Limited Live-time Measurements of Frequency Stability *

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1. Abstract

For common FM noises in oscillators, discontinuous measurements with dead-time, small sample statistics, and an assumed χ^2 distribution, the RMS frequency fluctuations vs. time-interval may be the only reportable measure of frequency stability. We show in a group of simulation trials that in a typical experimental scenario, the RMS frequency stability can seriously underestimate true flicker frequency (FLFM) and random-walk frequency (RWFM) noise and can have larger long-term uncertainty with respect to the zero dead-time Allan deviation and should not be reported as Allan deviation.

2. Introduction and Summary

This paper offers insights into the problem of interpreting the RMS frequency stability with limited live-time frequency measurements, that is, measurements having dead-time of more than 30%. Consider a pair of clocks or oscillators being compared which run continuously, however, measurement live time is a τ averaged frequency difference but the duration between these averages is T and thus no measurement is made during a dead time of $T - \tau$. A result is summarized as the square root of the mean-square of the frequency differences with a corresponding duration T. Thus the root-mean-square frequency error is a measurement of RMS frequency change vs. T and is presumed to be a suitable substitute for $\sigma_y(\tau)$, the Allan deviation. We show in a group of simulation trials involving common FM noises, dead-time, small sample statistics, and an assumed χ^2 distribution that this procedure underestimates the FLFM and RWFM noise level.

Measurements of frequency stability with dead-time are biased relative to the zero-dead-time Allan deviation for stochastic noise, and the usual relationship to the fractional-frequency spectral density is often lost. Although dead-time $T - \tau$ bias for given FM noise types can be removed in principle, it is often not

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removed because this is tedious and is easily susceptible to guesswork regarding the validity of important assumptions. For example, the selection of noise type to use for determining bias is obscured by degraded confidence intervals, a major problem in the presence of FLFM and RWFM.

Results may be reported as an RMS frequency change vs. T instead of as an Allan deviation plot. Even though this characterization of frequency stability might have been determined using limited livetime measurements, it may still be useful for some applications. However, it is not the Allan deviation and usually is not a fair estimate of characteristic oscillator FM noise level.

3. RMS Frequency Stability: $\psi_y(\tau, T)$

We define a generic two-sample frequency variance and compare it to the two-sample Allan variance. Frequency instability is generally regarded as an uncertainty on an oscillator's expected or predicted average frequency. At long averaging times, the dominant component of frequency prediction is often the error due to linear frequency drift. In general, a sample estimate of linear frequency drift between two oscillators is $\frac{\Delta^\tau \bar{y}(t)}{T}$ where $\Delta^\tau \bar{y}(t)$ is a *change* in a pair of measured values of frequency offset $\bar{y}_n$ and $\bar{y}_{n+1}$ separated by T, the span of time over which the change occured. τ is the averaging time used to compute each value of frequency offset. The Allan deviation $\sigma_y(\tau)$ is $(\Delta^\tau \bar{y}(t))_{rms}$ and division by a time interval τ is implied because *adjacent* values of $^\tau \bar{y}(t)$ must be used by definition, making $T = \tau$ [1]. Thus the Allan deviation can be interpreted as an uncertainty of a τ-sample estimate of systematic linear frequency drift. Some experimentalists measure quantities such as "rms frequency deviation," labeled as $(\Delta f)_{rms}$ vs. T and mean the RMS of measured values of $\left(\frac{\Delta \bar{y}(t)}{T}\right)$. The fact that $\tau < T$ is not an important consideration or is constrained by other measurement factors. In otherwords, some experimentalists often desire the same drift uncertainty measure as the Allan deviation, so compute $\left(\frac{\Delta \bar{y}(t)}{T}\right)_{rms}$. We will define this experimentalist's statistic more carefully as the

square root of $\psi_y^2(\tau,T)$, or psi-variance. Its square root, psi-deviation $\psi_y(\tau,T)$ is often taken to be the root Allan variance $\sigma_y(T = \tau)$ thinking that this function is close enough to serve as a counterpart. At the very least, it is easily confused with the Allan deviation $\sigma_y(\tau)$ which strictly is $(\Delta^\tau \bar{y}(t))_{rms}$ associated with $T \neq \tau$ and corrected based on ratio $r = \frac{T}{\tau}$. The effect of dead-time is always somewhat obscured by the restriction that the ratio r of dead-to-live time be constant. This restriction is necessary so that integer power-law noise processes have a constant slope (on a log-log scale) corresponding to a constant slope in the frequency domain [1,2]. $\psi_y^2(\tau,T)$ is a convenient experimental characterization in which the live measurements are a constant τ and the dead time is a free parameter $T - \tau$.

