of the wave. In calculating transmission conditions, therefore, the ionosphere around the middle of the path, for single-reflection transmission, and around the middle of each jump, for multireflection transmission, must be considered. The latter requirement means that usually, for practical calculations one must consider the condition of ionosphere over all the transmission path except a few hundred kilometers near the transmitter and receiver. This question has been discussed in detail elsewhere.[13] It must also be kept in mind that at times the *o* and *x* components are reflected from different geographical parts of the ionosphere.

XI. Determination of sec ϕ_0

Fig. 15 gives an alignment chart for the rapid determination of the factor sec ϕ_0 to be used in calculating the logarithmic transmission curves. To use the chart, place a straightedge so that it passes through the desired virtual height and the desired distance laid off on the distance scale at the lower left-hand edge of the chart (increasing distances lie to the left). The ordinate of the intersection of the straightedge with the vertical line corresponding to the same desired distance laid off on the main distance scale (increasing distances lie to the right) gives the value of sec ϕ_0. The relation of the point of intersection to the curved dashed lines of equal ψ gives the value of the angle of departure of the waves from the horizontal. A point of intersection falling above the $\psi=0$ degrees line indicates an impossible case, where the ray would have to depart at an angle below the horizon.

For example, a distance of 2400 kilometers and a virtual height of 300 kilometers corresponds to a sec ϕ_0 of 3.07, and an angle of departure of 8.02.

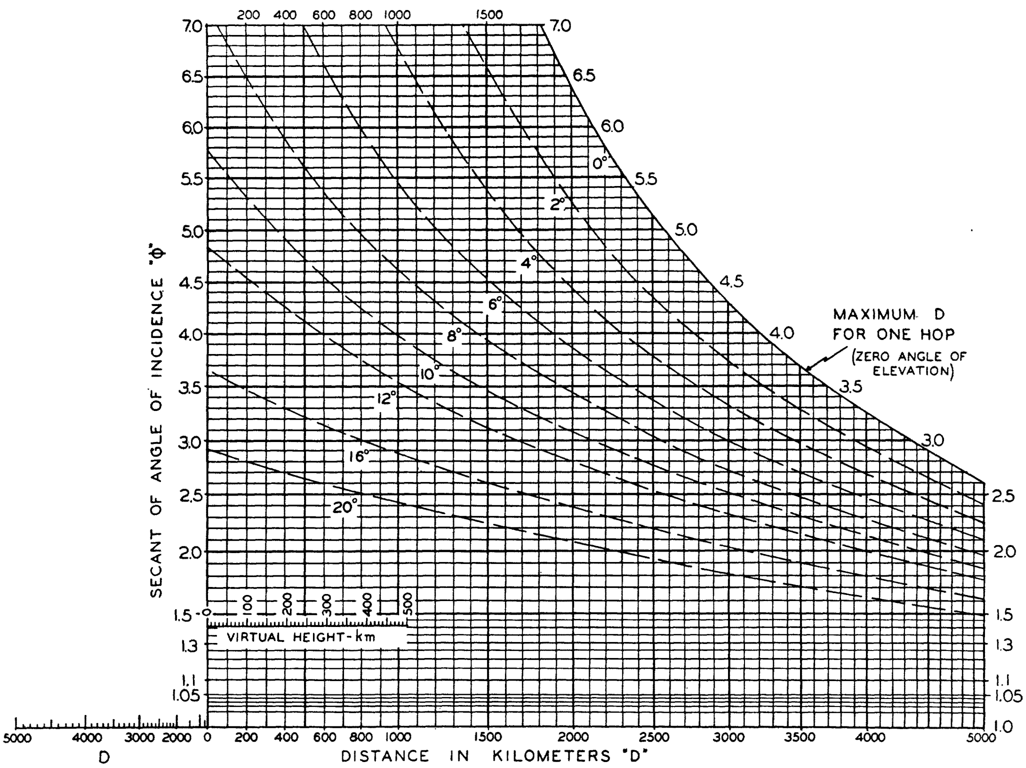

Fig. 15—Alignment chart for determining sec ϕ_0 and ψ, the angle which the ray makes with the horizontal at the transmitter or receiver. The curved dashed lines are lines of equal ψ.

[13] T. R. Gilliland, S. S. Kirby, N. Smith, and S. E. Reymer, "Characteristics of the ionosphere and their applications to radio transmission," Proc. I.R.E., vol. 25, pp. 823–840; July, (1937); *Nat. Bur. Stand. Jour. Res.*, vol. 18, pp. 645–668; June, (1937); the weekly radio broadcasts of the National Bureau of Standards on the ionosphere and radio transmission conditions; Letter Circular LC499.

XII. Transmission Factors

When average transmission conditions over a period of time or when a variety of transmission paths are to be considered, or when an estimate of the maximum usable frequencies is to be made without a precise knowledge of the ionosphere over the transmission path, it is convenient to have available a means by which the maximum usable frequencies may be quickly estimated from an approximate value of the vertical-incidence critical frequency.

The National Bureau of Standards is now beginning a compilation of factors by which the critical frequency for the *o* component, measured at vertical incidence, may be multiplied in order to obtain the maximum usable frequencies. These factors are based on average observations over a period of time, and may be applied either to average critical frequencies to give average transmission conditions, or to a given observation of a critical frequency to obtain approximate transmission conditions at a given time.

XIII. Conclusions

The type of transmission curves described above has been in use at the National Bureau of Standards for the past two years in studying the correlation of high-frequency radio transmission conditions with regular ionosphere observations. The results of continuous measurements of high-frequency broadcast stations and observations on other high-frequency signals, as well as the results of some specific experiments have been compared with vertical-incidence data. Practically all the available data agree with what would be expected on the basis of the theory outlined above, and the exceptions may in most cases be accounted for.