We show in simulation that psi-variance $\psi_y^2(\tau,T)$ is a suitable substitute for $\sigma_y^2(T)$ in the presence of white frequency (WHFM) noise but that its level is always too low with FLFM and RWFM noises which are likely to occur in long term. Coupled with the fact that these same long-term estimates may derive from only one or two measurements, they are subject to a negatively skewed probability distribution, hence the reported frequency stability may be significantly low.

4. Mathematical Details and Dilemma

Measurement samples of the time-error function $x(t)$ occur at a rate f_s having an interval $\tau_0 = \frac{1}{f_s}$. Given a sequence of time errors $\{x_n : n = 1, \ldots, N_x\}$ with a sampling period between adjacent observations given by τ_0, we define the $m\tau_0$-average fractional-frequency deviate as

$$\bar{y}_n(m) \equiv \frac{1}{m} \sum_{j=0}^{m-1} y_{n-j},$$

where $y_n = \frac{1}{\tau_0}(x_n - x_{n-1})$. Define psi-variance

$$\psi_y^2(\tau,T) = \frac{\tau}{T} < [^\tau \bar{y}(t) - ^\tau \bar{y}(t-T)]^2 >, \quad (1)$$

where $< \cdot >$ denotes an ensemble average and $^\tau \bar{y}(t)$ is the mean frequency over duration $\tau = m\tau_0$. Thus, in terms of $x(t)$, $^\tau \bar{y}(t) = \frac{x(t)-x(t-\tau)}{\tau}$. Definition (1) is based on taking sequential mean frequency measurements spaced T apart, differencing them, and computing the mean square. Figure 1 shows the sampling function associated with $\psi_y^2(\tau,T)$ acting on $\{y_n\}$. τ is called the averaging time and $T - \tau$ is the measurement dead time. $\psi_y^2(\tau,T)$ becomes twice the two-point standard (Allan) variance if $\tau = T$ [1]. $\psi_y^2(\tau,T)$ expressed in terms of the time-error function $x(t)$ is

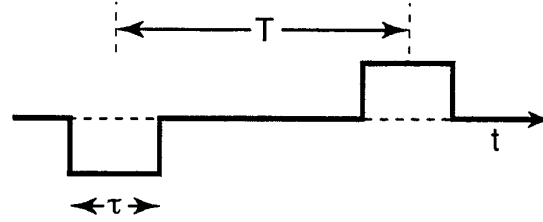$$\psi_y^2(\tau,T) = \frac{1}{\tau T} \psi_x^2(\tau,T), \quad (2)$$

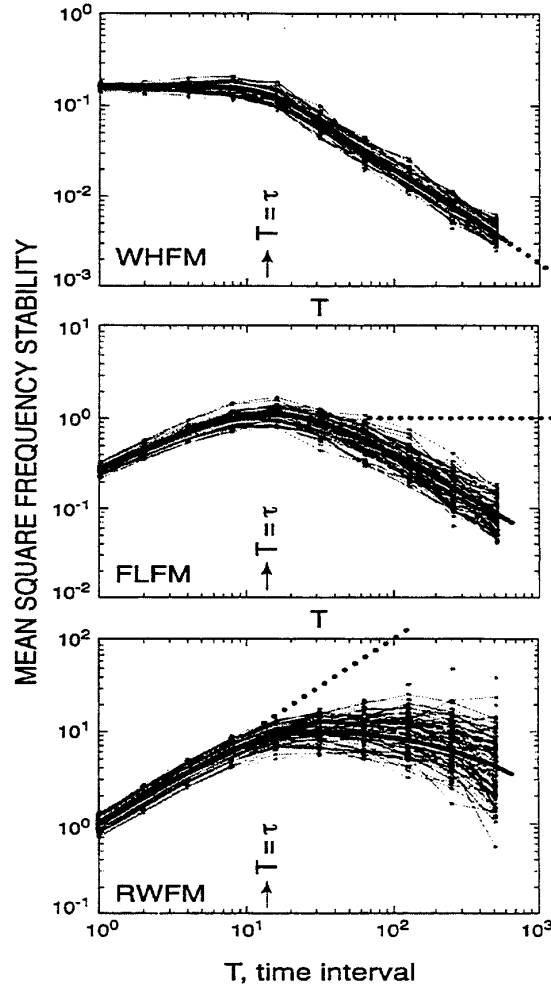Figure 1: Measurement sequence for mean-square measurements of frequency stability $\psi_y^2(\tau,T)$.

Figure 2: Mean-square frequency fluctuations ψ_y^2 with respect to T of 100 simulations of FM noises whose mean is the solid curve. $T > \tau$ to the right of the arrow are limited live-time measurements, dead time $= T - \tau$. Dashed line is the expected result if the zero dead-time Allan variance were used.

where $\psi_x^2(\tau,T) = < \{[x(t+\frac{\tau}{2})-x(t-\frac{\tau}{2})]-[x(t-T+\frac{\tau}{2})-x(t-T-\frac{\tau}{2})]\}^2 >$, the mean-square time error for Doppler radar [1] rewritten as a central difference. (We are not relating this writing to Doppler radar.) Thus,

$$\psi_y^2(\tau,T) = \frac{1}{\tau T}\left\langle \{[x(t+\frac{\tau}{2})-x(t-\frac{\tau}{2})]\right.$$
$$\left. -[x(t-T+\frac{\tau}{2})-x(t-T-\frac{\tau}{2})]\}^2\right\rangle. \quad (3)$$

How do we interpret "$\frac{1}{\tau T}$" in (3) when converting from a mean-square time error $\psi_x^2(\tau,T)$ to a mean-square frequency error $\psi_y^2(\tau,T)$? Terms in (3) can be rearranged and written as

$$\psi_y^2(\tau,T) = \frac{1}{\tau T}\left\langle \{[x(t+\frac{\tau}{2})-x(t-T+\frac{\tau}{2})]\right.$$
$$\left. -[x(t-\frac{\tau}{2})-x(t-T-\frac{\tau}{2})]\}^2\right\rangle. \quad (4)$$

This can be interpreted to mean that frequency differences that are *overlapped* by $T-\tau$, squared, averaged, and divided by T produce the same value as frequency differences that are *separated* by $T-\tau$, squared, averaged, and divided by τ. In the frequency domain, (3)-(4) act on the same underlying spectrum with two passband filters. The responses of (3) and (4) are identical but interpreted as filters centered at two different Fourier frequencies corresponding to $f_\tau = \frac{1}{2\tau}$ and $f_T = \frac{1}{2T}$.

Averages of frequency error are often made with constant τ and varying T, the result being reported as an RMS frequency error corresponding to T. At this point, there are several intricate issues involved in estimating FM noise level. For example, there is a cutoff frequency f_c between a maximum of f_s (the sampling frequency) and $\frac{1}{2m\tau_0}$ (half the reciprocal of the τ average used in determining $\bar{y}_n(m)$). There is also a measurement-system high-frequency cutoff f_h which must be accounted for. Most of the determinations of FM noise level with dead time depend on an assumption that this high-frequency cutoff f_h is modeled as a rectangular or "brickwall" filter. In theory, the relationship among the parameters f_τ, f_T, f_s, f_c, and f_h is known, but calculating noise levels and simultaneously verifying that the assumptions remain valid for a given experiment can make for an arduous task [2-5].

5. Simulation Study

For more direct insights into statistical errors from dead time, we used simulation trials acting on defined FM noises (WHFM, FLFM, and RWFM). We are in a position with (3) to measure the effect on $\psi_y^2(\tau,T)$ of varying T with respect to a constant τ. Figure 2 shows the result of $\psi_y^2(\tau,T)$ in simulation trials in which T is varied with a fixed value of τ indicated at
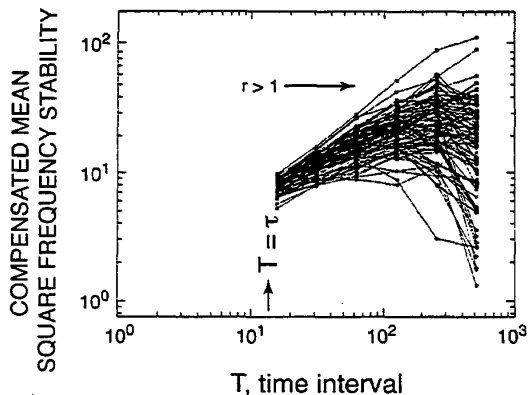

Figure 3: Compensated mean-square frequency fluctuations with respect to T of 100 simulations of RWFM noise (compensated for dead-time $r > 1$ bias). Note that the spread in the distribution is significant and can often indicate two possible FM noise types.

the arrow, that is, where $T = \tau$. The arrow points to $\psi_y^2(\tau,T) = 2\sigma_y(\tau)$, or twice the Allan variance. Now let $r = T/\tau$. To the right of the arrow are r-ratios corresponding to dead-time measurements $T > \tau$. Notice that if T is reported as if it were "τ" used in the Allan variance, then values will be underneath the expected τ^μ slopes indicated by the dashed lines with FLFM and RWFM. For WHFM, the result is not biased low, but the problem is that it would be an easy mistake to *interpret* the slope as WHFM even though the characteristic noise type is actually FLFM, or FLFM if the noise type is RWFM.

6. Correcting for Dead-time Bias

The proper procedure in experimental design is to fix r and then adjust the associated $\psi_y^2(\tau,T)$ result to remove bias in order to estimate $\sigma_y^2(\tau = T)$ due to r. Then translations to the FM noise coefficients h_α are straightforward [6]. But if r cannot be held constant, then an adjustment must be made for each estimate and all estimates subsequently averaged, a potentially tedious task.

The bias is determined from the fact that the variance of the process function $y(t)$ increases in proportion to $\ln T - \ln \tau$ and $T - \tau$ for FLFM and RWFM respectively and can be expressed in terms of r [1]. Compensating for this, we have

Flicker-y FLFM: $\psi_y^2(\tau = T) \propto (1 + 3\ln r) \cdot \psi_y^2(\tau,T)$,

for $1 < r < \sim 100$ and

Random-walk-y RWFM: $\psi_y^2(\tau = T) \propto r \cdot \psi_y^2(\tau,T)$.

Applying this compensation to the dead-time region $r > 1$ to simulation trials of RWFM noise yields the results illustrated in figure 3. Now T indeed can be used as "τ" in the Allan variance but there remains the problem of the spread in the distribution. For accurate measurements of long-term FM noise level, we may be unable to determine FM noise type because as many as all three noise types may be consistent with the error bar uncertainty at long term associated with a given measurement.

7. Uncertainty of Estimation

Dead time is likely to be a serious issue when we can make only a limited number of measurements. Lesage and Audoin were the first to theoretically consider the problem of small samples on estimating the Allan variance [8]. Yoshimura showed that the probability distribution in cases with dead time $r > 1$ is governed by χ^2 statistics for the FM noises considered here (but not PM noises) [9]. This means that in the case of limited numbers of measurements, the results are more likely to be below the underlying characteristic oscillator noise level than above it. In the case of one sample as in the very longest T value, the result is more than twice as likely to be too low. Furthermore, its most probable value is 0.

8. "First Results" are Likely to be Excellent

When faced with significant periods of dead time between separated measurements of average frequency difference between two oscillators, it is common to assign a confidence interval to each frequency measurement and to call the absolute value of the overall frequency difference an estimate of the Allan deviation. In this case, should frequency stability still be reported as an Allan deviation? No, because the uncertainty of $\psi_y(\tau, T)$ with significant dead time and only one or two estimates is usually greater than the uncertainty on each frequency measurement. Depending on the assumed power-law noise type, the uncertainty on each frequency measurement will usually underestimate the uncertainty of $\psi_y(\tau, T)$ [10]. Even if the ratio of live to dead time is held constant, long-term uncertainty is a major concern in noise typing. Should frequency stability be reported as an Allan deviation with bias removed? No, because this study shows that with limited live-time measurements, the RMS frequency stability in long-term is likely to fall below the actual noise level, and we can no longer reliably judge the noise type based on slopes of frequency stability $vs.\ \tau$ and finally, cannot feasibly correct for this because of increased uncertainty.

9. References

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